Candidates are admitted to the examination room ten minutes before the start of the examination. On admission to the examination room, you are permitted to acquaint yourself with the instructions below and to read the question paper.

Do not write anything until the invigilator informs you that you may start the examination. You will be given five minutes at the end of the examination to complete the front of any answer books used.

May/June 2013 **[MA3VC 2012/3 A 001]**

Any non-programmable calculator permitted

UNIVERSITY OF READING

[VECTOR CALCULUS (MA3VC)]

[2 hours]

[Answer ALL questions]

1. (a) Prove the vector differential identity:

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2 \mathbf{F} \, .$$

It is sufficient to prove the equality for the x-component of each side.

[7 marks]

(b) Demonstrate that the above identity holds for

$$\mathbf{F} = z^2 \hat{\mathbf{i}} + y^2 \hat{\mathbf{j}} + x^2 \hat{\mathbf{k}} \, .$$

[7 marks]

2. Demonstrate Green's theorem,

$$\int_R [\nabla \times \mathbf{F}]_z \, dA = \oint_{\partial R} \mathbf{F} \cdot d\mathbf{r} \, ,$$

by explicitly evaluating both sides of the equality for

$$\mathbf{F} = -x^2 y \hat{\mathbf{i}} + x y^2 \hat{\mathbf{j}} \, ,$$

where R is the unit circle defined by $x^2 + y^2 \leq 1$.
Hint: $dA = r \, d\theta \, dr$ in polar coordinates. Depending how you do the line integral, you may need: $\sin^2(t)\cos^2(t) = (1 - \cos(4t))/8$.

[12 marks]

3. (a) Prove that

$$\int_D \frac{\partial \phi}{\partial z} \, dV = \oint_{\partial D} \phi \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}} \, dS \, ,$$

for the special case where D is a z-simple domain defined by

$$a \leq x \leq b, \quad c \leq y \leq d, \quad \text{and} \quad f(x, y) \leq z \leq g(x, y) \, .$$

As usual, $\hat{\mathbf{n}}$ is the outward-pointing unit normal to the surface, ∂D.

[10 marks]

(b) Briefly explain how this proof is extended to regular domains.

[2 marks]

4. Demonstrate the identity,

$$\int_D \nabla \times \mathbf{F}\; dV = \oint_{\partial D} \hat{\mathbf{n}} \times \mathbf{F}\, dS\ ,$$

by evaluating both sides of the equality for

$$\mathbf{F} = xy\hat{\mathbf{k}}\ ,$$

where D is the unit cube defined by

$$0 \leq x \leq 1, \quad 0 \leq y \leq 1, \quad \text{and} \quad 0 \leq z \leq 1.$$

As usual, $\hat{\mathbf{n}}$ is the outward-pointing unit normal to the surface, ∂D.

[12 marks]

[End of Question Paper]