**On admission to the examination room, you should acquaint yourself with the instructions below. You must listen carefully to all instructions given by the invigilators. You may read the question paper, but must not write anything until the invigilator informs you that you may start the examination.**

**You will be given five minutes at the end of the examination to complete the front of any answer books used.**

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**May/June 2016** **MA2VC 2015/6 A 001**

**Any non-programmable calculator permitted**

**UNIVERSITY OF READING**

**VECTOR CALCULUS (MA2VC)**

Two hours

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Full marks can be gained from complete answers to **ALL** questions in Section A and **TWO** questions (out of four) from Section B. If more than two questions from Section B are attempted then marks from the **BEST** two section B questions will be used. If the exam mark calculated in this way is less than 40%, then marks from any other Section B questions which have been attempted will be added to the exam mark until 40% is reached.
Total marks: 100.

You may use the following identities and special coordinate systems in the solution:

$$\vec{\nabla}(fg) = f\vec{\nabla}g + g\vec{\nabla}f, \tag{1}$$
$$\vec{\nabla}\cdot(f\vec{\mathbf{G}}) = (\vec{\nabla}f)\cdot\vec{\mathbf{G}} + f\vec{\nabla}\cdot\vec{\mathbf{G}}, \tag{2}$$
$$\vec{\nabla}\cdot(\vec{\mathbf{F}}\times\vec{\mathbf{G}}) = (\vec{\nabla}\times\vec{\mathbf{F}})\cdot\vec{\mathbf{G}} - \vec{\mathbf{F}}\cdot(\vec{\nabla}\times\vec{\mathbf{G}}), \tag{3}$$
$$\vec{\nabla}\times(f\vec{\mathbf{G}}) = (\vec{\nabla}f)\times\vec{\mathbf{G}} + f\vec{\nabla}\times\vec{\mathbf{G}}, \tag{4}$$
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases} \qquad \begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \tag{5}$$

## **Section A** (Answer all questions)

1. Let us fix the vector field

$$\vec{\mathbf{F}} = x^2\hat{\boldsymbol{\imath}} + y^2\hat{\boldsymbol{\jmath}},$$

and the half-circle path

$$\Gamma = \left\{\vec{\mathbf{r}} \in \mathbb{R}^2,\ x^2 + y^2 = 1,\ x \geq 0\right\},$$

from the initial point $-\hat{\boldsymbol{\jmath}}$ to the end point $\hat{\boldsymbol{\jmath}}$.

(a) Write a curve $\vec{\mathbf{a}}(t)$, defined on a suitable interval $[t_I, t_F] \subset \mathbb{R}$, that parametrises $\Gamma$.
Hint: Note that $\Gamma$ is a part of the unit circle $\{x^2 + y^2 = 1,\ z = 0\}$, for which you know a parametrisation. To obtain the desired arc, you need to choose the correct interval $[t_I, t_F]$.

[6 marks]

(b) Compute the line integral of $\vec{\mathbf{F}}$ along $\Gamma$.

[8 marks]

(c) Compute the line integral of the first component $F_1$ of $\vec{\mathbf{F}}$ along $\Gamma$.

[10 marks]

Hint: Can any potential help you computing one of the two integrals?

2. (a) Prove the following integral identity:

$$\iiint_D \vec{\nabla} f \cdot \vec{\mathbf{G}}\,\mathrm{d}V = -\iiint_D f\vec{\nabla}\cdot\vec{\mathbf{G}}\,\mathrm{d}V + \oiint_{\partial D} f\vec{\mathbf{G}}\cdot \mathrm{d}\vec{\mathbf{S}}, \qquad (\star)$$

where $D \subset \mathbb{R}^3$ is a domain, $f : \mathbb{R}^3 \to \mathbb{R}$ a scalar field and $\vec{\mathbf{G}} : \mathbb{R}^3 \to \mathbb{R}^3$ a vector field.

<u>Hint</u>: Use a suitable integral theorem and one of the vector product rules (1)–(4).

[12 marks]

(b) Use identity ($\star$) to compute the flux of the product

$$(xz^3)(z\hat{\boldsymbol{\imath}} + x\hat{\boldsymbol{\jmath}} + y\hat{\boldsymbol{k}})$$

through the boundary of the unit cube

$$C = \left\{\vec{\mathbf{r}} \in \mathbb{R}^3,\ 0 < x < 1,\ 0 < y < 1,\ 0 < z < 1\right\}.$$

[14 marks]

(c) Use identity ($\star$) to show that, if $f(\vec{\mathbf{r}}) = 1$ for all $\vec{\mathbf{r}} \in \partial D$ (the boundary of $D$), then, for all vector fields $\vec{\mathbf{H}}$,

$$\iiint_D \vec{\nabla} f \cdot (\vec{\nabla} \times \vec{\mathbf{H}})\,\mathrm{d}V = 0.$$

<u>Hint</u>: You might need to use again an integral theorem.

[10 marks]

**Section B** (Choose two questions out of four)

3. Prove the product rule for the Laplacian

$$\Delta(fg) = (\Delta f)g + 2\vec{\nabla} f \cdot \vec{\nabla} g + f(\Delta g),$$

where $f$ and $g$ are smooth scalar fields.

<u>Hint</u>: You can either expand the identity in partial derivatives or use simpler vector identities such as the product rules (1)–(4) above.

[20 marks]

4. Use Stokes' theorem to compute the line integral of the vector field

$$\vec{\mathbf{M}} = zy\hat{\boldsymbol{\imath}} + 2xz\hat{\boldsymbol{\jmath}}$$

along the boundary of the graph surface

$$S = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + g(x,y)\hat{\boldsymbol{k}},\ x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in R\},$$

where

$$g(x,y) = xy, \qquad R = (0,1)^2 = \{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ 0 < x < 1,\ 0 < y < 1\}.$$

[20 marks]

5. Compute the triple integral of the scalar field $f = z/\sqrt{x^2+y^2}$ over

$$E = \{\vec{\mathbf{r}} \in \mathbb{R}^3,\ |\vec{\mathbf{r}}| < 1,\ x > 0,\ y > 0,\ z > 0\},$$

namely the intersection of the unit sphere centred at the origin and the first octant.

Hint: : You can use a system of special coordinates of those in (5).

[20 marks]

6. Prove that

$$\iint_R \frac{\partial f}{\partial y}\,\mathrm{d}A = -\oint_{\partial R} f\,\mathrm{d}x,$$

where $R \subset \mathbb{R}^2$ is the $y$-simple region defined by

$$R = \{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} \in \mathbb{R}^2,\ x_0 < x < x_1,\ a(x) < y < b(x)\}$$

and $f : R \to \mathbb{R}$ is a smooth scalar field.

As usual, the integral $\oint_{\partial R} f\,\mathrm{d}x$ is taken in the anti-clockwise direction.

[20 marks]

[End of Question Paper]