Vector calculus MA2VC and MA3VC 2015–16: Assignment 2 SOLUTIONS

(Exercise 1 — 6 marks) *Consider the square* $Q = (0,1)^2 = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 1, 0 < y < 1\}$ *and the change of variables*

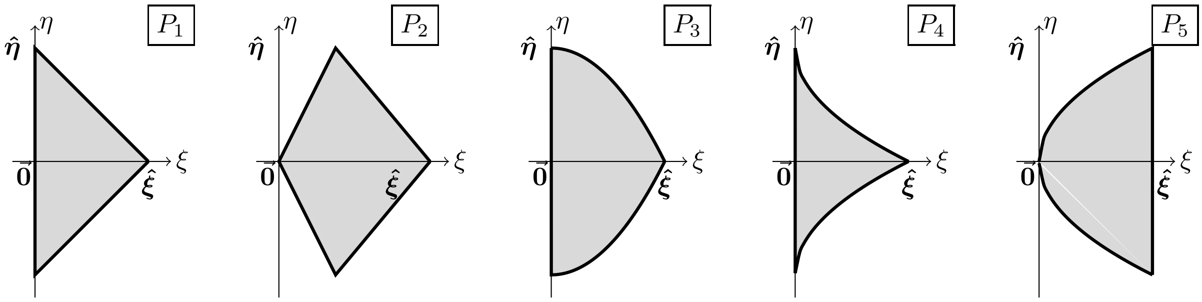
$$\vec{\mathbf{T}}(x,y) = \xi(x,y)\hat{\boldsymbol{\xi}} + \eta(x,y)\hat{\boldsymbol{\eta}} \qquad \textit{where} \qquad \xi(x,y) = xy, \qquad \eta(x,y) = y^2 - x^2.$$

1. *Compute the area of the transformed region* $\vec{\mathbf{T}}(Q)$.

   Hint: Recall example 2.28 in the notes.

2. *Which of the following regions corresponds to* $\vec{\mathbf{T}}(Q)$*? Justify your answer.*

   Hint: the equations of the sides of $\vec{\mathbf{T}}(Q)$*, obtained from those of the four sides of* Q*, may help.*



(1.) We compute the Jacobian determinant and the area as the integral of the constant field $f = 1$:

$$\frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -2x & 2y \end{vmatrix} = 2y^2 + 2x^2,$$

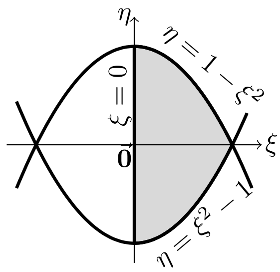
$$\text{Area}(\vec{\mathbf{T}}(Q)) = \iint_{\vec{\mathbf{T}}(Q)} 1 \,\mathrm{d}\xi \,\mathrm{d}\eta = \iint_Q \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| \mathrm{d}y \,\mathrm{d}x = \int_0^1 \int_0^1 (2x^2 + 2y^2) \,\mathrm{d}y \,\mathrm{d}x = \int_0^1 \Big(2x^2 + \frac{2}{3}\Big) \,\mathrm{d}x = \boxed{\frac{4}{3}}.$$

(2.) To understand the shape of $\vec{\mathbf{T}}(Q)$ we first compute its vertices by substituting the coordinates of the four vertices $\vec{\mathbf{0}}, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}, \hat{\boldsymbol{\jmath}}$ of the square Q in the change of variables:

$$\vec{\mathbf{T}}(\vec{\mathbf{0}}) = 0\hat{\boldsymbol{\xi}} + (0-0)\hat{\boldsymbol{\eta}} = \vec{\mathbf{0}}, \qquad \vec{\mathbf{T}}(\hat{\boldsymbol{\imath}}) = 0\hat{\boldsymbol{\xi}} + (0-1)\hat{\boldsymbol{\eta}} = -\hat{\boldsymbol{\eta}},$$
$$\vec{\mathbf{T}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) = 1\hat{\boldsymbol{\xi}} + (1-1)\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\xi}}, \qquad \vec{\mathbf{T}}(\hat{\boldsymbol{\jmath}}) = 0\hat{\boldsymbol{\xi}} + (1-0)\hat{\boldsymbol{\eta}} = \hat{\boldsymbol{\eta}}.$$

This rules out figures P_2 and P_5 whose boundaries do not contain $\pm\hat{\boldsymbol{\eta}}$. To decide between the remaining figures, we compute the image under $\vec{\mathbf{T}}$ of the lines of the four edges of Q:

$$\begin{aligned} \{x = 0\} &\mapsto \{\xi = 0\}, \\ \{y = 0\} &\mapsto \{\xi = 0\}, \\ \{x = 1\} &\mapsto \{\xi = y,\ \eta = y^2 - 1\} = \{\eta = \xi^2 - 1\}, \\ \{y = 1\} &\mapsto \{\xi = x,\ \eta = 1 - x^2\} = \{\eta = 1 - \xi^2\}. \end{aligned}$$



We see that two sides are mapped to the vertical line $\{\xi = 0\}$, which is part of the boundary of the regions in P_1, P_3 and P_4. The third line $\{x = 1\}$ becomes $\{\eta = \xi^2 - 1\}$, i.e. the graph of the parabola $\eta = \xi^2 - 1$, which is the lower side of the region in P_3, see figure above. Similarly $\{y = 1\}$ becomes $\{\eta = 1 - \xi^2\}$ i.e. the parabola at the upper side of the same region. Thus the answer is $\boxed{P_3}$.

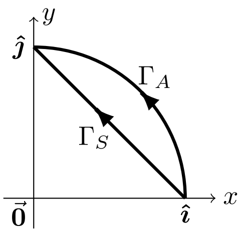
Alternatively, (after ruling out P_2 and P_5) one can verify that the point $\vec{\mathbf{p}} = \frac{1}{2}\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}$ on the boundary of Q (the mid point of the upper side) is mapped to $\vec{\mathbf{T}}(\vec{\mathbf{p}}) = \frac{1}{2}\hat{\boldsymbol{\xi}} + \frac{3}{4}\hat{\boldsymbol{\eta}}$. This point lies above the straight line $\eta = 1 - \xi$ through the points $\hat{\boldsymbol{\xi}}$ and $\hat{\boldsymbol{\eta}}$, so the upper side of $\vec{\mathbf{T}}(Q)$ must be convex (graph of a concave function).

Another alternative solution is to recall that the area of $\vec{\mathbf{T}}(Q)$ is $4/3 > 1$, as computed in the first part of the exercise, while $\text{Area}(P_1) = 1$ and $\text{Area}(P_4) < 1$, as they have the same vertices of P_3.

You can visualise the change of coordinates with the Matlab function VCplotter (available on the course web page) with the command: `VCplotter(6, @(x,y) x*y, @(x,y) y^2-x^2, 0,1,0,1);`

(Exercise 2 — 14 marks) *Let us fix the vector field* $\vec{\mathbf{F}} = x(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{k}}) + 2y\hat{\boldsymbol{\jmath}}$.

1. *Compute the line integral of* $\vec{\mathbf{F}}$ *on the straight segment* Γ_S *from* $\hat{\boldsymbol{\imath}}$ *to* $\hat{\boldsymbol{\jmath}}$.
   Hint: recall Remark 1.24 on the parametrisation of paths.
2. *Compute the line integral of* $\vec{\mathbf{F}}$ *on the arc* Γ_A *of the unit circle* $\{x^2 + y^2 = 1,\ z = 0\}$ *from* $\hat{\boldsymbol{\imath}}$ *to* $\hat{\boldsymbol{\jmath}}$.
3. *Prove that, for all paths* Γ *running from* $\hat{\boldsymbol{\imath}}$ *to* $\hat{\boldsymbol{\jmath}}$ *and lying in the xy-plane* $\{z = 0\}$, *the equality* $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$ *holds (where* Γ_S *is the segment from part 1 of the question).*
   Hint: what is special in the parametrisation of a path lying in the xy-plane?
4. *Find a path* Γ_V *from* $\hat{\boldsymbol{\imath}}$ *to* $\hat{\boldsymbol{\jmath}}$ *such that* $\int_{\Gamma_V} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} \neq \int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$.
   Hint: don't forget the statement shown in question 3 (even if you did not manage to prove it). Look for a simple path, you should be able to find one whose parametrisation's components are polynomials of degree at most two.



(1.–2.) We write the parametrisations of the paths Γ_S and Γ_A (using Remark 1.24) and compute the corresponding line integrals:

$$\vec{\mathbf{a}}_S(t) = \hat{\boldsymbol{\imath}} + t(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}) = (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} \qquad 0 \le t \le 1, \qquad \frac{d\vec{\mathbf{a}}_S}{dt}(t) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}},$$

$$\int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^1 (x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + x\hat{\boldsymbol{k}}) \cdot (-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})\,\mathrm{d}t = \int_0^1 (-x + 2y)\,\mathrm{d}t = \int_0^1 (-(1-t) + 2t)\,\mathrm{d}t = \int_0^1 (3t - 1)\,\mathrm{d}t = \boxed{\frac{1}{2}},$$

$$\vec{\mathbf{a}}_A(t) = \cos\tau\hat{\boldsymbol{\imath}} + \sin\tau\hat{\boldsymbol{\jmath}} \qquad 0 \le \tau \le \frac{\pi}{2}, \qquad \frac{d\vec{\mathbf{a}}_A}{d\tau}(\tau) = -\sin\tau\hat{\boldsymbol{\imath}} + \cos\tau\hat{\boldsymbol{\jmath}},$$

$$\int_{\Gamma_A} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^{\pi/2} (x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + x\hat{\boldsymbol{k}}) \cdot (-\sin\tau\hat{\boldsymbol{\imath}} + \cos\tau\hat{\boldsymbol{\jmath}})\,\mathrm{d}\tau = \int_0^{\pi/2} (-1+2)\sin\tau\cos\tau\,\mathrm{d}\tau = \frac{\sin^2\tau}{2}\bigg|_0^{\pi/2} = \boxed{\frac{1}{2}}.$$

(3.) Since $\vec{\nabla} \times \vec{\mathbf{F}} = -\hat{\boldsymbol{\jmath}} \neq \vec{\mathbf{0}}$, the field is not irrotational, thus $\boxed{\vec{\mathbf{F}} \text{ is not conservative}}$. So we cannot use directly the fundamental theorem of vector calculus and cannot expect that *all* paths from $\hat{\boldsymbol{\imath}}$ to $\hat{\boldsymbol{\jmath}}$ give the same line integral. However, question 3 asks to consider only paths lying in the plane $\{z = 0\}$. A path of this kind has parametrisation

$$\vec{\mathbf{a}}(t) = a_1(t)\hat{\boldsymbol{\imath}} + a_2(t)\hat{\boldsymbol{\jmath}}, \qquad t_I \le t \le t_F, \qquad \vec{\mathbf{a}}(t_I) = \hat{\boldsymbol{\imath}}, \qquad \vec{\mathbf{a}}(t_F) = \hat{\boldsymbol{\jmath}},$$

with no $\hat{\boldsymbol{k}}$ component. From this expression, it follows that also the total derivative has no $\hat{\boldsymbol{k}}$ component: $\frac{d\vec{\mathbf{a}}}{dt}(t) = \frac{da_1}{dt}(t)\hat{\boldsymbol{\imath}} + \frac{da_2}{dt}(t)\hat{\boldsymbol{\jmath}}$. Thus the integral along this path reads

$$\begin{aligned}
\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} &= \int_{t_I}^{t_F} (x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + x\hat{\boldsymbol{k}}) \cdot \Big(\frac{da_1}{dt}(t)\hat{\boldsymbol{\imath}} + \frac{da_2}{dt}(t)\hat{\boldsymbol{\jmath}}\Big)\mathrm{d}t && \text{from line integral formula (44),}\\
&= \int_{t_I}^{t_F} \Big(a_1(t)\frac{da_1}{dt}(t) + 2a_2(t)\frac{da_2}{dt}(t)\Big)\,\mathrm{d}t && x = a_1(t),\ y = a_2(t)\ \ \text{(note } x\hat{\boldsymbol{k}} \text{ does not contribute),}\\
&= \int_{t_I}^{t_F} \frac{d}{dt}\Big(\frac{1}{2}a_1^2(t) + a_2^2(t)\Big)\,\mathrm{d}t && \text{product/chain rule for functions } \big(a^2(t)\big)' = 2a(t)a'(t),\\
&= \frac{1}{2}a_1^2(t_F) + a_2^2(t_F) - \frac{1}{2}a_1^2(t_I) - a_2^2(t_I) && \text{fundamental theorem of calculus,}\\
&= 0 + 1 - \frac{1}{2} - 0 \ = \ \frac{1}{2} && \text{because } \Gamma \text{ runs from } \hat{\boldsymbol{\imath}} \text{ to } \hat{\boldsymbol{\jmath}}, \text{ so } \vec{\mathbf{a}}(t_I) = \hat{\boldsymbol{\imath}},\ \vec{\mathbf{a}}(t_F) = \hat{\boldsymbol{\jmath}}.
\end{aligned}$$

Thus the line integral on any Γ from $\hat{\boldsymbol{\imath}}$ to $\hat{\boldsymbol{\jmath}}$ lying in the xy-plane coincides with the integral found in question 1.

The key here is that the horizontal part of $\vec{\mathbf{F}}$ is conservative, while only the component in the direction $\hat{\boldsymbol{k}}$ gives a non-conservative contribution, namely $\vec{\mathbf{F}} = \vec{\nabla}(\frac{1}{2}x^2 + y^2) + x\hat{\boldsymbol{k}}$. Since the path considered lies in the xy-plane, the "non-conservative component" $x\hat{\boldsymbol{k}}$ of $\vec{\mathbf{F}}$ does not contribute to the integral.

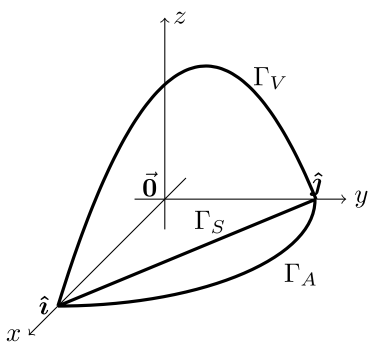
(4.) From the previous question it is clear that we need a path that does not lie in the xy-plane. How to find it? The simplest option is to start from $\vec{\mathbf{a}}_S(t) = (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}}$ from question 1, and add to it a third component $a_3(t)\hat{\boldsymbol{k}}$. This must satisfy $a_3(0) = a_3(1) = 0$ in order to connect $\hat{\boldsymbol{\imath}}$ to $\hat{\boldsymbol{\jmath}}$, so we can take $a_3(t) = t(1-t) = t - t^2$. Let us check if this gives an integral different from $\frac{1}{2}$ (it is not guaranteed):

$$\vec{\mathbf{a}}_V(t) := (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + t(1-t)\hat{\boldsymbol{k}}, \qquad 0 \le t \le 1, \qquad \frac{d\vec{\mathbf{a}}_V}{dt}(t) = -\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + (1-2t)\hat{\boldsymbol{k}}, \quad \vec{\mathbf{a}}_V(0) = \hat{\boldsymbol{\imath}},\ \vec{\mathbf{a}}_V(1) = \hat{\boldsymbol{\jmath}},$$

$$\int_{\Gamma_S} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_0^1 (x\hat{\boldsymbol{\imath}} + 2y\hat{\boldsymbol{\jmath}} + x\hat{\boldsymbol{k}}) \cdot \big(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}} + (1-2t)\hat{\boldsymbol{k}}\big)\,\mathrm{d}t = \int_0^1 \big(-x + 2y + x(1-2t)\big)\,\mathrm{d}t$$

$$= \int_0^1 \big(-(1-t) + 2t + (1-t)(1-2t)\big)\,\mathrm{d}t = \int_0^1 (2t^2)\,\mathrm{d}t = \frac{2}{3} \neq \frac{1}{2},$$

as desired. So the curve $\boxed{\vec{\mathbf{a}}_V(t) = (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + t(1-t)\hat{\boldsymbol{k}}}$ satisfies the request. The path Γ_V is shown in figure. (Actually, we can obtain any real number I as integral by choosing the curve $\vec{\mathbf{a}}_V(t) = (1-t)\hat{\boldsymbol{\imath}} + t\hat{\boldsymbol{\jmath}} + (6I-3)t(1-t)\hat{\boldsymbol{k}}$.) Of course, many other curves can be chosen, they all need to exit the xy-plane and satisfy $\vec{\mathbf{a}}_V(t_I) = \hat{\boldsymbol{\imath}}$, $\vec{\mathbf{a}}_V(t_F) = \hat{\boldsymbol{\jmath}}$.



(Exercise 3 — 5/10 marks) *Say which of the following statements are true.*
MA2VC: You do NOT need to justify your answer.
MA3VC: Justify your answer. (In case the statement is true, prove it, otherwise find a simple counterexample.)

1. *Let the path Γ be part of the graph of a function $y = g(x)$ and $\vec{\mathbf{F}}$ be a conservative field. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.*
   FALSE, for example $\vec{\mathbf{F}} = \hat{\boldsymbol{\imath}} = \vec{\nabla} x$ and Γ the segment $[\vec{\mathbf{0}}, \hat{\boldsymbol{\imath}}]$, graph of constant function $y = 0$, give $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 1$.

2. *Let Γ be a circle and $\vec{\mathbf{F}}$ an irrotational field defined in all of $\mathbb{R}^3$. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.*
   TRUE, $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ because by Theorem 2.18 $\vec{\mathbf{F}}$ is conservative and by the fundamental theorem of vector calculus 2.14 (or by Theorem 2.19) its integral on a loop is zero. Recall that $\mathbb{R}^3$ is star-shaped and that a circle is a loop. The fact that <u>$\vec{\mathbf{F}}$ is defined in all of $\mathbb{R}^3$</u> is crucial to ensure it is conservative.

3. *Let $\vec{\mathbf{F}}$ be a vector field perpendicular to the path Γ at each point. Then $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$.*
   TRUE, $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = 0$ because the line integral of $\vec{\mathbf{F}}$ is the integral of the tangential component of $\vec{\mathbf{F}}$, which is zero if $\vec{\mathbf{F}}$ is perpendicular to the path. In formulas: $\int_\Gamma \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}} = \int_\Gamma (\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}})\,\mathrm{d}s$ by equation (45), and $\vec{\mathbf{F}} \cdot \hat{\boldsymbol{\tau}} = 0$.

4. *Let $\vec{\mathbf{F}}$ be a vector field perpendicular to a surface S at each point. Then $\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = 0$.*
   FALSE, e.g. for any oriented surface $(S, \hat{\boldsymbol{n}})$ and $\vec{\mathbf{F}} = \hat{\boldsymbol{n}}$, the unit normal vector field on S, we have

   $$\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_S (\hat{\boldsymbol{n}} \cdot \hat{\boldsymbol{n}})\,\mathrm{d}S = \iint_S 1\,\mathrm{d}S = \mathrm{Area}(S) > 0, \qquad \text{by equation (72).}$$

   For a more specific example, take e.g. the graph $S_0 = \{0 < x < 1, 0 < y < 1, z = 0\}$ and $\vec{\mathbf{F}} = \hat{\boldsymbol{n}} = \hat{\boldsymbol{k}}$, $\iint_S \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{S}} = 1$.

5. *Let $\vec{\mathbf{X}}$ be a chart of a parametric surface S. Then $\iint_S \frac{\partial \vec{\mathbf{X}}}{\partial u} \cdot \mathrm{d}\vec{\mathbf{S}} = 0$.*
   TRUE, the vector field $\frac{\partial \vec{\mathbf{X}}}{\partial u}$ is tangent to S at each point, so its flux is zero. Using the flux formula (73) and the triple product property $\vec{\mathbf{p}} \cdot (\vec{\mathbf{p}} \times \vec{\mathbf{q}}) = \vec{\mathbf{q}} \cdot (\vec{\mathbf{p}} \times \vec{\mathbf{p}}) = 0$, we have $\iint_S \frac{\partial \vec{\mathbf{X}}}{\partial u} \cdot \mathrm{d}\vec{\mathbf{S}} = \iint_R \frac{\partial \vec{\mathbf{X}}}{\partial u} \cdot (\frac{\partial \vec{\mathbf{X}}}{\partial u} \times \frac{\partial \vec{\mathbf{X}}}{\partial w})\,\mathrm{d}A = 0$.

Recall: the flux of a *tangent* field through a *surface* is zero, the line integral of a field *perpendicular* to a *path* is zero, but the the flux of a perpendicular field and the line integral of a tangent field can take any value.