Space–time DG for the wave equation: quasi-Trefftz and sparse versions

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Initial–boundary value problem

First-order initial–boundary value problem (Dirichlet): find \((v, \sigma)\) s.t.

\[
\begin{aligned}
\nabla v + \partial_t \sigma &= 0 \\
\nabla \cdot \sigma + \frac{1}{c^2} \partial_t v &= f \\
v(\cdot, 0) &= v_0, \quad \sigma(\cdot, 0) = \sigma_0 \\
v(x, \cdot) &= g
\end{aligned}
\]

in \(Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}, \ n \in \mathbb{N},\)

in \(Q,\)

on \(\Omega,\)

on \(\partial \Omega \times (0, T).\)

From \(-\Delta u + c^{-2} \partial_t^2 u = f,\) choose \(v = \partial_t u\) and \(\sigma = -\nabla u.\)

Velocity \(c = c(x)\) piecewise smooth. \(\Omega \subset \mathbb{R}^n\) Lipschitz bounded.
Initial–boundary value problem

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\begin{align*}
\nabla v + \partial_t \sigma &= 0 & \text{in } Q = \Omega \times (0, T) \subset \mathbb{R}^{n+1}, \ n \in \mathbb{N}, \\
\nabla \cdot \sigma + \frac{1}{c^2} \partial_t v &= f & \text{in } Q, \\
v(\cdot, 0) &= v_0, \quad \sigma(\cdot, 0) = \sigma_0 & \text{on } \Omega, \\
v(x, \cdot) &= g & \text{on } \partial \Omega \times (0, T).
\end{align*}
\]

From \(-\Delta u + c^{-2} \partial^2_t u = f\), choose \(v = \partial_t u\) and \(\sigma = -\nabla u\).

Velocity \(c = c(x)\) piecewise smooth. \(\Omega \subset \mathbb{R}^n\) Lipschitz bounded.

- Neumann \(\sigma \cdot n = g\) & Robin \(\frac{\varrho}{c} v - \sigma \cdot n = g\) BCs
- more general coeff.’s \(-\nabla \cdot (\rho^{-1} \nabla u) + G \partial^2_t u = 0\)

Extensions:
- Maxwell equations
- elasticity
- 1\textsuperscript{st} order hyperbolic systems...
Space–time mesh and assumptions

Introduce space–time polytopic mesh $\mathcal{T}_h$ on $\Omega$. Assume: $c = c(\mathbf{x})$ smooth in each element.

Assume: each face $F = \partial K_1 \cap \partial K_2$ with normal $(\mathbf{n}_F^x, n_F^t)$ is either

- space-like: $c |\mathbf{n}_F^x| < n_F^t$, 
  or
- time-like: $n_F^t = 0$.

Usual DG notation with averages $\{ \cdot \}$, $\mathbf{n}_x$-normal space jumps $[ \cdot ]_N$, $n^t$-time jumps $[ \cdot ]_t$.

Lateral boundary $\mathcal{F}_h^\partial := \partial \Omega \times [0, T]$. 
DG elemental equation and numerical fluxes

Multiply PDEs with test field \((w, \tau)\) & integrate by parts on \(K \in \mathcal{T}_h\):

\[
- \int_K \left( v (\nabla \cdot \tau + c^{-2} \partial_t w) + \sigma \cdot (\nabla w + \partial_t \tau) \right) \, dV
\]
\[
+ \int_{\partial K} \left( (v \tau + \sigma w) \cdot n_K^x + (\sigma \cdot \tau + c^{-2} v w) n_K^t \right) \, dS = \int_K f w \, dV.
\]
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+ \int_{\partial K} \left( (v \tau + \sigma w) \cdot \mathbf{n}_K^x + \left( \sigma \cdot \tau + c^{-2} v w \right) n_K^t \right) \, dS = \int_K fw \, dV.
\]

Approximate skeleton traces of \((v, \sigma)\) with numerical fluxes \((\hat{v}_h, \hat{\sigma}_h)\), defined as

\[
\hat{v}_h := \begin{cases}
\begin{align*}
v_h^– \\
v_0 \\
\{v_h\} + \beta \left[ \sigma_h \right]_{N} \\
g
\end{align*}
\end{cases} \quad \hat{\sigma}_h := \begin{cases}
\begin{align*}
\sigma_h^– \\
\sigma_0 \\
\{\sigma_h\} + \alpha \left[ v_h \right]_{N} \\
\sigma_h - \alpha (v - g) \mathbf{n}_\Omega^x
\end{align*}
\end{cases}
\]

“upwind in time, elliptic-DG in space”.

\(\alpha = \beta = 0 \rightarrow \text{Kretzschmar–S.–T.–W.}, \quad \alpha \beta \geq \frac{1}{4} \rightarrow \text{Monk–Richter.}\)
Space–time DG formulation

Substitute the fluxes in the elemental equation, choose discrete space $V_p(T_h)$, sum over $K \rightarrow$ write $x_t$-DG as:

Seek $(v_h, \sigma_h) \in V_p(T_h)$ s.t., $\forall (w, \tau) \in V_p(T_h)$,

$A(v_h, \sigma_h; w, \tau) = \ell(w, \tau)$

where

$A(v_h, \sigma_h; w, \tau) := - \sum_{K \in T_h} \int_K \left( v_h \left( \nabla \cdot \tau + c^{-2} \partial_t w \right) + \sigma_h \cdot \left( \nabla w + \partial_t \tau \right) \right) dV$

$+ \int_{\mathcal{F}_h^{\text{space}}} \left( \frac{v_h \cdot [w]_t}{c^2} + \sigma_h \cdot [\tau]_t + v_h \cdot [\tau]_N + \sigma_h \cdot [w]_N \right) dS$

$+ \int_{\mathcal{F}_h^{\text{time}}} \left( \{v_h \cdot [\tau]_N + \{\sigma_h \cdot [w]_N + \alpha \{v_h \cdot [w]_N + \beta \{\sigma_h \cdot [\tau]_N \right) dS$

$+ \int_{\Omega \times \{T\}} (c^{-2} v_h w + \sigma_h \cdot \tau) dS + \int_{\mathcal{F}_h^{\partial}} (\sigma_h \cdot n_\Omega + \alpha v_h) w dS,$

$\ell(w, \tau) := \int_Q f w dV + \int_{\Omega \times \{0\}} (c^{-2} v_0 w + \sigma_0 \cdot \tau) dS + \int_{\mathcal{F}_h^{\partial}} g(\alpha w - \tau \cdot n_\Omega) dS.$

This is an “ultra-weak” variational formulation (UWVF).
Coercivity in DG semi-norm

Key property, from integration by parts:

\[ A(w, \tau; w, \tau) \geq \|\| (w, \tau) \|\|_{DG}^2 \]

where

\[
\|\| (w, \tau) \|\|_{DG}^2 := \frac{1}{2} \left\| \left( \frac{1 - \gamma}{n_F^t} \right)^{1/2} c^{-1} [w]_t \right\|_{L^2(\mathcal{F}_{h}^{\text{space}})}^2 + \frac{1}{2} \left\| \left( \frac{1 - \gamma}{n_F^t} \right)^{1/2} [\tau]_t \right\|_{L^2(\mathcal{F}_{h}^{\text{space}}) n}^2 \\
+ \frac{1}{2} \left\| c^{-1} w \right\|_{L^2(\mathcal{F}_{h}^{0} \cup \mathcal{F}_{h}^{T})}^2 + \frac{1}{2} \left\| \tau \right\|_{L^2(\mathcal{F}_{h}^{0} \cup \mathcal{F}_{h}^{T})}^2 \\
+ \left\| \alpha^{1/2} [w]_N \right\|_{L^2(\mathcal{F}_{h}^{\text{time}}) n}^2 + \left\| \beta^{1/2} [\tau]_N \right\|_{L^2(\mathcal{F}_{h}^{\text{time}}) n}^2 + \left\| \alpha^{1/2} w \right\|_{L^2(\mathcal{F}_{h}^{\partial})}^2 \\
\gamma := \frac{\|c\|_{C^0(F)} |n_F^\times|}{n_F^t} \in [0, 1) \sim \text{distance between space-like face } F \text{ & char. cone.}
\]

In general, \( \|\| (w, \tau) \|\|_{DG} \) is only a semi-norm.
Assume $c$ is constant in $K \subset \mathbb{R}^{n+1}$. Consider homogeneous wave eq. 
\[-\Delta u + c^{-2}\partial_t^2 u = 0\] in $K$.

Can choose Trefftz space of polynomials of deg. $\leq p$ on element $K$:

$$
\mathbb{U}^p(K) := \{ u \in \mathbb{P}^p(K), -\Delta u + c^{-2}\partial_t^2 u = 0 \},
$$

$$
\mathbb{W}^p(K) := \{(v, \sigma) = (\partial_t u, -\nabla u), u \in \mathbb{U}^{p+1}(K) \}.
$$
Special case: space–time Trefftz method

Assume \( c \) is constant in \( K \subset \mathbb{R}^{n+1} \).

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\]

► Basis functions easily constructed, e.g. \( b_{j,\ell} (x, t) = (d_{j,\ell} \cdot x - ct)^j \).

► Taylor \( T^{p+1}[u] \in \mathbb{U}^p(K) \Rightarrow \) orders of approximation in \( h \) are for free.

Much better accuracy for fewer DOFs:

\[
\dim(\mathbb{U}^p(K)) = O_{p \to \infty}(p^n) \ll \dim(\mathbb{P}^p(K)) = O_{p \to \infty}(p^{n+1}).
\]

► With Trefftz test fields, volume terms in \( xt \)-DG bilinear form vanish: quadrature on \( n \)-dimensional faces only.

► \( \| \cdot \|_{DG} \) is a norm: stability and error analysis. (M., PERUGIA 2018)
Global, implicit and explicit schemes

1. $xt$-DG formulation is **global in space–time domain** $\Omega$:
   - large linear system!
   - Might be good for adaptivity and DD.

If mesh is partitioned in time-slabs $\Omega \times (t_{j-1}, t_j)$, matrix is block lower-triangular:
- For each time-slab a system can be solved sequentially: implicit method.

If mesh is suitably chosen, DG solution can be computed with a sequence of local systems: explicit method, allows parallelism!


Trefftz requires quadrature on faces only: easier tent-pitching.

Versions 1–2–3 are algebraically equivalent (on the same mesh).
Global, implicit and explicit schemes

1. $xt$-DG formulation is **global in space–time domain** $\mathcal{Q}$: large linear system! Might be good for adaptivity and DD.

2. If mesh is partitioned in **time-slabs** $\Omega \times (t_{j-1}, t_j)$, matrix is **block lower-triangular**: for each time-slab a system can be solved sequentially: **implicit** method.

\[
\begin{array}{c}
\text{t} \\
\downarrow \\
\Omega \\
\downarrow \\
\text{x}
\end{array}
\]

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Global, implicit and explicit schemes

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3. If mesh is suitably chosen, DG solution can be computed with a sequence of **local systems:**
   - **explicit method,** allows parallelism!
   - Trefftz requires **quadrature on faces only:** easier tent-pitching.

Versions 1–2–3 are algebraically equivalent (on the same mesh).
Tent-pitched elements

Tent-pitched elements/patches obtained from regular space meshes in 2+1D give parallelepipeds or octahedra+tetrahedra:

More complicated shapes from unstructured meshes:

Simplices around a tent pole can be merged in macroelement.

Trefftz requires quadrature on faces only:
only the shape of space elements matters.

(from GOPALAKRISHNAN, SCHÖBERL, WINTERSTEIGER 2016)
Proposed \(xt\)-DG formulation comes from:

- \(\text{(Monk, Richter 2005)}\), linear symmetric hyperbolic systems, tent-pitched meshes, \(\mathbb{P}^p\) spaces, \(\alpha \beta \geq \frac{1}{4}\)
- \(\text{(Kretzschmar, Schnepp et al. 2014–16)}\), Maxwell eq.s, Trefftz
- \(\text{(M., Perugia 2018)}\), Trefftz error analysis
- \(\text{(Perugia, Schoeberl, Stocker, Wintersteiger 2020)}\), Trefftz & tents

This presentation:

- \(\text{(Bansal, M., Perugia, Schwab 2021)}\), tensor-product grids, corner singularities, sparse version

Related works:

- \(\text{(Barucq, Calandra, Diaz, Shishenina 2020)}\), elasticity
- \(\text{(Gómez, M. 2021 — arXiv:2106.04724)}\), Schrödinger
Part I

Quasi-Trefftz $xt$-DG

Imbert-Gérard, Moiola, Stocker
Trefftz doesn’t like smooth coefficients

Homogeneous wave equation \(-\Delta u + c^{-2} \partial_t^2 u = 0, \quad c = \text{wavespeed}\).

Trefftz-DG is clear for piecewise-constant \(c\):
basis functions are polynomial local solution of wave eq.

How to extend to piecewise-smooth \(c = c(\mathbf{x})\)?
No analytical solutions are available.
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Helmholtz equation: \(\Delta u + k^2 u = 0\).

- Constant wavenumber \(k \in \mathbb{R} \rightarrow \text{plane waves} \ b_J(x) = e^{ik \cdot d_j \cdot x}, |d_j| = 1\).
- Smooth wavenumber \(k = k(x)\)

IMBERT-GÉRARD, \(\approx 2013\): generalised plane waves \(b_J(x) = e^{P_j(x)} \) s.t.

\[ D_i(\Delta b_J + k^2 b_J)(x_K) = 0 \ \forall |i| < q \ (x_K = \text{centre of element } K). \]
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\[ D^{i}(\Delta b_{J} + k^{2}b_{J})(x_{K}) = 0 \quad \forall |i| < q \quad (x_{K} = \text{centre of element } K). \]

Order-\(q\) Taylor polynomial vanishes in a given point.

- Provides high-order \(h\)-convergence for DG.
- Basis construction, implementation, analysis are complicated.
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Order-\(q\) Taylor polynomial vanishes in a given point.

- Provides high-order \(h\)-convergence for DG.
- Basis construction, implementation, analysis are complicated.

Our goal: extend this idea to wave equation, without pain!
Define wave operator
\[ \Box_G u := \Delta u - G \partial_t^2 u, \quad G(x) = c^{-2} \text{ smooth.} \]

Fix \((x_K, t_K) \in K \subset \mathbb{R}^{n+1}\).

Define quasi-Trefftz (polynomial) space
\[ \mathcal{Q}U^p(K) := \{ u \in \mathbb{P}^p(K) : D^i \Box_G u(x_K, t_K) = 0, \quad \forall |i| \leq p - 2 \} \]
\[ \mathcal{Q}W^p(K) := \{ (\partial_t u, -\nabla u), u \in \mathcal{Q}U^{p+1}(K) \} \]
Quasi-Trefftz space

Define wave operator

\[ \Box_G u := \Delta u - G \partial_t^2 u, \quad G(\mathbf{x}) = c^{-2} \text{ smooth.} \]

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Define quasi-Trefftz (polynomial) space

\[
\begin{align*}
\mathbb{Q}U^p(K) &:= \left\{ u \in \mathbb{P}^p(K) : \ D^i \Box_G u(\mathbf{x}_K, t_K) = 0, \ \forall |i| \leq p - 2 \right\} \\
\mathbb{Q}W^p(K) &:= \left\{ (\partial_t u, -\nabla u), u \in \mathbb{Q}U^{p+1}(K) \right\}
\end{align*}
\]

Theorem: approximation properties

If \( u \in C^{p+1}(K), \quad \Box_G u = 0, \quad 0 \leq j \leq p, \quad K \text{ star-shaped wrt } (\mathbf{x}_K, t_K) \)

\[
\begin{align*}
\inf_{P \in \mathbb{Q}U^p(K)} \| u - P \|_{C^j(K)} &\leq h^{p+1-j} \frac{n^{p+1-j}}{(p + 1 - j)!} |u|_{C^{p+1}(K)}
\end{align*}
\]

Main idea: Taylor polynomial \( T_{(\mathbf{x}_K, t_K)}^{p+1} [u] \in \mathbb{Q}U^p(K) \).

In condition “\(|i| \leq q\)”, why \( q = p - 2 \)?
If \( q < p - 2 \), space is too big, larger than Trefftz for constant \( G \).
If \( q > p - 2 \), space loses approximation properties.
Generalised Trefftz basis

The local discrete space is clear. How to construct a basis for it?

Choose two \( x \)-only polynomial basis:

\[
\{ \hat{b}_J \}_{J=1,\ldots,(p+n)} \text{ for } \mathbb{P}^{p}(\mathbb{R}^n), \quad \{ \tilde{b}_J \}_{J=1,\ldots,(p-1+n)} \text{ for } \mathbb{P}^{p-1}(\mathbb{R}^n).
\]
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\]

Construct a basis for $\mathcal{Q}^p_u(K)$ “evolving” $\hat{b}_J$ and $\tilde{b}_J$ in time:

\[
\begin{cases}
  b_J \in \mathcal{Q}^p_u(K) : \\
  b_J(\cdot, t_K) = \tilde{b}_J, \quad \partial_t b_J(\cdot, t_K) = 0, \quad \text{for } J \leq (p+n) \\
  b_J(\cdot, t_K) = 0, \quad \partial_t b_J(\cdot, t_K) = \tilde{b}_{J-(p+n)}, \quad \text{for } (p+n) < J
\end{cases}
\]

for $J = 1, \ldots, (p+n) + (p-1+n)$.

We prove that this defines a basis and show how to compute $\{b_J\}$. 
Computation of basis coefficients

Fix $n = 1$ (for simplicity). Denote $G(x) = \sum_{m=0}^{\infty} g_m (x - x_K)^m$. $g_0 > 0$.

Monomial expansion of basis element:

$$b_J(x, t) = \sum_{i_x+i_t \leq p} a_{i_x, i_t} (x - x_K)^{i_x} (t - t_K)^{i_t},$$

Cauchy conditions $(b_J(\cdot, t_K), \partial_t b_J(\cdot, t_K))$ determine $a_{i_x,0}, a_{i_x,1}$. 
Computation of basis coefficients

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\]

Cauchy conditions \((b_J(\cdot, t_K), \partial_t b_J(\cdot, t_K))\) determine \(a_{i_x,0}, a_{i_x,1}\).

To be in \( \mathbb{QU}^p \), coeff.s have to satisfy:

\[
\partial_x^{i_x} \partial_t^{i_t} \square_G b_J(x_K, t_K) = (i_x + 2)! i_t! a_{i_x+2, i_t} - \sum_{j_x=0}^{i_x} i_x! (i_t + 2)! g_{i_x-j_x} a_{j_x, i_t+2} = 0
\]

Linear system for coeff.s \(a_{i_x, i_t}\).
Computation of basis coefficients

Fix $n = 1$ (for simplicity). Denote $G(x) = \sum_{m=0}^{\infty} g_m(x - x_K)^m$. $g_0 > 0$.

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$$b_J(x, t) = \sum_{i_x + i_t \leq p} a_{i_x, i_t} (x - x_K)^{i_x} (t - t_K)^{i_t},$$

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$$\partial_x^{i_x} \partial_t^{i_t} \square_G b_J(x_K, t_K) = (i_x + 2)! \, i_t! \, a_{i_x + 2, i_t} - \sum_{j_x=0}^{i_x} i_x! \, (i_t + 2)! \, g_{i_x - j_x} a_{j_x, i_t + 2} \not= 0$$

for $i_x + i_t \leq p - 2$

Linear system for coeff.s $a_{i_x, i_t}$.

Compute $a_{i_x, i_t + 2}$ from coefficients $\bullet$:

first loop across diagonals $\nearrow$, then along diagonals $\searrow$. 

```
\begin{tikzpicture}
\fill[yellow!20] (0,0) rectangle (6,6);
\draw[->,thick] (0,0) -- (7,0) node[below] {$i_x$};
\draw[->,thick] (0,0) -- (0,7) node[right] {$i_t$};
\node at (3,3) {$p$};
\draw[thick] (0,0) -- (7,0) -- (0,7) -- cycle;
\foreach \i in {0,...,3} {
  \foreach \j in {0,...,\i} {
    \filldraw[black] (\i,\j) circle (2pt);
  }
}
\foreach \i in {0,...,3} {
  \foreach \j in {\i,...,3} {
    \filldraw[red] (\i,\j) circle (4pt);
  }
}
\node at (1,1) {$a_{i_x, i_t + 2}$};
\node at (3,3) {$a_{i_x + 2, i_t}$};
\end{tikzpicture}
```
Basis construction: algorithm — $n = 1$

Data: $(g_m)_{m \in \mathbb{N}_0}, x_K, t_K, p$.

Choose favourite polynomial bases $\{\hat{b}_J\}, \{\tilde{b}_J\}$ in $\mathbf{x}$, \rightarrow coeff's $a_{kx,0}, a_{kx,1}$.

For each $J$ (i.e. for each basis function), construct $b_J$ as follows:

$$b_J(x, t) = \sum_{0 < k_x + k_t \leq p} a_{k_x, k_t} (x - x_K)^{k_x} (t - t_K)^{k_t}$$

for $\ell = 2$ to $p$ (loop across diagonals $\nearrow$) do

for $i_t = 0$ to $\ell - 2$ (loop along diagonals $\searrow$) do

set $i_x = \ell - i_t - 2$ and compute

$$a_{i_x, i_t + 2} = \frac{(i_x + 2)(i_x + 1)}{(i_t + 2)(i_t + 1)g_0} a_{i_x + 2, i_t} - \sum_{j_x = 0}^{i_x - 1} \frac{g_{i_x - j_x}}{g_0} a_{j_x, i_t + 2}$$

end

end
Basis construction: algorithm — $n > 1$

In higher space dimensions $n > 1$, with $G(x) = \sum_{i} (x - x_K)^i g_i$, the algorithm is the same with a further inner loop:

for $\ell = 2$ to $p$ (loop across $\{|i| + i_t = \ell - 2\}$ hyperplanes, ↑) do

for $i_t = 0$ to $\ell - 2$ (loop across constant-$t$ hyperplanes ↑) do

for $i_x$ with $|i_x| = \ell - i_t - 2$ do

$$a_{i_x, i_t+2} = \sum_{l=1}^{n} \frac{(i_{x_l} + 2)(i_{x_l} + 1)}{(i_t + 2)(i_t + 1)g_0} a_{i_x+2e_l, i_t} - \sum_{j_x < i_x} \frac{g_{i_x - j_x}}{g_0} a_{j_x, i_t+2}$$

end

end

end
Quasi-Trefftz $\mathbf{x}t$-DG

Use $\prod_{K \in T_h} \mathbb{Q}^p_\infty(K)$ with $\mathbf{x}t$-DG for IBVP with piecewise-smooth $c$.

Use idea of (IMBERT-GÈRARD, MONK 2017): add volume penalty term

$$\sum_{K \in T_h} \int_K \mu_1 (\nabla \cdot \sigma + c^{-2} \partial_t \mathbf{v}) (\nabla \cdot \tau + c^{-2} \partial_t \mathbf{w}) + \mu_2 (\partial_t \sigma + \nabla \mathbf{v}) \cdot (\partial_t \tau + \nabla \mathbf{w}).$$

- Coercivity in DG norm (with volume terms)
- Well-posedness
- Quasi-optimality
- Error bounds (high-order $h$-convergence, optimal rates, explicit)

$$||| (\mathbf{v}, \sigma) - (\mathbf{v}_h, \sigma_h) |||_{DG} \leq C \sup_{K \in T_h} h^{p+1/2}_{K,c} |u|_{C^{p+2}_c(K)}.$$

Same DOF saving as for Helmholtz or constant $c$ ($\mathcal{O}(p^n)$ vs $\mathcal{O}(p^{n+1})$).
More general IBVPs

Everything extends to 2 piecewise-smooth material parameters $\rho, G$:

$$
\nabla v + \rho \partial_t \sigma = 0, \quad \nabla \cdot \sigma + G \partial_t v = 0,
$$

Wavespeed is $c = (\rho G)^{-1/2}$. Second-order version:

$$
- \nabla \cdot \left( \frac{1}{\rho} \nabla u \right) + G \partial_t^2 u = 0 \quad (v = \partial_t u, \ \sigma = -\frac{1}{\rho} \nabla u).
$$

Basis coefficient algorithm needs some more terms.
More general IBVPs

Everything extends to 2 piecewise-smooth material parameters \( \rho, G \):

\[
\nabla v + \rho \partial_t \sigma = 0, \quad \nabla \cdot \sigma + G \partial_t v = 0,
\]

Wavespeed is \( c = (\rho G)^{-1/2} \).

Second-order version:

\[
-\nabla \cdot \left( \frac{1}{\rho} \nabla u \right) + G \partial_t^2 u = 0 \quad (v = \partial_t u, \sigma = -\frac{1}{\rho} \nabla u).
\]

Basis coefficient algorithm needs some more terms.

If the 1st-order IBVP does not come from a 2nd-order one, we use

\[
\mathbb{Q}_{\mathbb{P}}^p(K) := \left\{ (w, \tau) \in \mathbb{P}^p(K)^{n+1} \mid \begin{array}{c}
D^i(\nabla w + \rho \partial_t \tau)(x_K, t_K) = 0 \\
D^i(\nabla \cdot \tau + G \partial_t w)(x_K, t_K) = 0 \\
\forall |i| \leq p - 1
\end{array} \right\}
\]

This space is only slightly larger \( \approx \frac{n+1}{2} \times \), still \( O_{p \to \infty}(p^n) \) DOFs and allows the same analysis.
Numerics

- Implemented in NGSolve.
- Both Cartesian and tent-pitched meshes.
- Volume penalty term not needed in computations.
- DG flux coefficients $\alpha^{-1} = \beta = c$, but even $\alpha = \beta = 0$ works.
- Good conditioning.
- Monomial bases $\{\hat{b}_J\}, \{\tilde{b}_J\}$ outperform Legendre/Chebyshev.
Comparing quasi-Trefftz, full polynomials, and Trefftz ($c|_K = c(\mathbf{x}_K)$) spaces

\[ \text{QW}^p(\mathcal{T}_h) := \{ (w, \tau) \in H(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \text{QU}^{p+1}(K) \} \]

\[ \text{Y}^p(\mathcal{T}_h) := \{ (w, \tau) \in H(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \mathbb{P}^{p+1}(K) \} \]

\[ \text{W}^p(\mathcal{T}_h) := \{ (w, \tau) \in H(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \mathbb{P}^{p+1}(K), \]

\[ -\Delta u + c^{-2}(\mathbf{x}_K)\partial^2_t u = 0 \text{ in } K \}. \]
Numerics 1: convergence

Compare quasi-Trefftz, full polynomials, Trefftz \((c|_K = c(\mathbf{x}_K))\) spaces

\[
\begin{align*}
\mathcal{QW}^p(\mathcal{T}_h) & := \{(w, \tau) \in \mathbf{H}(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \mathcal{QU}^{p+1}(K)\} \\
\mathcal{Y}^p(\mathcal{T}_h) & := \{(w, \tau) \in \mathbf{H}(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \mathbb{P}^{p+1}(K)\} \\
\mathcal{W}^p(\mathcal{T}_h) & := \{(w, \tau) \in \mathbf{H}(\mathcal{T}_h) : w|_K = \partial_t u, \ \tau|_K = -\nabla u, \ u \in \mathbb{P}^{p+1}(K), \ -\Delta u + c^{-2}(\mathbf{x}_K)\partial^2_t u = 0 \text{ in } K\}.
\end{align*}
\]

DG-norm error: optimal order in \(h\), exponential in \(p\).

\[
\begin{align*}
n = 2, \quad G = (x_1 + x_2 + 1)^{-1}, \quad u = (x_1 + x_2 + 1)^{2.5}e^{-\sqrt{7.5}t}, \quad Q = (0, 1)^3.
\end{align*}
\]
Quasi-Trefftz wins > 1 order of magnitude against full polynomials:

\[ \| (u, \sigma) - (v_{hp}, \sigma_{hp}) \|_{DG} \]

\[ h = 2^{-3}, 2^{-4}, \quad p = 1, 2, 3, 4. \]
\[ n = 2, \quad G = x_1 + x_2 + 1, \quad u = A_i(-x_1 - x_2 - 1) \cos(\sqrt{2}t), \quad Q = (0, 1)^3. \]
Numerics 3: tent pitching

\((n = 2)\) Final-time error, computational time (sequential), speedup: 
\((\#\text{dof}^{-1/3} \sim h)\)
Numerics 4: energy conservation

Plane wave through medium with $G = 1 + x_2$ in $(0, 1)^3$:

\[ \mathcal{E} = \frac{1}{2} \int_{\Omega} (c^{-2} v^2 + |\sigma|^2) \, dS \]

DG scheme is (provably) dissipative. For $p = 3$, $h = 2^{-7}$, only 0.076% loss.
Part 1: summary

Quasi-Trefftz DG:

- Extend Trefftz scheme to piecewise-smooth coefficients. Basis are PDE solution “up to given order in $h$.”
- Simple construction of basis functions: same “Cauchy data” at element centre as for Trefftz.
- Use in $\mathbf{x}t$-DG, stability and error analysis. High orders of convergence in $h$, much fewer DOFs than standard polynomial spaces.

Part II

$xt$-DG with point singularities

Bansal, Moiola, Perugia, Schwab
Wave solutions on polygons are singular

Fix $n = 2$.

Piecewise-constant $c$, on polygonal partition of $\Omega$. Denote by $\{c_i\}_{i=1,\ldots,M}$ the vertices of this partition.
Wave solutions on polygons are singular

Fix $n = 2$.

Piecewise-constant $c$, on polygonal partition of $\Omega$.
Denote by $\{c_i\}_{i=1,\ldots,M}$ the vertices of this partition.

Even for smooth initial conditions & source term, homogeneous BCs, the IBVP solution in polygon $\times (0, T)$ lives in corner-weighthed spaces:

$$(v, \sigma) = (\partial_t u, -\nabla u) \in C^{k_t-1}([0, T]; H^{k_x+1,2}_\delta(\Omega)) \times C^{k_t}([0, T]; H^{k_x,1}_\delta(\Omega)^2)$$

$$\|u\|^2_{H^{k,\ell}_\delta(\Omega)} := \|u\|^2_{H^{\ell-1}(\Omega)} + \sum_{m=\ell}^{k} \int_\Omega (\prod_{i=1}^{M} |x - c_i|^\delta_i \sum_{\alpha \in \mathbb{N}^2_0, \alpha_1 + \alpha_2 = m} |D^\alpha u|^2)$$


- This means $v(\cdot, t) \notin H^2(\Omega), \sigma(\cdot, t) \notin H^1(\Omega)^2$.

+ Diffraction singularities are confined (in space) to the corners $c_i$ and have smooth time-dependence.
Wave solutions on polygons are singular

Fix \( n = 2 \).

Piecewise-constant \( c \), on polygonal partition of \( \Omega \).
Denote by \( \{c_i\}_{i=1,\ldots,M} \) the vertices of this partition.

Even for smooth initial conditions & source term, homogeneous BCs, the IBVP solution in polygon \( \times (0, T) \) lives in corner-weighted spaces:

\[
(v, \sigma) = (\partial_t u, -\nabla u) \in C^{k_t-1}([0, T]; H^{k_x+1,2}_\delta(\Omega)) \times C^{k_t}([0, T]; H^{k_x,1}_\delta(\Omega)^2)
\]

\[
\|u\|_{H^{k,\ell}_\delta(\Omega)}^2 := \|u\|_{H^{\ell-1}(\Omega)}^2 + \sum_{m=\ell}^k \int_\Omega (\prod_{i=1}^M |x - c_i|^{\delta_i} \sum_{\alpha \in \mathbb{N}_0^2 \atop \alpha_1 + \alpha_2 = m} |D\alpha u|^2)
\]


- This means \( v(\cdot, t) \notin H^2(\Omega), \sigma(\cdot, t) \notin H^1(\Omega)^2 \).

+ Diffraction singularities are confined (in space) to the corners \( c_i \) and have smooth time-dependence.

\( \rightarrow \) Suggests local mesh refinement in space only.
Locally-refined product meshes

Locally-refined mesh in space $\times$ quasi-uniform mesh in time:

Space-like faces are horizontal.

To avoid short time steps, corner elements will be "tall&thin": → implicit method.

Can’t use Trefftz spaces as they requires some $\textbf{x}t$-shape regularity.

$$V_p(\mathcal{T}_h) = \prod_{K=K_x \times I_n \in \mathcal{T}_h} \left( \mathbb{P}_{p_x,K} (K_x) \otimes \mathbb{P}_{p_t,K} (I_n) \right) \times \left( \mathbb{P}_{p_x,K} (K_x) \otimes \mathbb{P}_{p_t,K} (I_n) \right)^2.$$
Locally-refined product meshes

Locally-refined mesh in space $\times$ quasi-uniform mesh in time:

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DG semi-norm is not a norm on $V_p(\mathcal{T}_h)$: “coercivity analysis” is not enough for well-posedness.
Well-posedness

In general, assume that “PDEs map local discrete space into itself”:

\[
\left( \nabla \cdot \tau_h + c^{-2} \partial_t \omega_h, \nabla \omega_h + \partial_t \tau_h \right) \in V_p(T_h) \quad \forall (\omega_h, \tau_h) \in V_p(T_h).
\]

Holds, e.g., for \( V_p(T_h) \) with \( |p_{x,K}^\sigma - p_{x,K}^\nu| \leq 1, \quad p_{i,K}^\sigma = p_{i,K}^\nu \).

This ensures that the method is well-posed.
In general, assume that “PDEs map local discrete space into itself”:

\[
\left( \nabla \cdot \tau_h + c^{-2} \partial_t w_h, \nabla w_h + \partial_t \tau_h \right) \in \mathbf{V}_p(\mathcal{T}_h) \quad \forall (w_h, \tau_h) \in \mathbf{V}_p(\mathcal{T}_h).
\]

Holds, e.g., for \( \mathbf{V}_p(\mathcal{T}_h) \) with \( |p_{\sigma,K}^\sigma - p_{\nu,K}^\nu| \leq 1, \quad p_{\sigma,K}^\sigma = p_{\nu,K}^\nu \).

This ensures that the method is well-posed:

- Assume \( A((v_h, \sigma_h), (w_h, \tau_h)) = 0 \quad \forall (w_h, \tau_h) \in \mathbf{V}_p(\mathcal{T}_h) \).
- \( 0 = A((v_h, \sigma_h), (v_h, \sigma_h)) = \| (v_h, \sigma_h) \|_{DG}^2 \Rightarrow \) jump and boundary traces of \((v_h, \sigma_h)\) vanish.
- After IBP, only volume terms are left in \( A((v_h, \sigma_h), (w_h, \tau_h)) \):
  \[
  0 = A((v_h, \sigma_h), (w_h, \tau_h)) =
  - \sum_{K \in \mathcal{T}_h} \int_K \left( (\nabla \cdot \sigma_h + c^{-2} \partial_t v_h) w_h + (\nabla v_h + \partial_t \sigma_h) \cdot \tau_h \right) dV
  \]
- Choose \( w_h = \nabla \cdot \sigma_h + c^{-2} \partial_t v_h \) and \( \tau_h = \nabla v_h + \partial_t \sigma_h \):
  \((v_h, \sigma_h)\) solves homogeneous IBVP.
- \( \Rightarrow (v_h, \sigma_h) = (0, 0) \).
Quasi-optimality and unconditional stability

Under the same assumption, DG norm of error is controlled by error of $L^2$-projection on $V_p(T_h)$:

$$\|\|(v, \sigma) - (v_h, \sigma_h)\|_{DG} \leq (3 + p_{x, \square})\|\|(v, \sigma) - (\Pi_{L^2} v, \Pi_{L^2} \sigma)\|_{DG^+}$$

Here $\|\cdot\|_{DG^+}$ is a skeleton seminorm, stronger than $\|\cdot\|_{DG}$.

It includes $\|\alpha^{-1/2} (\sigma - \Pi_{L^2} \sigma) \cdot n_x \|_{L^2(F_t, L^1(F_x))}$ terms on time-like faces of corner elements, to accommodate $H^{1,1}_\delta$ arguments.

$p_{x, \square}$ is the polynomial degree in $x$ used in corner elements

(from inverse & trace estimates for $H^{1,1}_\delta$)
Quasi-optimality and unconditional stability

Under the same assumption, DG norm of error is controlled by error of $L^2$-projection on $V_p(\mathcal{T}_h)$:

$$
\frac{1}{2} \| c^{-1} (v - v_h) \|_{L^2(\Omega \times \{t_n\})} + \frac{1}{2} \| \sigma - \sigma_h \|_{L^2(\Omega \times \{t_n\})^2} \leq
\|(v, \sigma) - (v_h, \sigma_h)\|_{DG} \leq (3 + p_{x,\angle}) \|(v, \sigma) - (\Pi_{L^2} v, \Pi_{L^2} \sigma)\|_{DG^+}
$$

Here $\| \cdot \|_{DG^+}$ is a skeleton seminorm, stronger than $\| \cdot \|_{DG}$.

It includes $\| \alpha^{-1/2} (\sigma - \Pi_{L^2} \sigma) \cdot n_x \|_{L^2(F_t, L^1(F_x))}$ terms on time-like faces of corner elements, to accommodate $H^{1,1}_{\delta}$ arguments.

$p_{x,\angle}$ is the polynomial degree in $x$ used in corner elements (from inverse & trace estimates for $H^{1,1}_{\delta}$)

Bound controls also $L^2(\Omega)$ error at discrete times.
$L^2$-projection & Galerkin error bounds

To obtain concrete error bound, we need approximation bounds for the $L^2(K)$ projection on $\mathbb{P}^{p_x}(K_x) \times \mathbb{P}^{p_t}(t_{n-1}, t_n)$, in Bochner norms, via Peetre–Tartar lemma$^1$:

$$\|\varphi - \Pi L^2 \varphi\|_{L^2(I_n; L^2(K_x))} + h_n |\varphi - \Pi L^2 \varphi|_{H^1(I_n; L^2(K_x))} + h_{K_x} |\varphi - \Pi L^2 \varphi|_{L^2(I_n; H^1(K_x))} \lesssim h_n^{s_t+1} |\varphi|_{H^{s_t+1}(I_n; L^2(K_x))} + h_{K_x}^{s_x+1} |\varphi|_{L^2(I_n; H^{s_x+1}(K_x))},$$

and similarly for weighted spaces.

---

$^1$A : $X \rightarrow Y \text{ injective, } T : X \rightarrow Z \text{ compact, } \|x\|_X \lesssim \|Ax\|_Y + \|Tx\|_Z \Rightarrow \|x\|_X \lesssim \|Ax\|_Y$.

Here, $X = H^{s_t+1}(I; L^2(K_x)) \cap L^2(I; H^{s_x+1}(K_x)) \overset{T}{\hookrightarrow} L^2(K)$,

$$X \xrightarrow{A=(\Pi_{L^2}, \partial_t^{s_t+1}, D_x^{s_x+1})} (\mathbb{P}^{s_x}(K_x) \otimes \mathbb{P}^{s_t}(I)) \times L^2(K) \times L^2(K)^{s_x+2}.$$
To obtain concrete error bound, we need approximation bounds for the $L^2(K)$ projection on $\mathbb{P}^{p_x}(K_x) \times \mathbb{P}^{p_t}(t_{n-1}, t_n)$, in Bochner norms, via Peetre–Tartar lemma:\(^1\):

$$
\|\varphi - \Pi L^2 \varphi\|_{L^2(I_n; L^2(K_x))} + h_n \|\varphi - \Pi L^2 \varphi\|_{H^1(I_n; L^2(K_x))} + h_{K_x} \|\varphi - \Pi L^2 \varphi\|_{L^2(I_n; H^1(K_x))} \\
\lesssim h_n^{s+1} \|\varphi\|_{H^{s+1}(I_n; L^2(K_x))} + h_{K_x}^{s+1} \|\varphi\|_{L^2(I_n; H^{s+1}(K_x))},
$$

and similarly for weighted spaces.

For smooth solutions + quasi-uniform meshes + uniform degree $p$:

$$
\|c^{-1}(\nu - \nu_h)\|_{L^2(\Omega \times \{t_n\})} + \|\sigma - \sigma_h\|_{L^2(\Omega \times \{t_n\})^2} \lesssim h^{p+\frac{1}{2}}
$$

\(\frac{1}{2}\)-order suboptimal: $h^{p+1}$ from numerics.

\(^1\)A : $X \to Y$ injective, $T : X \to Z$ compact, $\|x\|_X \lesssim \|Ax\|_Y + \|Tx\|_Z \Rightarrow \|x\|_X \lesssim \|Ax\|_Y$.

Here, $X = H^{s+1}(I; L^2(K_x)) \cap L^2(I; H^{s+1}(K_x)) \overset{T}{\hookrightarrow} L^2(K)$,

\[
X \xrightarrow{A=(\Pi L^2, \partial_t^{s+1}, \partial_x^{s+1})} (\mathbb{P}^{p_x}(K_x) \otimes \mathbb{P}^{p_t}(I)) \times L^2(K) \times L^2(K)^{s+2}
\]
Error bounds: singular solutions & graded meshes

- \((v, \sigma) \in C^{k-1}([0, T]; H^{k+1,2}_\delta(\Omega)) \times C^k([0, T]; H^{k,1}_\delta(\Omega)^2),\)
  \(k_x \geq 1, k_t \geq 2,\)

- graded mesh \(\mathcal{T}_{h_x}^x\) in \(x\) (GASPOZ–MORIN), max size \(h_x\),
  refinement of uniform \(\mathcal{T}_0^x\) with \(# \mathcal{T}_{h_x}^x - # \mathcal{T}_0^x \leq C h_x^{-2}\)

- \(h_x \sim h_t \sim h\)

- uniform polynomial degrees \(p\) (in \(x\&t, v\&\sigma, K\))

- numerical flux parameters \(\alpha^{-1} = \beta = c \frac{h_{Fx}}{h_x} = c \frac{\text{local}}{\text{global}}\)

\[\Rightarrow \quad \| c^{-1}(v - v_h) \|_{L^2(\Omega \times \{t_n\})} + \| \sigma - \sigma_h \|_{L^2(\Omega \times \{t_n\})^2} \lesssim h^{\min\{k-\frac{1}{2}, p+\frac{1}{2}\}}\]

Again, numerics on \(L\)-shape give \(h^{p+1}\) rates.
Want to use a **sparse grid** approach in space–time.

Take initial mesh $\mathcal{T}_{0,0}$ of size $h_{0,x}, h_{0,t}$.

For $(l_x, l_t) \in \mathbb{N}_0^2$,

denote $\mathcal{T}_{l_x,l_t}$ a refinement of $\mathcal{T}_{0,0}$ with

\[ h_{l_x,x} = 2^{-l_x} h_{0,x}, \quad h_{l_t,t} = 2^{-l_t} h_{0,t}, \]

$w_{l_x,l_t}$ = corresponding DG solution (same polynomial space \( \forall \) element).
Sparse $x t$-DG

Want to use a sparse grid approach in space–time.

Take initial mesh $\mathcal{T}_{0,0}$ of size $h_{0,x}, h_{0,t}$.
For $(l_x, l_t) \in \mathbb{N}_0^2$, denote $\mathcal{T}_{l_x,l_t}$ a refinement of $\mathcal{T}_{0,0}$ with

$$h_{l_x,x} = 2^{-l_x} h_{0,x}, \quad h_{l_t,t} = 2^{-l_t} h_{0,t},$$

$w_{l_x,l_t} = \text{corresponding DG solution (same polynomial space $\forall$ element)}$.

Combination formula:

$$\hat{w}_L := \sum_{l=0}^{L} w_{l,L-l} - \sum_{l=0}^{L-1} w_{l,L-1-l}$$


We observe comparable accuracy for full-tensor $w_{L,L}$ and sparse $\hat{w}_L$:

$$\|(v, \sigma) - w_{L,L}\|_{L^2(\Omega \times \{T\})} \approx \|(v, \sigma) - \hat{w}_L\|_{L^2(\Omega \times \{T\})}.$$ 

Consistent with sparse grid theory, which we can’t apply here.

So why is it convenient?
We observe comparable accuracy for full-tensor $\mathbf{w}_{L,L}$ and sparse $\hat{\mathbf{w}}_{L}$:

$$\|(v, \sigma) - \mathbf{w}_{L,L}\|_{L^2(\Omega \times \{T\})} \approx \|(v, \sigma) - \hat{\mathbf{w}}_{L}\|_{L^2(\Omega \times \{T\})}.$$  

Consistent with sparse grid theory, which we can’t apply here.

So why is it convenient? Same accuracy but cheaper!

\[
\begin{align*}
\text{#DOFs}^{\text{full}} &= \mathcal{O}(p^3 2^{3L}) = \mathcal{O}(p^3 h_L^{-3L}), \\
\text{#DOFs}^{\text{sparse}} &= \mathcal{O}(p^3 2^{2L}) = \mathcal{O}(p^3 h_L^{-2L}).
\end{align*}
\]  

$(h_{0,x} = h_{0,t})$
Sparse vs full $xt$-DG: accuracy and $\#$DOFs

We observe comparable accuracy for full-tensor $w_{L,L}$ and sparse $\hat{w}_L$:

$$\|(u, \sigma) - w_{L,L}\|_{L^2(\Omega \times \{T\})} \approx \|(u, \sigma) - \hat{w}_L\|_{L^2(\Omega \times \{T\})}.$$ 

Consistent with sparse grid theory, which we can’t apply here.

So why is it convenient?

Same accuracy but cheaper!

$$\#\text{DOFs}^{\text{full}} = \mathcal{O}(p^3 2^{3L}) = \mathcal{O}(p^3 h_L^{-3L}),$$

$$\#\text{DOFs}^{\text{sparse}} = \mathcal{O}(p^3 2^{2L}) = \mathcal{O}(p^3 h_L^{-2L}).$$

(p=1 $\rightarrow$ $\leftarrow$ p=2)

Singular solution on $L$-shape, mesh locally refined in $x$.

$\rightarrow$ #DOFs is not where sparse scheme wins...
Sparse vs full $xt$-DG: complexity

Not only #DOFs differ but also sizes & numbers of linear systems.

**Full-tensor** $\mathbf{w}_{L,L}$ requires:

- $O(2^L) \times$ solves of size $O(2^{2L})$

**Sparse** $\hat{\mathbf{w}}_L$ requires:

- $O(1) \times$ solves of size $O(2^{2L})$
- $O(2) \times$ solves of size $O(2^{2(L-1)})$
- $\vdots$
- $O(2^L) \times$ solves of size $O(1)$

Total complexity is the same as a single elliptic solve in $\Omega(\subset \mathbb{R}^2) \times$ logarithmic terms.

Includes CFL-violating solves:

- requires unconditionally stable formulation.
Part 2: summary

- Unconditionally stable $\mathbf{x}t$-DG formulation, discrete functions are tensor-product polynomials.
- Well-posedness and error control also for solutions with point singularities.
- $h^{p+\frac{1}{2}}$ convergence rates for smooth solutions and quasi-uniform meshes, for singular solutions and refined meshes.
- Sparse version: same accuracy, fewer DOFs, lower complexity.

Main future work: sparse $\mathbf{x}t$-DG error analysis.

(BANSAL, M., PERUGIA, SCHWAB, IMA JNA, 2021)
Part 2: summary

- Unconditionally stable $\mathbf{x}t$-DG formulation, discrete functions are tensor-product polynomials.
- Well-posedness and error control also for solutions with point singularities.
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- Sparse version: same accuracy, fewer DOFs, lower complexity.

Main future work: sparse $\mathbf{x}t$-DG error analysis.

(BANSAL, M., PERUGIA, SCHWAB, IMA JNA, 2021)

Thank you!
Quasi-optimality

In non-Trefftz case, assume
\[
\left( \nabla \cdot \tau_h + c^{-2} \partial_t \omega_h, \ nabla \omega_h + \partial_t \tau_h \right) \in V_p(\mathcal{T}_h) \quad \forall (\omega_h, \tau_h) \in V_p(\mathcal{T}_h);
\]

Then
\[
\left| (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h) \right|_{DG(Q_n)}^2 \\
= A_{DG(Q_n)} \left( (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h); (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h) \right) \\
= A_{DG(Q_n)} \left( (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v, \sigma); (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h) \right) \\
\leq 2C_{\infty} \left| (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v, \sigma) \right|_{DG(Q_n)} + \left| (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h) \right|_{DG(Q_n)}.
\]

Last ineq. uses inverse inequality on corner elements and cancellation of volume terms due to choice of $L^2$ projection.

\[
\frac{1}{2} \left\| c^{-1} (v - v_h) \right\|_{L^2(\Omega \times \{t_n\})} + \frac{1}{2} \left\| \sigma - \sigma_h \right\|_{L^2(\Omega \times \{t_n\})}^2 \\
\leq \left| (v, \sigma) - (v_h, \sigma_h) \right|_{DG(Q_n)} \\
\leq \left| (v, \sigma) - (\Pi_{L^2} v, \Pi_{L^2} \sigma) \right|_{DG(Q_n)} + \left| (\Pi_{L^2} v, \Pi_{L^2} \sigma) - (v_h, \sigma_h) \right|_{DG(Q_n)} \\
\leq (1 + 2C_{\infty} |2|) \left| (v, \sigma) - (\Pi_{L^2} v, \Pi_{L^2} \sigma) \right|_{DG(Q_n)}. \]