**On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.**

**You will be given five minutes at the end of the examination to complete the front of any answer books used.**

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**May/June 2015** **MA3VC 2014/5 A 001**

**Any non-programmable calculator permitted**

**UNIVERSITY OF READING**

# VECTOR CALCULUS (MA3VC)

Two hours

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Full marks can be gained from complete answers to **ALL** questions in Section A and **TWO** questions (out of four) from Section B. If more than two questions from Section B are attempted then marks from the **BEST** two section B questions will be used. If the exam mark calculated in this way is less than 40%, then marks from any other Section B questions which have been attempted will be added to the exam mark until 40% is reached.
Total marks: 100.

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You may use the following identities in the solution of the exercises:

$$\begin{aligned}
\vec{\nabla}(fg) &= f\vec{\nabla}g + g\vec{\nabla}f, & (1)\\
\vec{\nabla}(\vec{\mathbf{F}}\cdot\vec{\mathbf{G}}) &= (\vec{\mathbf{F}}\cdot\vec{\nabla})\vec{\mathbf{G}} + (\vec{\mathbf{G}}\cdot\vec{\nabla})\vec{\mathbf{F}} + \vec{\mathbf{G}}\times(\vec{\nabla}\times\vec{\mathbf{F}}) + \vec{\mathbf{F}}\times(\vec{\nabla}\times\vec{\mathbf{G}}), & (2)\\
\vec{\nabla}\cdot(f\vec{\mathbf{G}}) &= (\vec{\nabla}f)\cdot\vec{\mathbf{G}} + f\vec{\nabla}\cdot\vec{\mathbf{G}}, & (3)\\
\vec{\nabla}\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) &= (\vec{\nabla}\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec{\nabla}\times\vec{\mathbf{G}}), & (4)\\
\vec{\nabla}\times(f\vec{\mathbf{G}}) &= (\vec{\nabla}f)\times\vec{\mathbf{G}} + f\vec{\nabla}\times\vec{\mathbf{G}}, & (5)\\
\vec{\nabla}\times(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) &= (\vec{\nabla}\cdot\vec{\mathbf{G}})\vec{\mathbf{F}} - (\vec{\nabla}\cdot\vec{\mathbf{F}})\vec{\mathbf{G}} + (\vec{\mathbf{G}}\cdot\vec{\nabla})\vec{\mathbf{F}} - (\vec{\mathbf{F}}\cdot\vec{\nabla})\vec{\mathbf{G}}, & (6)\\
\Delta(fg) &= (\Delta f)g + 2\vec{\nabla}f\cdot\vec{\nabla}g + f(\Delta g). & (7)
\end{aligned}$$

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**Section A** (Answer all questions)

1. (a) Let $f$, $\varphi$ and $\psi$ be smooth scalar fields. Prove the following identity:

$$\vec{\nabla}\cdot\big((\vec{\nabla}\varphi\times\vec{\nabla}\psi)f\big) = (\vec{\nabla}\varphi\times\vec{\nabla}\psi)\cdot\vec{\nabla}f.$$

You can use the vector differential identities proved in class, including those listed above.

[14 marks]

(b) Demonstrate the above identity for the following choice of the fields:

$$\varphi = \mathrm{e}^{xy+z}, \qquad \psi = x^2 + y^2, \qquad f = xyz.$$

[12 marks]

(c) Show that the field $\vec{\mathbf{G}} = ((\vec{\nabla}\varphi\times\vec{\nabla}\psi)\varphi)$ is solenoidal.
Find a vector potential for $\vec{\mathbf{G}}$.
<u>Hint</u>: Try to guess (and then verify) a vector potential constructed using some powers of $\varphi$ and/or $\psi$.

[14 marks]

2. Compute the line integral of the scalar field $f(\vec{\mathbf{r}}) = |\vec{\mathbf{r}}|^{-1/2}$ over the logarithmic spiral $\Gamma$ parametrised by

$$\vec{\mathbf{a}}(t) = \mathrm{e}^{-t}\cos t\,\hat{\boldsymbol{\imath}} + \mathrm{e}^{-t}\sin t\,\hat{\boldsymbol{\jmath}}, \qquad 0 \le t < \infty.$$

[20 marks]

**Section B** (Choose two questions out of four)

3. Consider the region

$$R = \left\{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 1 < \xi(x,y) < 2,\ 1 < \eta(x,y) < 2\right\},$$

defined by the change of variables

$$\xi(x,y) = \left(\frac{x^2}{y}\right)^{1/5}, \qquad \eta(x,y) = \left(\frac{y^3}{x}\right)^{1/5}.$$

Compute the double integral over $R$ of $f = (xy)^{-1}$.

Hint: Note that in order to evaluate the field $f$ in the variables $x$ and $y$ you need to express these as functions of $\xi$ and $\eta$.

[20 marks]

4. By computing the left- and the right-hand side of its assertion, demonstrate Green's theorem for the vector field $\vec{\mathbf{F}} = (x + 2y)(\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}})$ and the disc of radius $a > 0$

$$R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}},\ \text{such that } x^2 + y^2 < a^2\}.$$

Hint: Recall the formula for the area of a disc of arbitrary radius and the formula for the parametrisation of its boundary.

[20 marks]

5. Consider a surface $S$ with unit normal vector field $\hat{\boldsymbol{n}}$, an irrotational vector field $\vec{\mathbf{G}}$ and a scalar field $f$ that vanishes on the boundary of $S$.

   Prove that the flux of the vector product $(\vec{\nabla} f) \times \vec{\mathbf{G}}$ through $S$ is zero.

   Hint: Use a suitable theorem relating integrals and derivatives, and a suitable product rule for differential operators.

   Hint: Note that, since $f$ vanished on $\partial S$, then some path integrals involving $f$ will vanish.

[20 marks]

6. Consider a continuous vector field $\vec{\mathbf{F}} : \mathbb{R}^3 \to \mathbb{R}^3$ and fix a point $\vec{\mathbf{r}}_0 \in \mathbb{R}^3$. Assume that for all $\vec{\mathbf{r}} \in \mathbb{R}^3$ the line integral of $\vec{\mathbf{F}}$ from $\vec{\mathbf{r}}_0$ to $\vec{\mathbf{r}}$ is independent of the path of integration chosen, such that

$$\Phi(\vec{\mathbf{r}}) = \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}} \vec{\mathbf{F}} \cdot \mathrm{d}\vec{\mathbf{r}}$$

   is a well-defined scalar field.

   Show that $\vec{\nabla}\Phi = \vec{\mathbf{F}}$, which implies that $\vec{\mathbf{F}}$ is conservative and $\Phi$ is its scalar potential. (As done in class, you only need to show that $\frac{\partial \Phi}{\partial x} = F_1$.)

   Does this fact imply that all irrotational fields admit a scalar potential?

[20 marks]

[End of Question Paper]