Explicit bounds for electromagnetic transmission problems

Andrea Moiola
Maxwell equations in heterogeneous media

Given:

- wavenumber \( k > 0 \)
- sources \( \mathbf{J}, \mathbf{K} \in H(\text{div}^0; \mathbb{R}^3) \), compactly supported
- \( \epsilon_0, \mu_0 > 0 \)
- \( \epsilon, \mu \in L^\infty(\mathbb{R}^3; \text{SPD}) \) such that
  \[ \Omega_i := \text{int}(\text{supp}(\epsilon - \epsilon_0 \mathbb{1}) \cup \text{supp}(\mu - \mu_0 \mathbb{1})) \] is bounded and Lipschitz

Find \( \mathbf{E}, \mathbf{H} \in H_{\text{loc}}(\text{curl}; \mathbb{R}^3) \) such that

\[
\begin{align*}
    \text{i} k \epsilon \mathbf{E} + \nabla \times \mathbf{H} &= \mathbf{J} \quad \text{in } \mathbb{R}^3, \\
    -\text{i} k \mu \mathbf{H} + \nabla \times \mathbf{E} &= \mathbf{K} \quad \text{in } \mathbb{R}^3,
\end{align*}
\]

(\( \mathbf{E}, \mathbf{H} \)) satisfy Silver–Müller radiation condition

\[
|\sqrt{\epsilon_0} \mathbf{E} - \sqrt{\mu_0} \mathbf{H} \times \frac{\mathbf{x}}{||\mathbf{x}||}| = O_{||\mathbf{x}|| \to \infty} (||\mathbf{x}||^{-2}).
\]

Special case: “transmission problem”, i.e. homogeneous scatterer

\[
\begin{align*}
    \epsilon &= \begin{cases} 
        \epsilon_i & \text{in } \Omega_i \\
        \epsilon_0 & \text{in } \Omega_o := \mathbb{R}^3 \setminus \overline{\Omega_i}
    \end{cases} \\
    \mu &= \begin{cases} 
        \mu_i & \text{in } \Omega_i \\
        \mu_0 & \text{in } \Omega_o := \mathbb{R}^3 \setminus \overline{\Omega_i}
    \end{cases} \quad 0 < \epsilon_i, \epsilon_0, \mu_i, \mu_0 \text{ constant}.
\end{align*}
\]
Wave scattering

The example we have in mind is incident wave $E^{Inc}, H^{Inc}$ hitting $\Omega_i$:

$\rightarrow$ BVP with data supported on $\Omega_i$:

$J = i k^2 (\varepsilon_0 - \varepsilon) E^{Inc}$,

$K = i k^2 (\mu - \mu_0) H^{Inc}$.

Incoming field

$E^{Inc} = \sqrt{\frac{\mu_0}{\varepsilon_0}} A e^{i k \sqrt{\varepsilon_0 \mu_0} x \cdot d}$

$H^{Inc} = d \times A e^{i k \sqrt{\varepsilon_0 \mu_0} x \cdot d}$

Scattered field

$E$

$H$

Total field

$E + E^{Inc}$

$H + H^{Inc}$

datum

BVP solution

physical field
If $\epsilon, \mu$ are sufficiently regular then the problem is well-posed. From Fredholm theory we have

$$\left\| \begin{pmatrix} E \\ H \end{pmatrix} \right\|_{\partial \Omega_{i/o}} \leq C \left\| \begin{pmatrix} J \\ K \end{pmatrix} \right\|_{\partial \Omega_{i/o}}$$

Goal: find out how $C = C(k, \epsilon, \mu)$ depends on $k, \epsilon$ and $\mu$.

Why? In FEM & BEM analysis and in UQ for time-harmonic problems, explicit parameter dependence allows to control:

- Quasi-optimality & pollution effect
- Gmres iteration numbers
- Matrix compression
- $hp$-FEM & BEM (Melenk–Sauter)
- Shape differentiation & uncertainty quantification
- . . .
Who cares?

LAFONTAINE, SPENCE, WUNSCH, arXiv 2019: (Helmholtz)

The following is a non-exhaustive list of papers on the frequency-explicit convergence analysis of numerical methods for solving the Helmholtz equation where a central role is played by either the non-trapping resolvent estimate (1.5), or its analogue (with the same $k$-dependence) for the commonly-used approximation of the exterior problem where the exterior domain $O_+$ is truncated and an impedance boundary condition is imposed:

- conforming FEMs (including continuous interior-penalty methods) [72, Proposition 2.1], [74, Proposition 8.1.4], [56, Lemma 2.1], [77, Lemma 3.5], [78, Assumptions 4.8 and 4.18], [45, §2.1], [110, Theorem 3.1], [113, §3.1], [44, §3.2.1], [40, Remark 3.2], [41, Remark 3.1], [29, Assumption 1], [30, Definition 2], [55, Theorem 3.2], [50, Lemma 6.7], [14, Equation 4],
- least squares methods [33, Assumption A1], [10, Remark 1.2], [64, Assumption 1 and equation after Equation 5.37],
- DG methods based on piece-wise polynomials [46, Theorem 2.2], [47, Theorem 2.1], [39, Assumption 3], [48, §3], [62, Assumption A (Equation 4.5)], [76, Equation 4.4], [35, Remark 3.2], [32, Equation 2.4], [84, Equation 4.3], [112, Remark 3.1], [94, Theorem 2.2],
- plane-wave/Trefftz-DG methods [3, Theorem 1], [59, Equation 3.5], [60, Theorem 2.2], [2, Lemma 4.1], [61, Proposition 2.1],
- multiscale finite-element methods [51, Equation 2.3], [13, §1.2], [88, Assumption 5.3], [8, Theorem 1], [87, Assumption 3.8], [31, Assumption 1],
- integral-equation methods [71, Equation 3.24], [75, Equation 4.4], [24, Chapter 5], [53, Theorem 3.2], [111, Remark 7.5], [43, Theorem 2], [49, Theorem 3.2], [52, Assumption 3.2],

In addition, the following papers focus on proving bounds on the solution of Helmholtz boundary-value problems (with these bounds often called “stability estimates”) motivated by applications in numerical analysis: [36], [57], [26], [11], [7], [70], [98], [28], [6], [9], [27], [93], [54], [55], [83], [50]. Of these papers, all but [70], [6], [27], [11] are in nontrapping situations, [70], [6], [27] are in parabolic trapping scenarios, and [11] proves the exponential growth (1.7) under elliptic trapping.
What about Helmholtz?  

(M. & S. 2019)

Simplest heterogeneous Helmholtz problem:

\[ \Delta u + k^2 n u = f \quad \text{in } \mathbb{R}^d \]
+Sommerfeld radiation c.

\[ f \in L^2(\mathbb{R}^d), \quad n = \begin{cases} n_i & \text{constant in } \Omega_i, \\ 1 & \text{in } \Omega_o. \end{cases} \]

\[ \Omega_i \cup \text{supp } f \subset B_R \]

\[ \text{find } u \in H^1_{\text{loc}}(\mathbb{R}^d) \text{ s.t.} \]

\[ \Delta u + k^2 n u = f \quad \text{in } \mathbb{R}^d \]

\[ \Omega_i \cup \text{supp } f \subset B_R \]

\[ \| \nabla u \|_{L^2(B_R)}^2 + k^2 \| \sqrt{n} u \|_{L^2(B_R)}^2 \leq \left[ 4R^2 + \frac{1}{n_i} \left( 2R + \frac{d-1}{k} \right)^2 \right] \| f \|_{L^2(B_R)}^2 \]

Fully explicit, \( k \)-independent, shape-robust estimate.  
(For \( d = 2 \) it implies bounds for Maxwell TE/TM modes.)

\[ \text{If } 0 < n_i < 1, \ \Omega_i \text{ star-shaped} \]

\[ \text{If } n_i > 1, \ \Omega_i \text{ strictly convex & } C^\infty: \]

superalgebraic blow up in \( k \), quasi-resonances,  
ray trapping, creeping waves . . .

Dependence on parameters is complicated!  
Monotonicity of \( n \) & shape of \( \Omega_i \) are crucial.
Wavenumber-explicit bounds: a bit of history

- **Morawetz 1960s/70s:** introduced main tools (multipliers)
- **Meulen 1995:** 1st $k$-explicit bound for Helmholtz, bdd dom.
- **Chandler-Wilde, Monk 2008:** unbounded domains
- **Hiptmaur, Moiola, Perugia 2011:** Maxwell, bdd dom.

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- **Moiola, Spence 2019:** Helmholtz & piecewise-constant $n$
- **Graham, Pembery, Spence 2019:** Helmholtz & general coeff.
- **Verfürth 2019:** Maxwell & impedance

Plenty of other related contributions exist!

Barucq, Chaumont-Frelet, Feng, Hetmaniuk, Lorton, Peterseim, Sauter, Torres, Wieners & Wohlmuth, (your name here), . . .

Our goal: extend (Graham, Pembery, Spence 2019) to Maxwell eq.s.
Bound #1: transmission problem

Single homogeneous scatterer:

\[ \epsilon = \begin{cases} \epsilon_i & \text{in } \Omega_i \\ \epsilon_0 & \text{in } \Omega_0 \end{cases}, \quad \mu = \begin{cases} \mu_i & \text{in } \Omega_i \\ \mu_0 & \text{in } \Omega_0 \end{cases}, \quad 0 < \epsilon_i, \epsilon_0, \mu_i, \mu_0 \text{ constant.} \]

If \( \epsilon_i \leq \epsilon_0 \), \( \mu_i \leq \mu_0 \), \( \Omega_i \) star-shaped, \( \Omega_i \cup \text{supp } J \cup \text{supp } K \subset B_R \), then

\[ \epsilon_i \|E\|_{B_R}^2 + \mu_i \|H\|_{B_R}^2 \leq 4R^2 \left( \frac{\epsilon_0}{\epsilon_i} + \frac{\mu_0}{\mu_i} \right) \left( \epsilon_0 \|K\|_{B_R}^2 + \mu_0 \|J\|_{B_R}^2 \right). \]

Equivalent to wavenumber-independent \( H(\text{curl}; B_R) \) bound for \( E \).

If \( \epsilon_i \) is (constant) SPD matrix, same holds if \( \max \text{eig}(\epsilon_i) \leq \epsilon_0 \) and with \( \epsilon_i \) substituted by \( \min \text{eig}(\epsilon_i) \) in the bound.

Same for \( \mu_i \).
Bound #2: more general $\epsilon, \mu$

Assume $\epsilon, \mu \in W^{1,\infty}(\Omega_i; SPD), \Omega_i$ Lipschitz,

- $\Omega_i$ star-shaped
- $\|\epsilon_i\|_{L^\infty(\partial\Omega_i)} \leq \epsilon_0$, $\|\mu_i\|_{L^\infty(\partial\Omega_i)} \leq \mu_0$, i.e. jumps are “upwards” on $\partial\Omega_i$
- $\epsilon_* := \text{ess inf}_{x \in \Omega_i} (\epsilon + (x \cdot \nabla)\epsilon) > 0$, $\mu_* := \text{ess inf}_{x \in \Omega_i} (\mu + (x \cdot \nabla)\mu) > 0$
  “weak monotonicity” in radial direction, avoid trapping of rays
- “extra regularity” ($E, H \in H^1(\Omega_i \cup \Omega_o)^3$ or $\epsilon, \mu \in C^1(\Omega_i)$ or $W^{1,\infty}(\mathbb{R}^3)$)

Then we have explicit wavenumber-independent bound:

$$\epsilon_* \|E\|^2_{BR} + \mu_* \|H\|^2_{BR} \leq 4R^2 \left( \frac{\|\epsilon\|^2_{L^\infty(B_R)}}{\epsilon_*} + \frac{\epsilon_0 \mu_0}{\mu_*} \right) \|K\|^2_{BR} + 4R^2 \left( \frac{\|\mu\|^2_{L^\infty(B_R)}}{\mu_*} + \frac{\epsilon_0 \mu_0}{\epsilon_*} \right) \|J\|^2_{BR}. $$

Expect (from Helmholtz analogy) superalgebraic blow up in $k$ if any of the first 3 assumptions is lifted.

Similar results when $\mathbb{R}^3$ is truncated with impedance BCs.
How our bound was obtained

First consider smooth case $E, H \in C^1(\mathbb{R}^3; \mathbb{C}^3)$.

(i) Multiply the 2 PDEs by the “test fields” (Morawetz multipliers)

\[
\begin{align*}
(\epsilon E \times x + R\sqrt{\epsilon \mu} H) & \quad \& \quad (\mu H \times x - R\sqrt{\epsilon \mu} E), \\
(\epsilon_0 E \times x + r\sqrt{\epsilon_0 \mu_0} H) & \quad \& \quad (\mu_0 H \times x - r\sqrt{\epsilon_0 \mu_0} E).
\end{align*}
\]

in $B_R \supset \Omega_i$,

(ii) integrate by parts in $\Omega_i, B_R \setminus \overline{\Omega_i}$ and $\mathbb{R}^3 \setminus B_R$,

(iii) sum 3 contributions, (iv) take real part, (v) have fun!

\[
\int_{B_R} E \cdot (\epsilon + (x \cdot \nabla)\epsilon) E + H \cdot (\mu + (x \cdot \nabla)\mu) H \geq \epsilon_* \text{ by assumpt.} \\
\geq \mu_* \text{ by assumpt.}
\]

Using PDEs & $\nabla \cdot [\epsilon E] = \nabla \cdot [\mu H] = 0$

\[
= 2 \int_{B_R} \Re \left\{ K \cdot (\epsilon E \times x + \sqrt{\epsilon_0 \mu_0} R H) + J \cdot (\mu H \times x - \sqrt{\epsilon_0 \mu_0} R E) \right\}
\]

+ $\int_{\partial \Omega_i} \left[ \text{terms from IBP} \right] \leq 0 \text{ by } \epsilon_i \leq \epsilon_0, \mu_i \leq \mu_0, \quad n \cdot x \geq 0, \quad [E_T, H_T, (\epsilon E)_N, (\mu H)_N] = 0$

+ $\int_{\partial B_R} \left[ \text{terms from IBP} \right] \leq 0 \text{ by S–M radiation c.}$

Conclude by Cauchy–Schwarz.
Rough coefficients, regularity and density

Proof in previous slide only uses elementary results if \( E, H \in C^1(\mathbb{R}^3; \mathbb{C}^3) \).

For general case we need density of inclusion

\[
C^\infty(D) \subset \left\{ v \in H(\text{curl}; D), \nabla \cdot [A\mathbf{v}] \in L^2(D), A\mathbf{v} \cdot \hat{n} \in L^2(\partial D), v_T \in L^2_T(\partial D) \right\}
\]

for \( A = \epsilon \) & \( A = \mu \), \( D \) Lipschitz bdd.

If \( A \in C^1(\Omega_i; \text{SPD}) \), this density is non-trivial but follows from regularity results for layer potentials on manifolds (Mitrea, Taylor 1999).

- Equivalent step for Helmholtz was much simpler.
- Constant scalar \( \epsilon \) & \( \mu \): density proved in Costabel, Dauge 1998.
- If \( E, H \in H^1_{\text{loc}}(\mathbb{R}^3; \mathbb{C}^3) \) then no density is needed. E.g. ensured if \( \epsilon, \mu \in W^{1,\infty}(\mathbb{R}^3; \text{SPD}) \) (no jumps).
- What about \( A \in W^{1,\infty}(\Omega_i; \text{SPD}) \)?
Summary

Time-harmonic Maxwell eq.s in $\mathbb{R}^3$ with heterogeneous inclusion:
- fully explicit bounds on $\|E\|_{H(\text{curl},B_R)}$ if $\epsilon, \mu$ “radially growing”
- also for impedance BVPs in star-shaped domains
- extends Helmholtz results from [GRAHAM, PEMBERY, SPENCE 2019]

Some open questions:
- resonance-free strip in complex $k$ plane?
- presence of quasi-resonances blow up for “wrong” coefficients?
- rougher ($W^{1,\infty}(\Omega_i; SPD), L^\infty$) coefficients?
- relation with shape-differentiation and UQ?

Preprint coming soon...