Scattering of time-harmonic acoustic waves: Helmholtz equation, boundary integral equations and BEM

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This is a preliminary and incomplete version of the notes.
Please let me know if you find any error or have any suggestion.

1 The Helmholtz equation

The main character in this course is the Helmholtz equation, which is the following partial differential equation (PDE):

$$\boxed{\Delta u + k^2 u = 0.} \tag{1}$$

Here $\Delta = \sum_{j=1}^{n} \partial_{x_j}^2$ is the Laplace operator in n variables for $n \in \{1, 2, 3\}$, $k > 0$ is a real parameter called wavenumber, and u is the unknown of the equation, a scalar field defined on a subset of $\mathbb{R}^n$. We will mostly consider the two-dimensional case $n = 2$.

In the following, we first introduce the derivation of the Helmholtz equation from different physical phenomena, showing that it can be used to model different kinds of linear wave problems. Then we show some simple special solutions of the equation. We describe some boundary value problems (BVPs) and focus on one of them, the exterior Dirichlet problem. We show how to reformulate this as a boundary integral equation (BIE) and introduce numerical methods to approximate its solution.

In the first few sections the approach will be more "physical" than "mathematical", so we will not make precise assumptions and will gloss over some issues such as the regularity of the objects involved or the admissibility of some operations.

At the end of this document a few useful references are listed. [CJ77] is classical book that describes very clearly and succinctly many kinds of wave phenomena, developing both physical intuition and mathematical formalism. Several Helmholtz (and Maxwell) BVPs and the corresponding boundary integral equations (BIEs) are analysed in mathematically rigorous way in [CK1, §3], [CK2, §1–3], [Néd01, §2–3]. [Spence14] is a survey of several variational formulations for Helmholtz and Laplace BVPs, the corresponding BIEs and the numerical methods for their discretisation; it is a very clear introduction to Helmholtz problems and the related literature. The lecture notes [Sayas15] and [Sayas06] are very good introductions to BIEs for Laplace and Helmholtz equations, respectively, and their discretisations with the boundary element method (BEM).

1.1 Derivation of the Helmholtz equation

1.1.1 Acoustics

We want to describe the propagation of a sound wave in a fluid. We denote by $\rho(\mathbf{x}, t)$ the **density**, by $p(\mathbf{x}, t)$ the **pressure** and by $\mathbf{v}(\mathbf{x}, t)$ the **velocity** of the fluid in a point $\mathbf{x} \in \mathbb{R}^n$ at time $t \in \mathbb{R}$. We denote by ∇, div (or $\nabla\cdot$) and Δ the gradient, the divergence and the Laplacian in the space coordinate $\mathbf{x}$ only (i.e. without derivatives in t). Conservation of mass gives the **continuity equation**

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \mathbf{v}) = 0$$

and conservation of momentum gives **Euler's equation**

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \frac{1}{\rho}\nabla p = \mathbf{0}$$

where $[(\mathbf{v}\cdot\nabla)\mathbf{v}]_j = \sum_{m=1}^n v_m \partial_{x_m} v_j$. Both PDEs are non-linear as they contain quadratic terms (products of ρ and $\mathbf{v}$ or $\mathbf{v}$ and its derivatives). We assume that all three quantities considered are small perturbations of the constant[1] static values:

$$\rho(\mathbf{x},t) = \rho_0 + \rho_\approx(\mathbf{x},t), \quad p(\mathbf{x},t) = p_0 + p_\approx(\mathbf{x},t), \quad \mathbf{v}(\mathbf{x},t) = \mathbf{v}_0 + \mathbf{v}_\approx(\mathbf{x},t) \quad \text{and that} \quad \mathbf{v}_0 = \mathbf{0}.$$

$p_\approx$ is called **acoustic pressure** or **excess pressure**. Linearising both equations around the static values we obtain

$$\frac{\partial \rho_\approx}{\partial t} + \rho_0 \operatorname{div}(\mathbf{v}_\approx) = 0 \quad \text{and} \quad \frac{\partial \mathbf{v}_\approx}{\partial t} + \frac{1}{\rho_0}\nabla p_\approx = \mathbf{0}. \tag{2}$$

The pressure is a function of the density $p = f(\rho)$ with $p_0 = f(\rho_0)$. Linearising this relation and denoting $c^2 := \frac{\partial f}{\partial \rho}(\rho_0)$ we have $p_0 + p_\approx = f(\rho_0+\rho_\approx) \approx f(\rho_0) + c^2\rho_\approx$, thus $p_\approx = c^2\rho_\approx$. (We can call the equations (2) together with $p_\approx = c^2\rho_\approx$ the "first-order acoustic wave equations".) Using this relation in the two linearised PDEs (2) we obtain that the pressure satisfies the **wave equation**:

$$\frac{1}{c^2}\frac{\partial^2 p_\approx}{\partial t^2} - \Delta p_\approx = 0.$$

(Here we use that the divergence of the gradient is the Laplacian, $\Delta u = \operatorname{div}\nabla u$.) Since $\rho_\approx = \frac{1}{c^2}p_\approx$, also $\rho_\approx$ satisfies the same equation.

Exercise 1.1.
- Show that the velocity $\mathbf{v}_\approx$ satisfies the vector wave equation $\frac{1}{c^2}\frac{\partial^2 \mathbf{v}_\approx}{\partial t^2} - \nabla(\nabla\cdot\mathbf{v}_\approx) = \mathbf{0}$.
- Fix $n = 3$. Prove that for all vector fields $\mathbf{F} \in C^2(\mathbb{R}^3)^3$ the following vector calculus identity holds

$$\nabla(\nabla\cdot\mathbf{F}) = \boldsymbol{\Delta}\mathbf{F} + \operatorname{curl}\operatorname{curl}\mathbf{F}. \tag{3}$$

  Here $\boldsymbol{\Delta}$ is the vector Laplacian (defined componentwise) and $\operatorname{curl}\mathbf{F} = \nabla\times\mathbf{F} = (\frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3}, \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1}, \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2})$ is the usual curl operator.
- Deduce that if $\mathbf{v}_\approx$ is irrotational, i.e. curl-free ($\operatorname{curl}\mathbf{v}_\approx = \mathbf{0}$), then each Cartesian component $v_{\approx,1}, v_{\approx,2}, v_{\approx,3}$ is solution of the scalar wave equation: $\frac{1}{c^2}\frac{\partial^2 v_{\approx,j}}{\partial t^2} - \Delta v_{\approx,j} = 0$, $j = 1,2,3$.
- Show that if $\mathbf{v}_\approx$ is irrotational at some given time t_0, then it remains irrotational for all $t > t_0$.

Exercise 1.2 (Velocity potential)**.** Assume that, at some initial time $t = t_0$, the velocity $\mathbf{v}_\approx(\mathbf{x},t_0)$ is gradient of a (time-independent) scalar field $\frac{1}{\rho_0}\phi_0(\mathbf{x})$. Show that the scalar field $\phi(\mathbf{x},t) := \phi_0(\mathbf{x}) - \int_{t_0}^t p_\approx(\mathbf{x},s)\,\mathrm{d}s$, called **velocity potential**, satisfies $-\frac{\partial\phi}{\partial t} = p_\approx$ and $\frac{1}{\rho_0}\nabla\phi = \mathbf{v}_\approx$ (using the linearised Euler equation and the fundamental theorem of calculus). Show that the velocity potential satisfies the wave equation $\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = 0$.

We have seen that several quantities (the acoustic pressure $p_\approx$, the density $\rho_\approx$, and, under suitable assumptions, the velocity potential ϕ and the components of the velocity $v_{\approx,j}$) satisfy the same wave equation, so we write it for a general scalar field U:

$$\boxed{\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0.} \tag{4}$$

This is the prototype of second-order, linear hyperbolic PDEs.

Exercise 1.3. Show that for any smooth function $F : \mathbb{R} \to \mathbb{R}$ and any unit vector $\mathbf{d} \in \mathbb{R}^n$, $|\mathbf{d}| = 1$, the field $U(\mathbf{x},t) = \boxed{F(\mathbf{x}\cdot\mathbf{d} - ct)}$ is a solution of the wave equation (4).

Exercise 1.3 shows that any wave profile move across space–time with speed c, which is thus called **wave speed**. Indeed, $c = \sqrt{\frac{p_\approx}{\rho_\approx}}$ and the square root of the ratio between a pressure and a mass density has the dimension of a velocity ($\sqrt{\frac{\mathit{kg}\,m^{-1}s^{-2}}{\mathit{kg}\,m^{-3}}} = \frac{m}{s}$).

Exercise 1.4 (Damped wave equation)**.** The **damped wave equation** (or equation of telegraphy, see [CJ77, §9]) with damping parameter $\gamma > 0$ is

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} + \frac{\gamma}{c^2}\frac{\partial U}{\partial t} - \Delta U = 0. \tag{5}$$

[1] In some applications, the static (time-independent) background quantities ρ_0, p_0 and $\mathbf{v}_0$ are not constant. E.g. in ocean acoustics the background density and pressure typically depend on the position; in aero-acoustics the velocity of the fluid is non-zero and variable. This leads to the presence of extra terms in the linearised differential equations.

Assume that γ is small so that γ^2 can be neglected. Show that wave profiles are damped in time with rate $\frac{\gamma}{2}$ while they propagate: for any smooth function $F : \mathbb{R} \to \mathbb{R}$ and any unit vector $\mathbf{d} \in \mathbb{R}^n$, $|\mathbf{d}| = 1$ the field $U(\mathbf{x}, t) = F(\mathbf{x} \cdot \mathbf{d} - ct)\mathrm{e}^{-\frac{\gamma}{2}t}$ is a solution of the damped wave equation (5) up to a factor $-\frac{\gamma^2}{4c^2}U \approx 0$.

Conversely, show that if U is solution of the damped wave equation (5), then $W(\mathbf{x}, t) := \mathrm{e}^{\frac{\gamma}{2}t}U(\mathbf{x}, t)$ is solution of the wave equation (4) up to a factor $\frac{\gamma^2}{4c^2}W \approx 0$.

When the acoustic waves hits a (smooth) obstacle $D \subset \mathbb{R}^n$ through which it cannot propagate, on the interface between the obstacle and the fluid some boundary conditions have to be imposed. Depending on the nature of the obstacle and of the fluid, different conditions can be imposed.

- If the obstacle is "**sound-soft**" then the acoustic pressure on its boundary vanishes, i.e. $p_{\approx} = 0$.
- If the obstacle is "**sound-hard**" then the normal velocity on the boundary vanishes, i.e. $\mathbf{v}_{\approx} \cdot \mathbf{n} = 0$, where $\mathbf{n}$ is the unit normal vector on the boundary of D.

  Both sound-soft and sound-hard boundary conditions reflect all the energy carried by the wave.
- A simple way to model a more realistic absorbing boundary condition is to impose that the normal velocity is proportional to the pressure: $\mathbf{v}_{\approx} \cdot \mathbf{n} = \frac{\vartheta}{c\rho_0}p_{\approx}$ for some $\vartheta > 0$ that represents how easily the obstacle yields to the acoustic pressure. Here we have assumed that $\mathbf{n}$ points outwards of the domain where the waves propagates and into the obstacle D. We have divided by $c\rho_0$ to ensure that ϑ is dimensionless (check this fact). Deriving this relations and using the linearised Euler's equation (2), we obtain a relation involving $p_{\approx}$ only: $\mathbf{n} \cdot \nabla p_{\approx} + \vartheta c^{-1}\frac{\partial p_{\approx}}{\partial t} = 0$. This is called "**impedance**" boundary condition.
- If the obstacle is made of a different fluid, then instead of imposing boundary conditions we consider two copies of the wave equations in the two fluids, with different values of c. The two equations are coupled by suitable "transmission conditions", i.e. by imposing the continuity of the pressure and the normal displacement across the interface.

  Similarly, if the obstacle is an elastic solid, acoustic waves in the fluid generates elastic waves in the solid and vice versa. This is modelled by coupling the acoustic wave equation (4) with the (more complicated) elastodynamic wave equation, whose unknown is the point displacement, through appropriate transmission conditions. We will briefly describe the equations of elastodynamics below.

When a source of acoustic disturbance is present in the bulk of the fluid, this is modelled by the **inhomogeneous wave equation**:

$$\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = F, \tag{6}$$

where $F(\mathbf{x}, t)$ is the source term.

1.1.2 Time-harmonic behaviour

A **time-harmonic** function is a scalar field whose time-dependence is prescribed to be sinusoidal, in the form[2]

$$\boxed{U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}} = \Re\{u(\mathbf{x})\}\cos\omega t + \Im\{u(\mathbf{x})\}\sin\omega t \tag{7}$$

for a **time frequency** $\omega > 0$ and a complex-valued field u which depends on the position in space $\mathbf{x}$ but not on the time variable t. (Here $\Re\{\cdot\}$ and $\Im\{\cdot\}$ denote real and imaginary parts, and i is the imaginary unit.) A sound wave in the form (7) is a "pure tone".

By taking the Laplacian and the second time-derivative of (7) we obtain the following crucial fact.

If $U(\mathbf{x}, t)$ is a time-harmonic (7) solution of the wave equation (4),
then $u(\mathbf{x})$ is solution of the Helmholtz equation (1) with **wavenumber** $k := \omega/c > 0$.

This is **the main reason of the interest in the Helmholtz equation: it describes all time-harmonic solutions of the wave equation**. Any solution of the Helmholtz equation has to be interpreted via (7): multiplying by $\mathrm{e}^{-\mathrm{i}\omega t}$ and taking the real part we obtain a "physical" field depending on space and time.

The higher the frequency ω and the wavenumber k, the more oscillatory are the solutions of the Helmholtz equation.

Sound-soft conditions translate to Dirichlet boundary conditions $u = 0$, sound-hard to Neumann $\mathbf{n} \cdot \nabla u = 0$, and impedance to Robin $\mathbf{n} \cdot \nabla u - \mathrm{i}k\vartheta u = 0$.

[2] Unfortunately, several references use the opposite convention $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{\mathrm{i}\omega t}\}$, with a different sign at the exponent. This causes changes in the signs and conjugation in all formulas in the following.

Exercise 1.5. Let U be time-harmonic as in (7) and be a solution of the inhomogeneous wave equation (6) with $F(\mathbf{x},t) = \Re\{f(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$. Show that u satisfies the inhomogeneous Helmholtz equation $-\Delta u - k^2 u = f$.

Exercise 1.6. Show that complex conjugation of Helmholtz solutions "reverses time": if U is the space–time wave solution associated to the Helmholtz solution u, then the space–time wave solution W associated to $\overline{u} = \Re w - \mathrm{i}\Im w$ satisfies $W(\mathbf{x},t) = U(\mathbf{x},-t)$.

Exercise 1.7 (Helmholtz equation with complex wavenumber)**.** Show that if U is a time-harmonic solution of the damped wave equation (5) then it is solution of the Helmholtz equation with complex wavenumber k, such that $k^2 = \omega(\omega + \mathrm{i}\gamma)/c^2$. (We always choose the root k with $\Re k > 0$ and $\Im k \geq 0$).

This shows that the solutions of the Helmholtz equation with complex wavenumber k can be understood as waves that are attenuated while they propagate, i.e. they are absorbed by the medium through which they propagate. The larger the imaginary part of the wavenumber, the stronger the damping. A negative imaginary part of k corresponds to $\gamma < 0$ in (5) and to waves increasing in time, which is an unphysical situation.

Remark 1.8. We have assumed that the medium through which the wave propagate is uniform. In the more general case of an acoustic wave propagating through heterogeneous materials, both the sound speed c and the static density ρ_0 depend on the position $\mathbf{x}$. Repeating the reasoning done above, one obtains the wave equation $\frac{1}{\rho_0(\mathbf{x})c^2(\mathbf{x})}\frac{\partial^2 p_\approx}{\partial t^2} - \operatorname{div}(\frac{1}{\rho_0(\mathbf{x})}\nabla p_\approx) = 0$. Assuming time-harmonic behaviour (7) for $U = p_\approx$ we have the Helmholtz equation with variable coefficients $\operatorname{div}(\frac{1}{\rho_0(\mathbf{x})}\nabla u) + \frac{\omega^2}{\rho_0(\mathbf{x})c^2(\mathbf{x})}u = 0$, which is often written as $\operatorname{div}(\frac{1}{\rho_0(\mathbf{x})}\nabla u) + k^2 n(\mathbf{x})u = 0$ and n is called refractive index. In the following we will not consider this more general problem and we will stick to the constant-coefficients case; see e.g. [CK2, §8] for more details on this problem.

Remark 1.9 (Helmholtz equation = wave equation + Fourier transform)**.** Fourier analysis tells us that any "reasonable" (e.g. square-integrable) time-dependent field U can be written as a continuous linear combination of time-harmonic fields $\mathrm{e}^{\mathrm{i}\omega t}\hat{U}(\mathbf{x},\omega)$ with different frequencies $\omega \in \mathbb{R}$, where $\hat{U}$ is its Fourier transform (in time):

$$U(\mathbf{x},t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}\omega t}\hat{U}(\mathbf{x},\omega)\,\mathrm{d}\omega \qquad \text{with} \qquad \hat{U}(\mathbf{x},\omega) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}\omega t}U(\mathbf{x},t)\,\mathrm{d}t.$$

Reasoning as above, we can verify that, if U is solution of the wave equation with wave speed c, then its Fourier transform $\hat{U}$ evaluated at a given frequency ω, i.e. $u(\mathbf{x}) = \hat{U}(\mathbf{x},\omega)$, is solution of the Helmholtz equation with wavenumber $k = \omega/c$. Thus in principle any solution of the wave equation is linear combination of infinitely many solutions of the Helmholtz equation at different wavenumbers. Numerically, often one approximates a wave equation solution by solving several Helmholtz problems. This is an important reason for studying the Helmholtz equation, even if we were not interested in problems at a fixed frequency.

When we study U and the wave equation we say that we work "in time domain"; when we study $\hat{U}$ or u and the Helmholtz equation we say that we work "in frequency domain".

1.1.3 Electromagnetism

Although the Helmholtz equation is usually associated to acoustic waves, it is important also in the modelling of other kinds of linear waves, e.g. electromagnetic ones. In this section and in the next one we fix $n = 3$, i.e. we consider 3D problems. Electromagnetic waves in a homogeneous material, in the absence of charges, are described by the **Maxwell's equations**:

$$\operatorname{curl}\mathcal{E}(\mathbf{x},t) + \mu\frac{\partial\mathcal{H}}{\partial t}(\mathbf{x},t) = \mathbf{0}, \qquad \operatorname{curl}\mathcal{H}(\mathbf{x},t) - \epsilon\frac{\partial\mathcal{E}}{\partial t}(\mathbf{x},t) - \sigma\mathcal{E}(\mathbf{x},t) = \mathbf{0}, \tag{8}$$

where $\mathcal{E}$ is the electric field, $\mathcal{H}$ the magnetic field, ϵ the electric permittivity, μ the magnetic permeability, and σ the conductivity. The parameters ϵ, μ and σ represent the properties of the material through which the wave propagates. As we consider an homogeneous, isotropic medium, ϵ and μ are positive constants, σ is a non-negative constant (0 in a dielectric, i.e. a perfect insulator, and positive in a conducting medium). The first equation is called Faraday law, the second one Ampère law.

If both the electric and the magnetic field are time-harmonic, i.e. $\mathcal{E}(\mathbf{x},t) = \Re\{\mathbf{E}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$ and $\mathcal{H}(\mathbf{x},t) = \Re\{\mathbf{H}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$ for some $\omega > 0$, then the time-independent fields $\mathbf{E}$ and $\mathbf{H}$ satisfy the **time-harmonic Maxwell's equations**:

$$\operatorname{curl}\mathbf{E}(\mathbf{x}) - \mathrm{i}\omega\mu\mathbf{H}(\mathbf{x}) = \mathbf{0}, \qquad \operatorname{curl}\mathbf{H}(\mathbf{x}) + \mathrm{i}\omega\epsilon\mathbf{E}(\mathbf{x}) - \sigma\mathbf{E}(\mathbf{x}) = \mathbf{0}. \tag{9}$$

These are two vector-valued PDEs with two vector fields as unknowns. Eliminating $\mathbf{H}$, we obtain the **second-order time-harmonic Maxwell's equations** for the electric field:

$$\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = \mathbf{0} \qquad \text{with} \quad k^2 = \omega^2\epsilon\mu + \mathrm{i}\omega\sigma\mu. \tag{10}$$

Since $\operatorname{div}\operatorname{curl}\mathbf{v} = 0$ for any vector field $\mathbf{v}$, any solution of (10) is divergence-free (solenoidal). Then the expansion (3) of the curl curl operator implies that each component of the solution of the second-order Maxwell's equations (10) is solution of the Helmholtz equation with (possibly complex, if $\sigma > 0$) wavenumber k:

$$\Delta E_j + k^2 E_j = 0 \qquad \text{for} \quad j = 1, 2, 3.$$

The speed of propagation of electromagnetic waves (e.g. of light) is $c = \frac{1}{\sqrt{\epsilon\mu}} > 0$ and the damping factor (as in Exercise 1.7) is $\gamma = \frac{\sigma}{\epsilon} \geq 0$.

As for any PDE, time-harmonic Maxwell's equations are complemented by boundary conditions. When the domain under consideration is surrounded by a metal, through which electric field do not penetrate, then typically one imposes the "**perfect electric conductor**" (PEC) boundary conditions, which impose that the *tangential* component of the electric field vanishes. In formulas this is $\mathbf{E}\times\mathbf{n} = \mathbf{0}$, where $\times$ denote the vector product and $\mathbf{n}$ is the unit normal vector on the boundary. In terms of the magnetic field, the PEC boundary conditions correspond to the vanishing of the *normal* component: $\mathbf{H}\cdot\mathbf{n} = 0$. This is easy to verify for a plane boundary, e.g. $\Pi = \{x_1 = 0\}$: in this case $\mathbf{E}\times\mathbf{n} = \mathbf{E}\times(1,0,0) = (0, E_3, -E_2)$ so $E_2 = E_3 = 0$ on the whole plane Π, and $\mathbf{H}\cdot\mathbf{n} = \frac{1}{\mathrm{i}\omega\mu}\operatorname{curl}\mathbf{E}\cdot(1,0,0) = \frac{1}{\mathrm{i}\omega\mu}(\frac{\partial E_3}{\partial x_2} - \frac{\partial E_2}{\partial x_3}) = 0$.

We also often encounter **impedance boundary conditions**: $\mathbf{H}\times\mathbf{n} - \vartheta(\mathbf{n}\times\mathbf{E})\times\mathbf{n} = \frac{1}{\mathrm{i}k}\mathbf{g}$, or equivalently $\mu^{-1}\operatorname{curl}\mathbf{E}\times\mathbf{n} - \mathrm{i}k\vartheta(\mathbf{n}\times\mathbf{E})\times\mathbf{n} = \mathbf{g}$, for a positive parameter ϑ and a boundary source term $\mathbf{g}$. Here $(\mathbf{n}\times\mathbf{E})\times\mathbf{n} = \mathbf{E} - (\mathbf{E}\cdot\mathbf{n})\mathbf{n}$ is the tangential component of $\mathbf{E}$.

Exercise 1.10. Complete the proof of the following statement. For $k \in \mathbb{C}$, $k \neq 0$, a vector field $\mathbf{v}$ is solution of $\operatorname{curl}\operatorname{curl}\mathbf{v} - k^2\mathbf{v} = \mathbf{0}$ if and only if it is divergence-free and each of its three components is solution of the Helmholtz equation $\Delta v_j + k^2 v_j = 0$, $j = 1, 2, 3$.

Exercise 1.11. We have shown that the components of the time-harmonic solutions of the Maxwell's equations (8) are Helmholtz solutions. Show again the same fact performing the same operations in different order. First eliminate $\mathcal{H}$ from (8) obtaining second-order Maxwell's equations in time-domain. Then verify that each component of $\mathcal{E}$ satisfies the wave equation (4). Finally assume that $\mathcal{E}$ is time-harmonic.

Remark 1.12. Often the conductivity term $\sigma\mathcal{E}(\mathbf{x}, t)$ in the time-domain Maxwell's equation is modelled as a *given* current density $\mathcal{J}(\mathbf{x}, t)$ and treated as a datum. If this is assumed to be time-harmonic $\mathcal{J}(\mathbf{x}, t) = \Re\{\mathbf{J}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$ we obtain the inhomogeneous time-harmonic Ampère law $\operatorname{curl}\mathbf{H} + \mathrm{i}\omega\epsilon\mathbf{E} = \mathbf{J}$ and the second-order equation $\operatorname{curl}\frac{1}{\mu}\operatorname{curl}\mathbf{E} - \omega^2\epsilon\mathbf{E} = \mathrm{i}\omega\mathbf{J}$. In absence of charges, the current density is divergence free: $\operatorname{div}\mathbf{J} = 0$ (more generally we would have the continuity equation $\operatorname{div}\mathbf{J} = -\frac{\partial\rho}{\partial t}$, where ρ is the charge density), so the component of the electric field satisfy the inhomogeneous Helmholtz equation $\Delta E_j + k^2 E_j = -\mathrm{i}\omega\mu J_j$.

Remark 1.13. We have seen that time-harmonic Maxwell solutions are componentwise Helmholtz solutions. However, in general one cannot reduce the solution of a boundary value problem for the Maxwell equations (10) to three independent Helmholtz problems for E_1, E_2, E_3, because the boundary conditions required are different.

For instance, when we impose PEC boundary conditions, only the tangential component of the electric field vanishes. This is equivalent to the imposition of two scalar boundary conditions (e.g. on two Cartesian components if the domain is a cube) for three unknown scalar fields and three scalar PDEs (Helmholtz); the boundary value problem is closed by the condition $\operatorname{div}\mathbf{E} = \mathbf{0}$ which intertwines the three components.

We see in the next remark that the decoupling of the scalar components and the reduction of Maxwell's problems to Helmholtz ones can be performed when symmetries are present.

Remark 1.14 (TE and TM modes)**.** The Helmholtz equation is important in dimensional reductions of the Maxwell's equations. Maxwell's equations simplify when we assume that the dependence on one of the Cartesian variables of all components of the fields is a given complex exponential, i.e.

$$\mathbf{E}(x_1, x_2, x_3) = \widetilde{\mathbf{E}}(x_1, x_2)\mathrm{e}^{\mathrm{i}\eta x_3}, \qquad \mathbf{H}(x_1, x_2, x_3) = \widetilde{\mathbf{H}}(x_1, x_2)\mathrm{e}^{\mathrm{i}\eta x_3}. \tag{11}$$

This is relevant when we consider the propagation of waves through very long objects such as optical fibers. In this case the curl becomes

$$\operatorname{curl}\mathbf{H} = \mathrm{e}^{\mathrm{i}\eta x_3}\Big(\frac{\partial\widetilde{H}_3}{\partial x_2} - \mathrm{i}\eta\widetilde{H}_2,\ \ \mathrm{i}\eta\widetilde{H}_1 - \frac{\partial\widetilde{H}_3}{\partial x_1},\ \ \frac{\partial\widetilde{H}_2}{\partial x_1} - \frac{\partial\widetilde{H}_1}{\partial x_2}\Big).$$

Maxwell's equations (9) (with $\sigma = 0$) become

$$\frac{\partial \widetilde{E}_3}{\partial x_2} - \mathrm{i}\eta \widetilde{E}_2 - \mathrm{i}\omega\mu \widetilde{H}_1 = 0, \qquad \frac{\partial \widetilde{H}_3}{\partial x_2} - \mathrm{i}\eta \widetilde{H}_2 + \mathrm{i}\omega\epsilon \widetilde{E}_1 = 0,$$
$$\mathrm{i}\eta \widetilde{E}_1 - \frac{\partial \widetilde{E}_3}{\partial x_1} - \mathrm{i}\omega\mu \widetilde{H}_2 = 0, \qquad \mathrm{i}\eta \widetilde{H}_1 - \frac{\partial \widetilde{H}_3}{\partial x_1} + \mathrm{i}\omega\epsilon \widetilde{E}_2 = 0,$$
$$\frac{\partial \widetilde{E}_2}{\partial x_1} - \frac{\partial \widetilde{E}_1}{\partial x_2} - \mathrm{i}\omega\mu \widetilde{H}_3 = 0, \qquad \frac{\partial \widetilde{H}_2}{\partial x_1} - \frac{\partial \widetilde{H}_1}{\partial x_2} + \mathrm{i}\omega\epsilon \widetilde{E}_3 = 0.$$

If $\widetilde{E}_3 = 0$, with some manipulation one can see that all other field components can be computed from $\widetilde{H}_3$ ($\widetilde{E}_1 = (-\mathrm{i}\omega\epsilon + \frac{\mathrm{i}\eta^2}{\omega\mu})^{-1} \frac{\partial \widetilde{H}_3}{\partial x_2}$, $\widetilde{E}_2 = (\mathrm{i}\omega\epsilon - \frac{\mathrm{i}\eta^2}{\omega\mu})^{-1} \frac{\partial \widetilde{H}_3}{\partial x_1}$, $\widetilde{H}_1 = -\frac{\eta}{\omega\mu}\widetilde{E}_2$, $\widetilde{H}_2 = \frac{\eta}{\omega\mu}\widetilde{E}_1$) and that $\widetilde{H}_3$ itself is solution of the 2D Helmholtz equation $\Delta \widetilde{H}_3 + (\omega^2\epsilon\mu - \eta^2)H_3 = 0$. These solutions are called "**transverse-electric (TE) modes**", since the electric field is perpendicular to the x_3 axis, along which the wave propagates. Similarly, the "**transverse-magnetic (TM) modes**" are solutions with $\widetilde{H}_3 = 0$, which can be computed by solving the same Helmholtz equation for $\widetilde{E}_3$.

Now assume that we want to compute the solutions of Maxwell's equation in an infinite cylinder $\Omega \times \mathbb{R} = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in \Omega, x_3 \in \mathbb{R}\}$, where $\Omega \subset \mathbb{R}^2$, and PEC boundary conditions on $\partial\Omega \times \mathbb{R}$. A given TE mode will satisfy the PEC conditions if the Neumann condition $\mathbf{n} \cdot \nabla \widetilde{H}_3 = 0$ holds, while a TM mode has to satisfy the Dirichlet one $\widetilde{E}_3 = 0$. Thus there exists an electromagnetic wave propagating through the "waveguide" $\Omega \times \mathbb{R}$ with frequency η in the x_3 direction only if the 2D Helmholtz problem admits a non-trivial solution. This is the same as saying that $\omega^2\epsilon\mu - \eta^2$ is either a Dirichlet (TM) or a Neumann (TE) eigenvalue for the 2D Laplacian in Ω. 3D Maxwell's problems have been reduced to 2D Helmholtz ones.

Exercise 1.15 (TEM modes)**.** Assume that the (non-trivial) pair $(\mathbf{E}, \mathbf{H})$ is a "TEM mode", which means that it is simultaneously a TE and a TM mode: it is in the form (11) with $\widetilde{E}_3 = \widetilde{H}_3 = 0$. Show the following facts.

- $\eta^2 = \omega^2\epsilon\mu$, i.e. $\eta = k$: the wavenumber in the x_3 direction coincide with the free-space wavenumber.
- $\mathbf{H} = \sqrt{\epsilon/\mu}\, \mathbf{e}_3 \times \mathbf{E}$, where $\mathbf{e}_3 = (0, 0, 1)$. This means that $\mathbf{E}$ and $\mathbf{H}$ are orthogonal vectors in the x_1x_2-plane.
- If ϕ is a 2D harmonic function ($\Delta\phi = 0$), then $\mathbf{E} = \nabla\phi(x_1, x_2)\mathrm{e}^{\mathrm{i}kx_3}$ and $\mathbf{H} = \sqrt{\epsilon/\mu}\, \mathbf{e}_3 \times \mathbf{E}$ constitute a TEM mode.
- If the domain $\Omega \subset \mathbb{R}^2$ is simply connected then there is no non-trivial TEM mode with PEC conditions propagating through $\Omega \times \mathbb{R}$. (This is a main motivation for the use of coaxial cables.)

Remark 1.16 (Complications)**.** The setting considered in this section is a special case of much more general ones, which are needed in many applications. If different materials are present in the region considered, or the properties of the material vary in space, then ϵ, μ, σ are function of position. In this case, for instance, to obtain (10) we cannot simply move μ to the second term and find componentwise solution of the Helmholtz equation, but we obtain some more general elliptic equations. If the material is anisotropic, then the coefficients are modelled by symmetric positive definite matrices (semi-definite in case of σ). Since the polarisation of a material given an impinging electromagnetic field is not immediate, the multiplications $\epsilon\mathcal{E}$ and $\mu\mathcal{H}$ in (8) are more precisely modelled as convolutions in time between $\mathcal{E}/\mathcal{H}$ and suitable kernels; however in frequency-domain these give rise to standard products $\epsilon(\omega)\mathbf{E}$ and $\mu(\omega)\mathbf{H}$ where now the coefficients depends on the frequency ω. In some materials and regimes (e.g. in lasers) the coefficients ϵ and μ need to be modelled as nonlinear operators acting on $\mathbf{E}$ and $\mathbf{H}$: this is the field of nonlinear optics.

1.1.4 Elastodynamics

Waves propagating in solids have more complicated behaviour than those in fluids, as two different types of waves can be present. The PDE describing small-amplitude time-harmonic waves in (homogeneous, isotropic) solids is **Navier's equations**:

$$(\lambda + 2\mu)\nabla \operatorname{div} \mathbf{u} - \mu \operatorname{curl} \operatorname{curl} \mathbf{u} + \omega^2\rho\, \mathbf{u} = \mathbf{0}. \tag{12}$$

Here $\mathbf{u}$ is the **displacement** vector field, describing the position of a point of the object with respect to the rest position; as before, ω is the angular frequency; the positive parameters λ and μ are the **Lamé constants**, describing the elastic properties of the material; and $\rho > 0$ is the density of the medium. We define the wavenumber of pressure (longitudinal) and shear (transverse) waves, respectively, as:

$$k_P := \omega\sqrt{\frac{\rho}{\lambda + 2\mu}}, \qquad k_S := \omega\sqrt{\frac{\rho}{\mu}}.$$

We define the scalar and vector potential, respectively, as

$$\chi := -\frac{\lambda+2\mu}{\omega^2\rho}\operatorname{div}\mathbf{u} = -\frac{\operatorname{div}\mathbf{u}}{k_P^2}, \qquad \boldsymbol{\psi} := \frac{\mu}{\omega^2\rho}\operatorname{curl}\mathbf{u} = \frac{\operatorname{curl}\mathbf{u}}{k_S^2}. \tag{13}$$

From (12), we can use these potentials to represent $\mathbf{u}$:

$$\mathbf{u} \quad = \quad -\frac{\lambda+2\mu}{\omega^2\rho}\nabla\operatorname{div}\mathbf{u} + \frac{\mu}{\omega^2\rho}\operatorname{curl}\operatorname{curl}\mathbf{u} \quad = \quad \nabla\chi + \operatorname{curl}\boldsymbol{\psi}, \tag{14}$$

which is a Helmholtz decomposition of the displacement field. With some manipulation we obtain

$$\Delta\chi + k_P^2\chi \overset{(13),\Delta=\operatorname{div}\nabla}{=} -\operatorname{div}\nabla\frac{\operatorname{div}\mathbf{u}}{k_P^2} - \operatorname{div}\mathbf{u} \overset{(12)}{=} -\frac{1}{k_P^2}\operatorname{div}\left(\frac{\mu}{\lambda+2\mu}\operatorname{curl}\operatorname{curl}\mathbf{u} - k_P^2\mathbf{u}\right) - \operatorname{div}\mathbf{u} \overset{\operatorname{div}\operatorname{curl}=0}{=} 0,$$

$$\operatorname{curl}\operatorname{curl}\boldsymbol{\psi} - k_S^2\boldsymbol{\psi} \overset{(13)}{=} \operatorname{curl}\operatorname{curl}\frac{\operatorname{curl}\mathbf{u}}{k_S^2} - \operatorname{curl}\mathbf{u} \overset{(12)}{=} \frac{1}{k_S^2}\operatorname{curl}\left(\frac{\lambda+2\mu}{\mu}\nabla\operatorname{div}\mathbf{u} + k_S^2\mathbf{u}\right) - \operatorname{curl}\mathbf{u} \overset{\operatorname{curl}\nabla=\mathbf{0}}{=} \mathbf{0}.$$

This means that the scalar and vector potentials satisfy Helmholtz and Maxwell's equations, respectively.

The decomposition (14) shows that any solution $\mathbf{u}$ of Navier's equations (12) is sum of two terms. The first one is a curl-free, longitudinal, time-harmonic wave propagating at speed $c_P = \frac{\omega}{k_P} = \sqrt{\frac{\lambda+2\mu}{\rho}}$; this is called **pressure wave** (P-wave). The second one is a divergence-free, transverse, time-harmonic wave propagating at (lower) speed $c_S = \frac{\omega}{k_S} = \sqrt{\frac{\mu}{\rho}}$; this is called **shear wave** (S-wave).

In particular, all time-harmonic elastic waves can be 'assembled' from solutions of two copies of the Helmholtz equation with different wavenumbers. In some applications, such as seismic imaging for oil retrieval, Navier's equations are sometimes approximated by the scalar Helmholtz equation, neglecting shear waves.

The limit $\mu \to 0$ corresponds to a fluid material, elasticity reduces to acoustics and shear waves disappear: Navier's equations tend to $\nabla\operatorname{div}\mathbf{u} + k_P^2\mathbf{u} = \mathbf{0}$, which is the equation satisfied by the acoustic displacement and the acoustic velocity.

All the reasoning done here could be done in time-domain, as opposed to frequency-domain, as well.

Remark 1.17. Using identity (3), equation (12) can also be written as $(\lambda+\mu)\nabla\operatorname{div}\mathbf{u} + \mu\boldsymbol{\Delta}\mathbf{u} + \omega^2\rho\mathbf{u} = \mathbf{0}$. We denote by $\mathbf{D}\mathbf{v}$ the Jacobian of the vector field $\mathbf{v}$, by $\mathbf{D}^S\mathbf{v} := \frac{1}{2}(\mathbf{D}\mathbf{v} + \mathbf{D}^\top\mathbf{v})$ the symmetric gradient (or Cauchy's strain tensor), by $\mathbf{div}$ the (row-wise) vector divergence of matrix fields, and by Id the 3×3 identity matrix. Using the identity $2\,\mathbf{div}\,\mathbf{D}^S = \nabla\operatorname{div} + \boldsymbol{\Delta} = 2\nabla\operatorname{div} - \operatorname{curl}\operatorname{curl}$, equation (12) can be written in the form $\mathbf{div}\,\boldsymbol{\sigma} + \omega^2\rho\mathbf{u} = \mathbf{0}$, where $\boldsymbol{\sigma} := 2\mu\mathbf{D}^S\mathbf{u} + \lambda(\operatorname{div}\mathbf{u})\,\mathrm{Id}$ is the Cauchy stress tensor.

Remark 1.18. The Helmholtz equation, possibly with varying coefficients, is used also to model water waves on the sea surface, under some conditions. In this setting it is sometimes called "Berkhoff equation".

1.2 Particular solutions of the Helmholtz equation

We now focus on the construction of some simple analytical solutions of the Helmholtz equation $\Delta u + k^2 u = 0$ in 2D and we study some of their qualitative properties. This is useful to understand some typical features of all Helmholtz solutions.

Plots and time-harmonic animation are available on the course webpage
http://matematica.unipv.it/moiola/MNAPDE2019/MNAPDE2019anim.html

1.2.1 The one-dimensional case

We begin with the (boring) simpler case of one space dimension ($n = 1$). In this case, the Helmholtz equation reduces to the ordinary differential equation: $u'' + k^2 u = 0$. All solutions are in the form

$$u(x) = c_1\cos(kx) + c_2\sin(kx) \qquad \text{for some } c_1, c_2 \in \mathbb{C}.$$

Equivalently

$$u(x) = C_1 e^{\mathrm{i}kx} + C_2 e^{-\mathrm{i}kx} \qquad \text{for some } C_1, C_2 \in \mathbb{C}.$$

All 1D Helmholtz solutions are periodic with period $\lambda = \frac{2\pi}{k}$; this value is called wavelength.

Let us fix $c = 1$, so $\omega = kc = k$. When we expand the time-dependence of the corresponding solutions (7) of the wave equation, we see that $u(x) = e^{\mathrm{i}kx}$ corresponds to $U(x,t) = \Re\{e^{\mathrm{i}kx - \mathrm{i}kt}\} = \cos(k(x-t))$, which is a wave propagating to the right. On the other hand, $u(x) = \cos(kx)$ corresponds to $U(x,t) = \Re\{\cos(kx)e^{-\mathrm{i}kt}\} = \cos(kx)\cos(kt)$, which oscillates in time but maintains the same space profile and does not propagate. See Figure 1 and the animations.

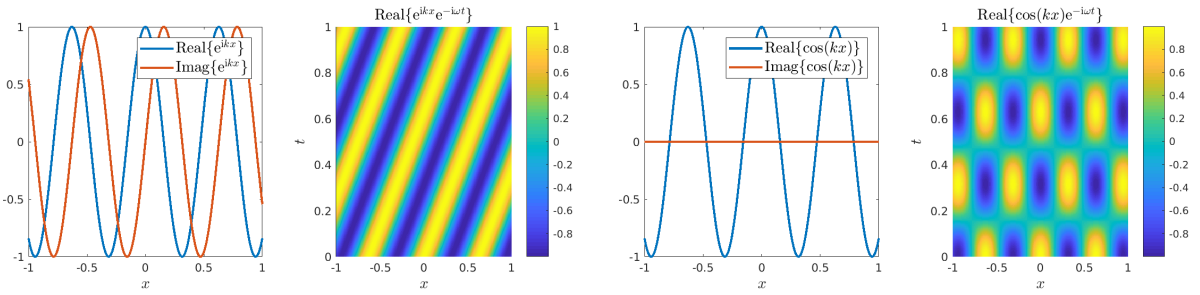


Figure 1: Panel 1: the real and the imaginary part of the one-dimensional propagative wave $u(x) = \mathrm{e}^{\mathrm{i}kx}$ *over the interval* $(-1, 1)$*, with* $k = 10$*.*
Panel 2: the real part of the corresponding solution of the wave equation $U(x,t) = \Re\{\mathrm{e}^{\mathrm{i}kx}\mathrm{e}^{-\mathrm{i}\omega t}\}$*, plotted as function of* x *(horizontal axis) and* t *(vertical axis). Here we are taking* $\omega = k$ *and* $c = 1$*. The wave propagates towards the right endpoint of the space interval.*
Panels 3 and 4: the same plots for the stationary wave $u(x) = \cos(kx) = \frac{1}{2}(\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx})$*. The waves oscillates in time but does not propagate: the peaks (yellow parts) appear at the same locations in space.*

1.2.2 Plane waves

We have seen in Exercise 1.3 that the space–time field $U(\mathbf{x}, t) = F(\mathbf{x}\cdot\mathbf{d} - ct)$, propagating in the direction of $\mathbf{d}$ at speed c, is solution of the wave equation (here $\mathbf{d} \in \mathbb{R}^2$ is a unit vector and F a smooth real function). To have a Helmholtz solution, we want U to be time-harmonic, i.e. $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$. A simple way to reconcile these two expressions is to choose $F(z) = \Re\{\mathrm{e}^{\mathrm{i}kz}\}$ so that $U(\mathbf{x}, t) = \Re\{\mathrm{e}^{\mathrm{i}(k\mathbf{x}\cdot\mathbf{d}-\omega t)}\} = \cos(k\mathbf{x}\cdot\mathbf{d} - \omega t)$ (recalling that $\omega = kc$) and

$$u(\mathbf{x}) = \boxed{\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}} = \cos(k\mathbf{x}\cdot\mathbf{d}) + \mathrm{i}\sin(k\mathbf{x}\cdot\mathbf{d}).$$

This is a time-harmonic **propagative plane wave**, which propagates in the direction $\mathbf{d}$. Plane waves are probably the simplest solutions of the Helmholtz equation.

Another way to obtain plane waves is to look for Helmholtz solutions that are independent of one of the Cartesian variables. If $u(x_1, x_2) = \tilde{u}(x_1)$, then $\tilde{u}$ has to satisfy $\tilde{u}'' + k^2\tilde{u} = 0$, so $\tilde{u}(x_1) = c_1\cos(kx_1) + c_2\sin(kx_2)$ for some $c_1, c_2 \in \mathbb{C}$. Propagative plane waves correspond to the choice $c_1 = 1$, $c_2 = \mathrm{i}$.

Plane waves have constant **amplitude** $|u(\mathbf{x})|$ and are constant on the lines perpendicular to $\mathbf{d}$. Their complex argument $\arg(u(\mathbf{x})) = k\mathbf{x}\cdot\mathbf{d}$ in a point $\mathbf{x}$ is called **phase**. Plane waves are periodic in the direction $\mathbf{d}$ with period (the distance in space between two peaks) $\boxed{\lambda = \frac{2\pi}{k}}$; this value is called **wavelength**. A translation along a vector $\mathbf{v}$ corresponds to a multiplication by a complex factor of absolute value 1, i.e. it is a phase shift: $u(\mathbf{x} + \mathbf{v}) = \mathrm{e}^{\mathrm{i}k(\mathbf{x}+\mathbf{v})\cdot\mathbf{d}} = \mathrm{e}^{\mathrm{i}k\mathbf{v}\cdot\mathbf{d}}u(\mathbf{x})$.

The sum and the difference of two plane waves with opposite directions are called **stationary**, or **standing**, plane waves:

$$\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} + \mathrm{e}^{-\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} = 2\cos(k\mathbf{x}\cdot\mathbf{d}), \qquad \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} - \mathrm{e}^{-\mathrm{i}k\mathbf{x}\cdot\mathbf{d}} = 2\mathrm{i}\sin(k\mathbf{x}\cdot\mathbf{d}).$$

As in the one-dimensional case of §1.2.1, the reason why these are called stationary while $\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ is called propagative is clear if one looks at the evolution in time of the corresponding time-domain wave $U(\mathbf{x}, t) = \Re\{u(\mathbf{x})\mathrm{e}^{-\mathrm{i}\omega t}\}$; see Figure 2 and the animations.

Exercise 1.19. Show that the complex-conjugate of a plane wave is a plane wave propagating in the opposite direction, in accordance with Exercise 1.6.

Exercise 1.20. Show that the vector plane wave $\mathbf{E}(\mathbf{x}) = \mathbf{A}\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ is solution of Maxwell's equations $\operatorname{curl}\operatorname{curl}\mathbf{E} - k^2\mathbf{E} = \mathbf{0}$ if and only if $\mathbf{d}\cdot\mathbf{d} = 1$ and $\mathbf{d}\cdot\mathbf{A} = 0$. This means that the amplitude vector is orthogonal to the propagation direction, i.e. electromagnetic plane waves are transverse waves. (The formula $\mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v})$ might help.)

Show that Navier's equations (12) support both transverse plane waves $\mathbf{A}\mathrm{e}^{\mathrm{i}k_S\mathbf{x}\cdot\mathbf{d}}$, with $\mathbf{d}\cdot\mathbf{A} = 0$, and longitudinal ones $\mathbf{d}\mathrm{e}^{\mathrm{i}k_P\mathbf{x}\cdot\mathbf{d}}$. Longitudinal elastic waves are faster and have longer wavelengths than transverse ones.

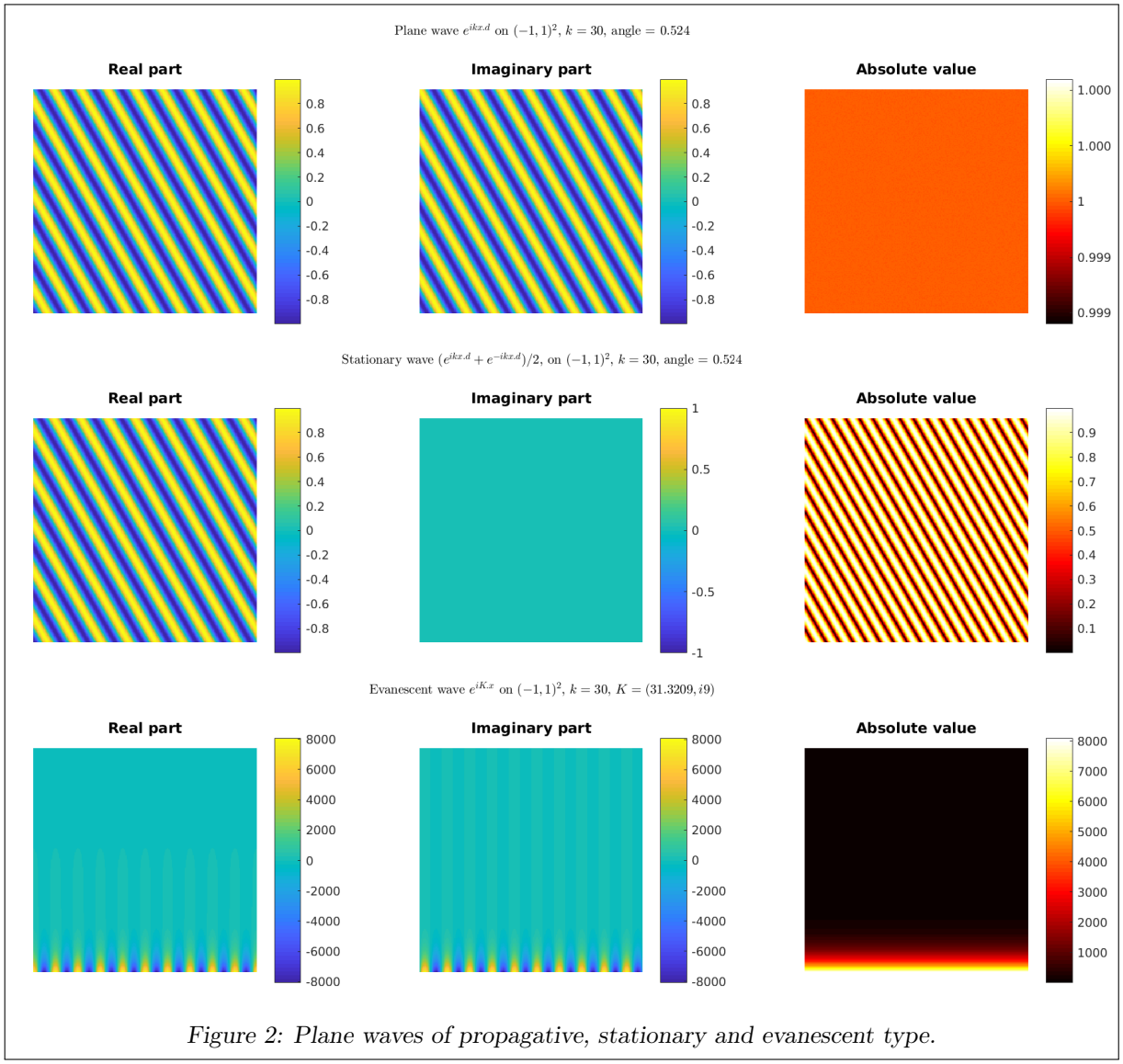


Figure 2: Plane waves of propagative, stationary and evanescent type.

1.2.3 Evanescent plane waves

Propagative and stationary waves are not the only solutions of the Helmholtz equation that are separable in Cartesian coordinates. If we look for functions in the form $u(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \mathrm{e}^{\mathrm{i}(k_1 x_1 + k_2 x_2)}$ satisfying $\Delta u + k^2 u = 0$, we see that we need a "wavevector" $\mathbf{k} \in \mathbb{C}^2$ with $\mathbf{k}\cdot\mathbf{k} = k_1^2 + k_2^2 = k^2$. If both k_1 and k_2 are real then we obtain again the plane waves. If at least one of the two is not real then we have a new kind of waves, called evanescent (plane) waves. Expanding $\mathbf{k} = \mathbf{k}_R + \mathrm{i}\mathbf{k}_I$ with $\mathbf{k}_R, \mathbf{k}_I \in \mathbb{R}^2$, we have $\boxed{u(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{x}} = \mathrm{e}^{\mathrm{i}\mathbf{k}_R\cdot\mathbf{x}}\mathrm{e}^{-\mathbf{k}_I\cdot\mathbf{x}}}$: this field oscillates in the direction $\mathbf{k}_R$ with wavenumber $|\mathbf{k}_R| \geq k$ and decays exponentially in the orthogonal direction $\mathbf{k}_I$ ($|u(\mathbf{x})| = \mathrm{e}^{-\mathbf{k}_I\cdot\mathbf{x}}$). The orthogonality of $\mathbf{k}_R$ and $\mathbf{k}_I$ is a consequence of $\mathbf{k}\cdot\mathbf{k} \in \mathbb{R}$. Evanescent waves typically appear at the interface between different materials. See Figure 2 for a representations.

Exercise 1.21. Verify the statements made in the paragraph.

Exercise 1.22. Show that all 2D plane waves, either propagative or evanescent, can be written in the form $\mathrm{e}^{\mathrm{i}k(x_1\cos\theta + x_2\sin\theta)} = \mathrm{e}^{\frac{k}{2}(\mathrm{i}(\nu+\frac{1}{\nu})x_1 + (\nu-\frac{1}{\nu})x_2)}$, parametrised by $0 \neq \nu \in \mathbb{C}$ or $\theta \in \mathbb{C}$, with $\nu = \mathrm{e}^{\mathrm{i}\theta}$.

1.2.4 Circular waves

We have seen Helmholtz solutions that are separable in Cartesian coordinates, we now look for those that are separable in the polar coordinates $(x_1, x_2) = (r\cos\theta, r\sin\theta)$. The 2D Laplacian in polar coordinates

reads

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}.$$

If we have a separable Helmholtz solution $u(\mathbf{x}) = f(r)g(\theta)$, the functions f, g have to satisfy

$$f''(r)g(\theta) + \frac{1}{r}f'(r)g(\theta) + \frac{1}{r^2}f(r)g''(\theta) + k^2 f(r)g(\theta) = \Delta u + k^2 u = 0.$$

The angular component g has to be periodic of period 2π, so we take the circular harmonic $g(\theta) = \mathrm{e}^{\mathrm{i}\ell\theta}$, for $\ell \in \mathbb{Z}$. Then $g''(\theta) = -\ell^2 g(\theta)$, so we can cancel g from the expression above, multiply by r^2, and obtain that f satisfies

$$r^2 f''(r) + r f'(r) + (r^2 k^2 - \ell^2) f(r) = 0. \tag{15}$$

For $k = 1$, this is called **Bessel differential equation**: it is a linear, second-order ODE with variable coefficients, it depends on the index ℓ and degenerates at $r = 0$. Two linearly independent real-valued solutions are the **Bessel functions of the first kind** and order ℓ, denoted $\boxed{J_\ell(r)}$, and the **Bessel function of the second kind** (or Neumann functions) and order ℓ, denoted $\boxed{Y_\ell(r)}$. Explicit expressions (e.g. as power series or integral representations), plenty of useful formulas and graphs can be found on the "NIST Digital Library of Mathematical Functions" at https://dlmf.nist.gov/

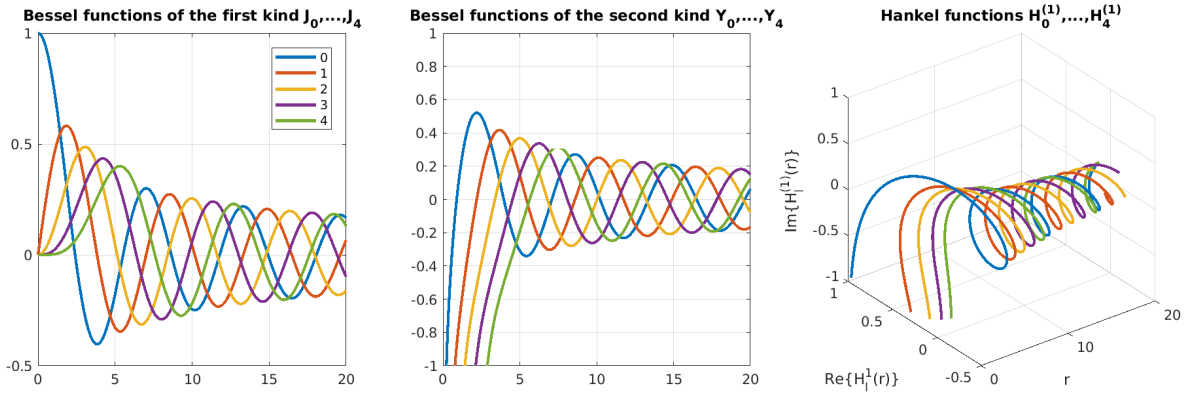


Figure 3: The Bessel functions of first and second kind, and the Hankel functions, for $\ell = 0, \dots, 4$.

The Bessel functions of the first and second kind for $\ell = 0, \dots, 4$ are plotted in Figure 3. We see that both families of functions oscillates around 0 and decays slowly for $r \to \infty$. The distance between two successive zeros of either J_ℓ or Y_ℓ is slightly shorter than π for $\ell = 0$ and slightly longer than π for $\ell \neq 0$. The main difference is that the $J_\ell(r)$s are smooth over $\mathbb{R}$, while the $Y_\ell(r)$s have a singularity at $r = 0$; the higher ℓ the stronger the singularity. Useful formulas are $J_{-\ell}(r) = (-1)^\ell J_\ell(r)$ and $Y_{-\ell}(r) = (-1)^\ell Y_\ell(r)$.

The **Hankel functions** (sometimes called Bessel functions of the third kind) are the complex-valued linear combinations

$$\boxed{H_\ell^{(1)}(r)} := J_\ell(r) + \mathrm{i}Y_\ell(r), \qquad \boxed{H_\ell^{(2)}(r)} := J_\ell(r) - \mathrm{i}Y_\ell(r) = \overline{H_\ell^{(1)}(r)}. \tag{16}$$

The right panel of Figure 3 shows the first few Hankel functions: the argument r is one of the axis, the real and the imaginary parts of $H_\ell^{(1)}(r)$ are on the other two axes. An important property of the Hankel functions is that the magnitude $r \mapsto |H_\ell^{(1)}(r)|$ is a monotonically decreasing function ($|H_\ell^{(1)}(r)| \approx \sqrt{2/(\pi r)}$ for large r). For increasing r the complex number $H_\ell^{(1)}(r)$ spirals clockwise towards the origin.

Bessel and Hankel functions can be used in Matlab with the commands `besselj`, `bessely` and `besselh`.

Exercise 1.23. Verify that if f_1 is solution of (15) for $k = 1$, then, for any $k > 0$, $f_k(r) := f_1(kr)$ solves (15).

Exercise 1.24. Compare numerically the plots of the Bessel functions against the asymptotics for small and large (positive) arguments (from https://dlmf.nist.gov/10.7):

$$J_\ell(z) \sim \frac{z^\ell}{\ell!\, 2^\ell} \quad \ell \in \mathbb{N}_0, \qquad Y_0(z) \sim \frac{2}{\pi}\log z, \quad Y_\ell(z) \sim -\frac{(\ell-1)!\, 2^\ell}{\pi z^\ell} \quad \ell \in \mathbb{N}, \qquad z \to 0,$$

$$J_\ell(z) \sim \sqrt{\frac{2}{\pi z}}\cos\Big(z - \frac{\ell\pi}{2} - \frac{\pi}{4}\Big), \quad Y_\ell(z) \sim \sqrt{\frac{2}{\pi z}}\sin\Big(z - \frac{\ell\pi}{2} - \frac{\pi}{4}\Big) \quad \ell \in \mathbb{N}_0, \qquad z \to \infty.$$

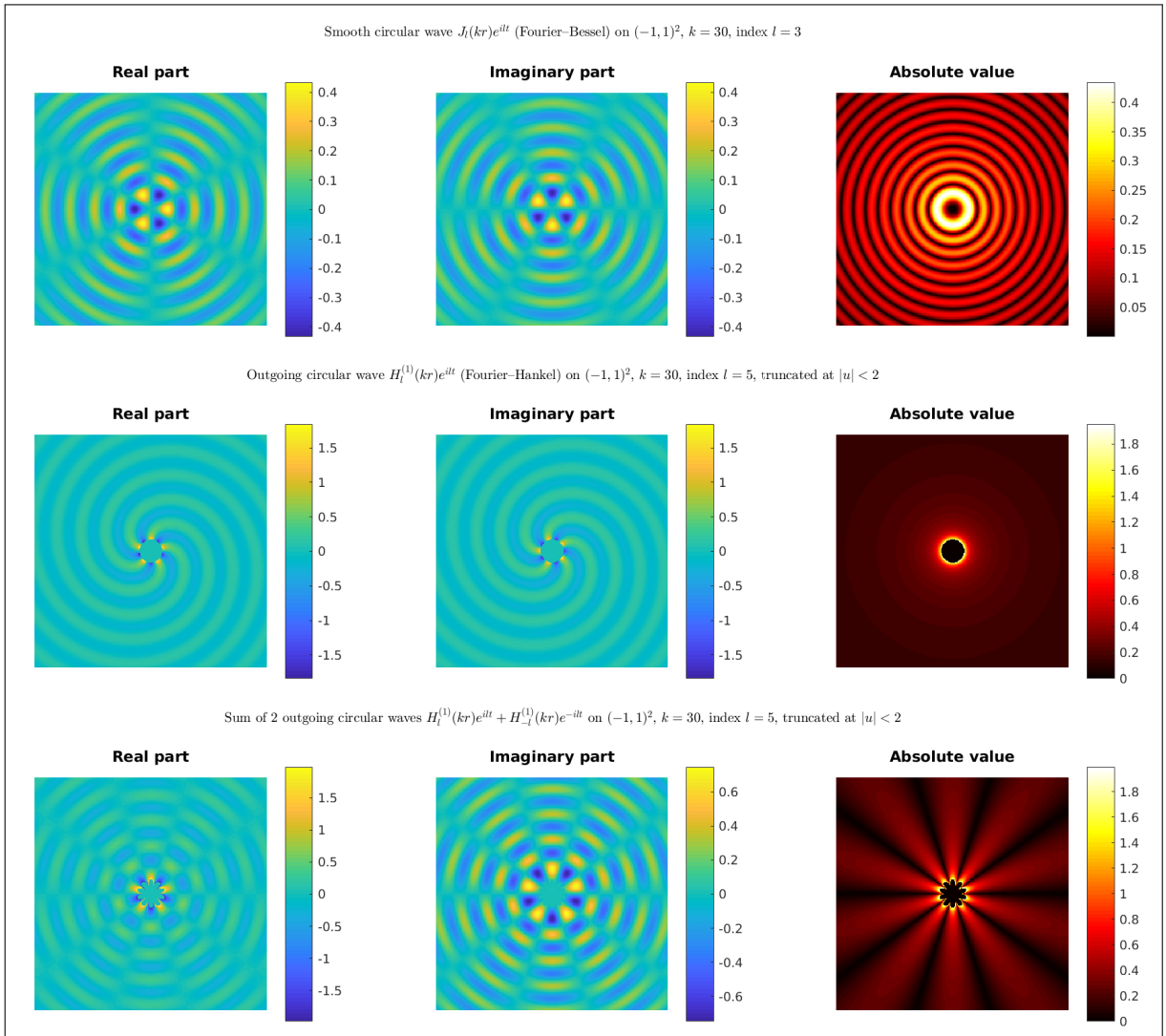


Figure 4: The Fourier–Bessel function $J_3(kr)\mathrm{e}^{\mathrm{i}3\theta}$, the Fourier–Hankel function $H_5^{(1)}(kr)\mathrm{e}^{\mathrm{i}5\theta}$, and the sum of two Fourier–Hankel functions with opposite indices $H_5^{(1)}(kr)\mathrm{e}^{\mathrm{i}5\theta}+H_{-5}^{(1)}(kr)\mathrm{e}^{-\mathrm{i}5\theta}$. Since the Hankel functions are unbounded at the origin, the field has been set to 0 where the value is larger than 2.

From what we have said, we deduce that for any $\ell\in\mathbb{Z}$ the two fields

$$\boxed{J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta},\qquad Y_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}}$$

and their linear combinations are the solutions of the Helmholtz equations that are separable in polar coordinates. They are called **circular waves** or **Fourier–Bessel** functions. Of all the elements of the 2-dimensional space $\mathrm{span}\{J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta},Y_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}\}$, only $J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ is defined in the whole of $\mathbb{R}^2$, while all the others are defined in the punctured plane $\mathbb{R}^2\setminus\{0\}$. They are all of class C^∞ in their domain of definition. From the angular dependence, we see that all these function are invariant under rotations of an angle multiple of $2\pi/|\ell|$; a rotation by an angle α corresponds to a multiplication by a complex factor $\mathrm{e}^{\mathrm{i}\ell\alpha}$.

Special circular waves are the **Fourier–Hankel** functions, namely

$$\boxed{H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}}=J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}+\mathrm{i}Y_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta},\qquad\boxed{H_\ell^{(2)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}}=J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}-\mathrm{i}Y_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta},\qquad\ell\in\mathbb{Z}.$$

We will see soon why the the Fourier–Hankel functions $H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ are important for problems posed in unbounded domains.

Plotting the time evolution (7) of these fields, one notes that the Fourier–Bessel functions $J_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ and $Y_\ell(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ rotate around the origin (anticlockwise if $\ell>0$, clockwise if $\ell<0$) and do not propagate in the radial direction. The Fourier–Hankel functions $H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ rotate and move towards infinity, while

the $H_\ell^{(2)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ towards the origin. This can be seen in the animations on the course webpage. Figure 4 shows some circular waves.

Remark 1.25 (Herglotz functions)**.** For $g \in L^2(0,2\pi)$, the field $u(\mathbf{x}) = \int_0^{2\pi} g(\theta)\mathrm{e}^{\mathrm{i}k(x_1\cos\theta + x_2\sin\theta)}\,\mathrm{d}\theta \in C^\infty(\mathbb{R}^2)$ is called **Herglotz function** with kernel g. It can be thought as a continuous linear combination of plane waves with different directions $(\cos\theta, \sin\theta)$ weighted by $g(\theta)$. Some interesting cases are the following.

- When g approximates a Dirac δ function in θ then u approximates the corresponding plane wave.
- When g is constant in a small interval of $(0,2\pi)$ and 0 otherwise, then u approximates a plane wave in a strip of the plane and decays away from it. In some applications this is more realistic than a plane wave, which has an infinite propagating front. You can see the plot of such a Herglotz function in Figure 5.
- When g is a circular harmonic $g(\theta) = \mathrm{e}^{\mathrm{i}\ell\theta}$ we obtain a Fourier–Bessel function $u(\mathbf{x}) = (2\pi\mathrm{i}^\ell)J_\ell(k|\mathbf{x}|)\mathrm{e}^{\mathrm{i}\ell\theta_\mathbf{x}}$, where $\theta_\mathbf{x}$ is the angular coordinate of $\mathbf{x}$.

  (Prove this fact using the Jacobi–Anger formula $\mathrm{e}^{\mathrm{i}z\cos\alpha} = \sum_{\ell\in\mathbb{Z}} \mathrm{i}^\ell J_\ell(z)\mathrm{e}^{\mathrm{i}\ell\alpha}$ and the $L^2(0,2\pi)$-orthogonality of circular harmonics.)

Plot with Matlab some Herglotz functions with different kernels.

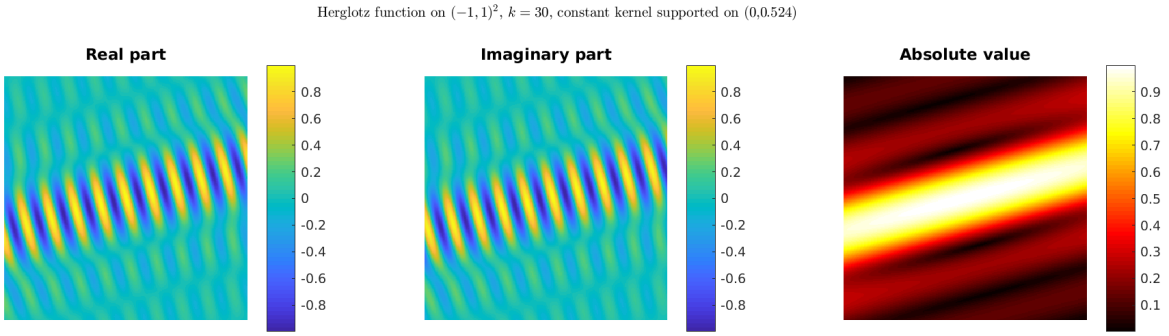


Figure 5: The Herglotz function with kernel $g(\theta) = 1$ if $0 < \theta < \frac{\pi}{6}$ and 0 otherwise, on $(-1,1)^2$. See Remark 1.25.

Remark 1.26 (Special Helmholtz solutions in 3 dimensions)**.** Plane waves on $\mathbb{R}^3$ are defined exactly as in 2D.

The 3D analogous of circular waves are called "spherical waves". In their expression, Bessel and Hankel functions are substituted by the similar "spherical Bessel functions" and "spherical Hankel functions", denoted $j_\ell, y_\ell, h_\ell^{(1)}, h_\ell^{(2)}$. The angular component $\mathrm{e}^{\mathrm{i}\ell\theta}$ is substituted by the "spherical harmonics" Y_ℓ^m, which are smooth functions defined on the unit sphere and indexed by two indices ℓ and m. All these functions are described in details in e.g. [Néd01, §2.4, 2.6] or [CK2, §2.3, 2.4]. In the definition of the Herglotz functions, the circle is substituted by the unit sphere.

2 Boundary value problems for the Helmholtz equation

2.1 Plane waves reflected by a straight line

To understand what happens when a wave hits an impenetrable obstacle we start with a very simple case that can be solved analytically. Let $u^{\mathrm{Inc}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\mathbf{d}}$ be a plane wave with $|\mathbf{d}| = 1$, $d_1 \geq 0$ and $d_2 \leq 0$ (i.e. propagating rightward and downward in the plane). This is called the "**incoming field**", or "**incident field**". Assume that we truncate the domain of propagation to the upper half plane $\Omega^+ = \{x_2 > 0\}$ and on the horizontal line $\Gamma = \{x_2 = 0\}$ we impose some boundary conditions that reflect the impinging wave. We call u^{Scat} the reflected wave, i.e. the "**scattered field**", and $u^{\mathrm{Tot}} = u^{\mathrm{Inc}} + u^{\mathrm{Scat}}$ the "**total field**". u^{Tot} is the physical field we would measure in a point of the half plane.

Given u^{Inc}, which is a datum, we now want to find u^{Tot} that satisfies the Helmholtz equation in the upper half plane, and satisfies some desired homogeneous boundary conditions on Γ. This is the same as saying that we want u^{Scat} that satisfies the Helmholtz equation in the same region, and satisfies boundary conditions that depend on u^{Inc} on Γ.

By the law of reflection, we expect u^{Scat} to propagate upwards and to make with the horizontal line Γ the same angle as u^{Inc}. This means that u^{Scat} is a plane wave with direction $\widetilde{\mathbf{d}} = (d_1, -d_2)$:

$u^{\text{Scat}}(\mathbf{x}) = Ae^{ik\tilde{\mathbf{d}}\cdot\mathbf{x}} = Ae^{ik(x_1d_1-x_2d_2)}$ for some reflection coefficient $A \in \mathbb{C}$ that gives the amplitude and the phase of u^{Scat} itself. The coefficient A depends on the particular type of boundary condition chosen.

- When the line Γ is sound-soft, the Dirichlet trace of the total field u^{Tot} vanishes on Γ:
$$0 = u^{\text{Tot}}(x_1, 0) = u^{\text{Inc}}(x_1, 0) + u^{\text{Scat}}(x_1, 0) = e^{ikx_1d_1} + Ae^{ikx_1d_1} \qquad \forall x_1 \in \mathbb{R} \qquad \Rightarrow\ A = -1.$$
- When the line Γ is sound-hard, the Neumann trace of the total field u^{Tot}, i.e. its normal derivative, vanishes on Γ:
$$0 = \frac{\partial u^{\text{Tot}}}{\partial x_2}(x_1, 0) = \frac{\partial}{\partial x_2}\Big(u^{\text{Inc}}(x_1, 0) + u^{\text{Scat}}(x_1, 0)\Big) = ikd_2e^{ikx_1d_1} - Aikd_2e^{ikx_1d_1} \quad \forall x_1 \in \mathbb{R} \quad \Rightarrow\ A = 1.$$
- To impose the impedance boundary condition $\frac{\partial u^{\text{Tot}}}{\partial \mathbf{n}} - ik\vartheta u^{\text{Tot}} = 0$ with $\vartheta > 0$, we recall that $\mathbf{n}$ is the outward-pointing unit normal vector on Γ so $\mathbf{n} = (0, -1)$. In this case we have
$$0 = \frac{\partial u^{\text{Tot}}}{\partial \mathbf{n}} - ik\vartheta u^{\text{Tot}} = -\frac{\partial u^{\text{Tot}}}{\partial x_2} - ik\vartheta u^{\text{Tot}} = \Big(ikd_2(-1 + A) - ik\vartheta(1 + A)\Big)e^{ikx_1d_1}.$$
This vanishes for $d_2(-1 + A) = \vartheta(1 + A)$, i.e. $A = \frac{d_2+\vartheta}{d_2-\vartheta}$.

Summarising, the fields are

$$\begin{aligned} u^{\text{Inc}}(\mathbf{x}) &= e^{ik(x_1d_1+x_2d_2)}, \\ u^{\text{Scat}}(\mathbf{x}) &= Ae^{ik(x_1d_1-x_2d_2)}, \\ u^{\text{Tot}}(\mathbf{x}) &= e^{ik(x_1d_1+x_2d_2)} + Ae^{ik(x_1d_1-x_2d_2)}, \end{aligned} \qquad \text{with } A = \begin{cases} -1 & \text{for sound-soft } \Gamma, \\ 1 & \text{for sound-hard } \Gamma, \\ \frac{d_2+\vartheta}{d_2-\vartheta} & \text{for impedance } \Gamma. \end{cases}$$

The plane waves reflected by sound-soft or sound-hard interfaces have the same amplitude of the incoming plane waves and either opposite (sound-soft) or the same (sound-hard) phase. On the other hand, since $|A| < 1$, the waves reflected by an impedance line have amplitude smaller than the incoming wave: the impedance boundary absorbs some of the wave energy. The amount of wave that is reflected and the amount that is absorbed depend on the direction of the incoming wave; in particular, if $\vartheta \le 1$ the impedance boundary does not reflect (but absorbs completely) the impinging waves propagating in direction $\mathbf{d} = (\sqrt{1-\vartheta^2}, -\vartheta)$. An impedance boundary with $\vartheta = 1$ does not reflect the waves hitting perpendicularly.

For $\vartheta \to 0$ the impedance boundary condition converges to the sound-hard one, and consistently $A \to 1$; for $\vartheta \to \infty$ it converges to the sound-soft boundary condition and $A \to -1$.

Here we have considered a wave with infinite front hitting an infinite obstacle: clearly this is not a very realistic problem, but it helps to get an intuition of what happens when a wave hits an impenetrable obstacle.

Remark 2.1. When we solve a well-posed boundary value problem the solution is typically determined by the PDE and by the boundary datum. Here we have used something more. Let us look for example at the sound-soft case. Given u^{Inc}, *any* combination $u_\lambda^{\text{Scat}}(\mathbf{x}) = -\lambda e^{ik(x_1d_1+x_2d_2)} - (1-\lambda)e^{ik(x_1d_1-x_2d_2)}$ satisfies the Helmholtz equation and the boundary condition $u_\lambda^{\text{Scat}} = -u^{\text{Inc}}$ on Γ. We have chosen the case $\lambda = 0$ because from the law of reflection we expect the scattered field to propagate upwards. We will see that in all problems posed on unbounded domains we need to select the waves propagating in the desired direction, and that this is equivalent to imposing conditions "at infinity".

Remark 2.2. The same reasoning made above shows that if u^{Inc} is any wave propagating downwards, then the reflected wave $u_D^{\text{Scat}}(\mathbf{x}) = -u^{\text{Inc}}(x_1, -x_2)$ propagates upwards and $u^{\text{Inc}} + u_D^{\text{Scat}}$ vanishes on Γ. So $u_D^{\text{Tot}} = u^{\text{Inc}} + u_D^{\text{Scat}}$ is the field reflected by a sound-soft line.

You can see the reflection of the Herglotz function $u^{\text{Inc}}(\mathbf{x}) = \int_{-\pi/3}^{-\pi/6} e^{ik(x_1\cos\theta + x_2\sin\theta)}\,d\theta$ by a sound-soft horizontal line in Figure 6 and in the animation on the course webpage.

Similarly, $u_N^{\text{Scat}}(\mathbf{x}) = u^{\text{Inc}}(x_1, -x_2)$ propagates upwards and $\frac{\partial}{\partial \mathbf{n}}(u^{\text{Inc}} + u_N^{\text{Scat}})$ vanishes on Γ, so $u_N^{\text{Tot}} = u^{\text{Inc}} + u_N^{\text{Scat}}$ is the reflection of any downward u^{Inc} by a sound-hard line.

For an impedance line Γ, the argument is slightly more complicated: since the reflection coefficient A depends on the direction of the incoming wave, to compute u^{Scat} we need to be able to decompose u^{Tot} in plane waves with different directions and reflect each one of them with its own coefficient. This is possible if u^{Inc} is a Herglotz function with kernel supported in the lower half of the unit circle ($g(\theta) = 0$ for $0 < \theta < \pi$).

Exercise 2.3. Plot the total field when the incoming field is the same Herglotz function as in Figure 6 and in Remark 2.2, but the horizontal line Γ acts as a Neumann boundary. Start from the Matlab file provided.

Repeat the same for an impedance boundary.

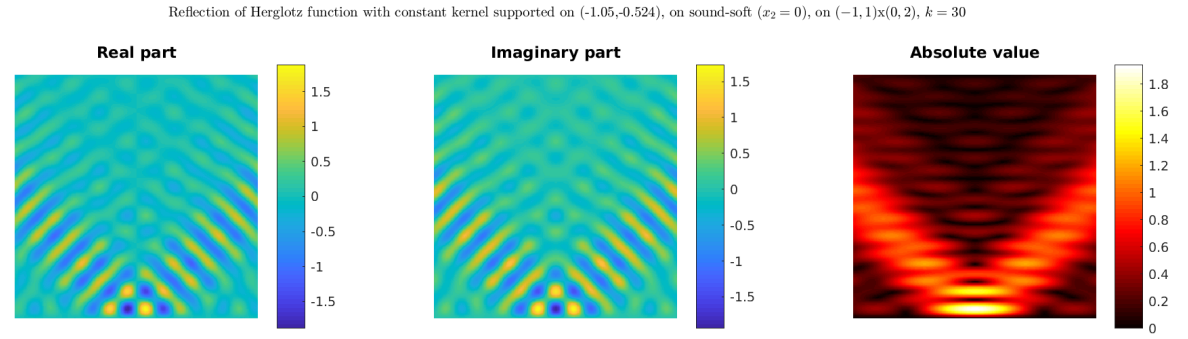


Figure 6: Reflection of the Herglotz function with kernel $g(\theta) = \chi_{(-\frac{\pi}{3},-\frac{\pi}{6})}(\theta)$ by the sound-soft line $\{x_2 = 0\}$. See Remark 2.2.

2.2 Tools from analysis

2.2.1 Lipschitz domains

We say that an open set $\Omega \subset \mathbb{R}^2$ with bounded boundary is Lipschitz if (1) there is a finite family of open sets $\{W_j\}_{j=1,\dots,J}$ that cover $\partial\Omega$ (i.e. $\partial\Omega \subset \bigcup_{j=1,\dots,J} W_j$), (2) there is a family of rotated Lipschitz hypographs $\{\Omega_j\}_{j=1,\dots,J}$ (i.e. $\Omega_j = \{(x_1', x_2') : x_2' < f_j(x_1')\}$, where (x_1', x_2') is a system of rotated Cartesian coordinates in $\mathbb{R}^2$ and f_j is a Lipschitz function), and (3) $W_j \cap \Omega = W_j \cap \Omega_j$.

Intuitively, for each $\mathbf{x} \in \partial\Omega$, there is a neighbourhood where the boundary can be represented as the graph of a Lipschitz function, and Ω lies only on one side of $\partial\Omega$. Smooth domains and polygons are Lipschitz. Domains with cusps (such as $\{0 < \sqrt{x_2} < x_1 < 1\}$) or cracks (such as $\{|\mathbf{x}| < 1, x_2 \neq 0\}$) are not allowed.

We also say that Ω is of class C^m, $m = 0, 1, \dots, \infty$, if the functions f_j are of class C^m.

An important property of Lipschitz domains is that the unit normal vector field is defined almost everywhere on their boundary (a.e. with respect to the 1-dimensional measure).

2.2.2 Function spaces on Lipschitz domains

To study boundary value problems we need some function spaces. Let $\Omega \subset \mathbb{R}^2$ be an open, Lipschitz set. We denote by $\boxed{\mathscr{D}(\Omega)}$ the space of the "test functions": complex-valued C^∞ functions defined on Ω whose support is compactly contained in Ω.

We denote by $\boxed{L^2(\Omega)}$ the usual Lebesgue space of complex-valued, square-integrable functions. This is an Hilbert space with inner product $(v, w)_{L^2(\Omega)} = \int_\Omega v\overline{w}\,\mathrm{d}\mathbf{x}$ and norm $\|v\|^2_{L^2(\Omega)} = \int_\Omega |v|^2\,\mathrm{d}\mathbf{x}$. Since we are dealing with complex-valued functions, the inner product is a sesquilinear form (with a complex conjugation on the second entry) and the norm requires the use of the absolute value of the argument.

Definition 2.4 ($H^1(\Omega)$ and $H^1_0(\Omega)$)**.** The Sobolev space $\boxed{H^1(\Omega)}$ is space of complex-valued $L^2(\Omega)$ functions, whose first (distributional) partial derivatives are in $L^2(\Omega)$. It is provided with the following seminorm, norm, and inner product:

$$|v|^2_{H^1(\Omega)} := \|\nabla v\|^2_{L^2(\Omega)^2} = \left\|\frac{\partial v}{\partial x_1}\right\|^2_{L^2(\Omega)} + \left\|\frac{\partial v}{\partial x_2}\right\|^2_{L^2(\Omega)}, \qquad \|v\|^2_{H^1(\Omega)} := |v|^2_{H^1(\Omega)} + \|v\|^2_{L^2(\Omega)},$$

$$(v, w)_{H^1(\Omega)} := \left(\frac{\partial v}{\partial x_1}, \frac{\partial w}{\partial x_1}\right)_{L^2(\Omega)} + \left(\frac{\partial v}{\partial x_2}, \frac{\partial w}{\partial x_2}\right)_{L^2(\Omega)} + (v, w)_{L^2(\Omega)} = \int_\Omega (\nabla v \cdot \nabla\overline{w} + v\overline{w})\,\mathrm{d}\mathbf{x}.$$

The space $\boxed{H^1_0(\Omega)}$ is the subspace of $H^1(\Omega)$ of the elements that can be approximated in $H^1(\Omega)$ norm by a sequence of elements of $\mathscr{D}(\Omega)$.

Remark 2.5. When we say that a partial derivative "in the sense of distributions" $\frac{\partial v}{\partial x_1}$ of an $L^2(\Omega)$ function is in $L^2(\Omega)$, we mean that there is a function $w \in L^2(\Omega)$ such that $\int_\Omega v\frac{\partial\varphi}{\partial x_1} = -\int_\Omega w\varphi$ for all $\varphi \in \mathscr{D}(\Omega)$.

The two terms in the definition of the $H^1(\Omega)$ norm are not dimensionally homogeneous, so when dealing with Helmholtz problems we often weigh the norm with the wavenumber $k > 0$ as $\boxed{\|v\|^2_{H^1_k(\Omega)}} := |v|^2_{H^1(\Omega)} + k^2 \|v\|^2_{L^2(\Omega)}$.

We define also the subspace of $H^1(\Omega)$ of the functions with square-integrable (distributional) Laplacian: $\boxed{H^1(\Omega;\Delta)} = \{v \in H^1(\Omega) : \Delta v \in L^2(\Omega)\}$.

Finally, we say that $u \in \boxed{H^1_{\text{loc}}(\overline{\Omega})}$ if the restriction of u to any bounded open subset D of Ω belongs to $H^1(D)$, [Sayas06, p. 12]. If Ω is bounded, then $H^1_{\text{loc}}(\overline{\Omega}) = H^1(\Omega)$, while if Ω is unbounded then the "local space" $H^1_{\text{loc}}(\overline{\Omega})$ includes functions that do not decay at infinity. For instance, all plane and circular waves belong to $H^1_{\text{loc}}(\overline{\Omega}) \setminus H^1(\Omega)$ if Ω is the complement of a bounded set (which must contain the origin, if the circular waves are the singular ones). We haven't defined a norm on $H^1_{\text{loc}}(\overline{\Omega})$, so this is not a Hilbert space; on the other hand the $H^1(D)$ norms are seminorms on $H^1_{\text{loc}}(\overline{\Omega})$. Similarly, $u \in H^1_{\text{loc}}(\overline{\Omega};\Delta)$ if the restriction of u to any bounded open subset D of Ω belongs to $H^1(D;\Delta)$.

The elements of $H^1(\Omega)$ are in general not continuous, so their point evaluation is not well-defined. However we will see that their values on the boundary of Ω, or any other Lipschitz curve, are well-defined.

2.2.3 Spaces on boundaries

We will need spaces of functions defined on boundary of Lipschitz sets with regularity weaker than H^1 and stronger than L^2. How to define functions with "half derivative"?

Let $\mathbb{S}^1 = \{\mathbf{x} \in \mathbb{R}^2, \|\mathbf{x}\| = 1\}$ be the unit circle. For a function v defined on $\mathbb{S}^1$ we write $v(\theta)$, with $\theta \in [0, 2\pi]$, for its value in polar coordinates. We say that $v \in L^2(\mathbb{S}^1)$ if $\|v\|^2_{L^2(\mathbb{S}^1)} := \int_0^{2\pi} |v|^2 \,\mathrm{d}\theta < \infty$ and $v \in H^1(\mathbb{S}^1)$ if $\|v\|^2_{H^1(\mathbb{S}^1)} := \int_0^{2\pi} (|v|^2 + |v'|^2) \,\mathrm{d}\theta < \infty$, where v' is the derivative in the angular coordinate. The expansion of v in circular harmonics is $v(\theta) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}$, for a sequence of coefficients $\hat{v}_\ell \in \mathbb{C}$. We can compute the norms using this expansion, exploiting their orthogonality $\int_0^{2\pi} \mathrm{e}^{\mathrm{i}\ell\theta}\mathrm{e}^{-\mathrm{i}\ell'\theta} \,\mathrm{d}\theta = 2\pi\delta_{\ell,\ell'}$ and the derivation formula $\frac{\partial}{\partial\theta}\mathrm{e}^{\mathrm{i}\ell\theta} = \mathrm{i}\ell\mathrm{e}^{\mathrm{i}\ell\theta}$:

$$\|v\|^2_{L^2(\mathbb{S}^1)} = \int_0^{2\pi} |v|^2 \,\mathrm{d}\theta = 2\pi \sum_{\ell\in\mathbb{Z}} |\hat{v}_\ell|^2, \qquad \|v\|^2_{H^1(\mathbb{S}^1)} = \int_0^{2\pi} (|v|^2 + |v'|^2) \,\mathrm{d}\theta = 2\pi \sum_{\ell\in\mathbb{Z}} |\hat{v}_\ell|^2 (1+\ell^2).$$

(Write down the intermediate computations.) Thus, a function defined on the circle is in $L^2(\mathbb{S}^1)$ if the sequence of its Fourier coefficients is an element of the sequence space $l^2(\mathbb{Z}) = \{(a_\ell), \ell \in \mathbb{Z}, \|(a_\ell)\|^2_{l^2} := \sum_{\ell\in\mathbb{Z}} |a_\ell|^2 < \infty\}$, and in $H^1(\mathbb{S}^1)$ if its Fourier coefficients weighted with $(1+\ell^2)^{1/2}$ are in $l^2(\mathbb{Z})$.

This suggests a way to define Sobolev spaces with other regularities:

$$\boxed{\|v\|^2_{H^s(\mathbb{S}^1)}} := 2\pi \sum_{\ell\in\mathbb{Z}} |\hat{v}_\ell|^2 (1+\ell^2)^s, \qquad \boxed{H^s(\mathbb{S}^1)} := \Big\{ v(\theta) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta} : \|v\|_{H^s(\mathbb{S}^1)} < \infty \Big\}, \quad \forall s \in \mathbb{R}. \tag{17}$$

For $s = 0$ and $s = 1$ we recover $L^2(\mathbb{S}^1)$ and $H^1(\mathbb{S}^1)$ as defined above. High-order Fourier coefficients correspond to rapidly oscillating components: imposing that they decay faster in ℓ is the same as imposing some regularity on v. So the higher s the smoother are the elements of $H^s(\mathbb{S}^1)$: for any real $s_- < s_+$, $H^{s_+}(\mathbb{S}^1) \subset H^{s_-}(\mathbb{S}^1)$ and $\|v\|_{H^{s_-}(\mathbb{S}^1)} \le \|v\|_{H^{s_+}(\mathbb{S}^1)}$. If $s > 1/2$, the elements of $H^s(\mathbb{S}^1)$ are continuous functions, if $s \ge 0$ they are simply $L^2(\mathbb{S}^1)$ classes of equivalence, for $s < 0$ they can only be understood as distributions.

Given a Lipschitz bounded domain Ω, if there is a bi-Lipschitz map $\Phi : \overline{B}_1 = \{|\mathbf{x}| \le 1\} \to \overline{\Omega}$ that maps $\mathbb{S}^1$ in $\partial\Omega$, we can define the space $\boxed{H^s(\partial\Omega)}$ as the space of functions v defined on $\partial\Omega$ whose pullback $v^*(\mathbf{x}) = v(\Phi(\mathbf{x}))$ is an element of $H^s(\mathbb{S}^1)$. For $-1 \le s \le 1$ this gives a well-defined space $H^s(\partial\Omega)$ independently of the choice of Φ. Different maps Φ give equivalent norms. If we do not have such a map Φ, $H^s(\partial\Omega)$ (for $-1 \le s \le 1$) can be defined in a slightly more complicated way, using the cover of $\partial\Omega$ from the definition of a Lipschitz set and a so-called "partition of unity".

Several other definitions of the spaces $H^s(\partial\Omega)$ exist and can be found in the literature. The norms obtained with different definitions are equivalent (for $-1 \le s \le 1$) but not necessarily equal.

We will need $H^s(\partial\Omega)$ only for $s = \pm 1/2$: $H^{\frac{1}{2}}(\partial\Omega) \subset L^2(\partial\Omega) \subset H^{-\frac{1}{2}}(\partial\Omega)$.

An important property of these spaces is that $H^{-s}(\partial\Omega)$ can be identified to the anti-dual space of $H^s(\Omega)$, i.e. the space of anti-linear continuous functionals on $H^s(\partial\Omega)$. Let us look at what this means in

the case of $\mathbb{S}^1$. If $v \in H^s(\mathbb{S}^1)$ and $w \in H^{-s}(\mathbb{S}^1)$ then we can define the **duality product**

$$\langle v, w\rangle_{\mathbb{S}^1} = \Big\langle \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}, \sum_{\ell\in\mathbb{Z}} \hat{w}_\ell \mathrm{e}^{\mathrm{i}\ell\theta} \Big\rangle_{\mathbb{S}^1} := 2\pi \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \overline{\hat{w}_\ell}.$$

This series is bounded: by Cauchy–Schwarz inequality in $l^2(\mathbb{Z})$,

$$|\langle v, w\rangle_{\mathbb{S}^1}| \le 2\pi \sum_{\ell\in\mathbb{Z}} (1+\ell^2)^{s/2} |\hat{v}_\ell| (1+\ell^2)^{-s/2} |\hat{w}_\ell| \le \|v\|_{H^s(\mathbb{S}^1)} \|w\|_{H^{-s}(\mathbb{S}^1)}.$$

Moreover, if $v, w \in L^2(\mathbb{S}^1)$, then the duality product coincides with the L^2 inner product: $\langle v, w\rangle_{\mathbb{S}^1} = \int_{\mathbb{S}^1} v(\theta)\overline{w}(\theta)\,\mathrm{d}\theta$.

Similarly, it is possible to define a duality product $\langle\cdot,\cdot\rangle_{\partial\Omega}$ on $\partial\Omega$ such that

$$|\langle v, w\rangle_{\partial\Omega}| \le C \|v\|_{H^{\frac12}(\partial\Omega)} \|w\|_{H^{-\frac12}(\partial\Omega)} \qquad \forall v \in H^{\frac12}(\partial\Omega),\ w \in H^{-\frac12}(\partial\Omega), \text{ and}$$

$$\langle v, w\rangle_{\partial\Omega} = \int_{\partial\Omega} v\overline{w}\,\mathrm{d}s, \qquad \text{if } w \in L^2(\partial\Omega).$$

Because of this, we sometimes write the duality product simply as $\int_{\partial\Omega} v\overline{w}\,\mathrm{d}s$, even when one of the two distributions is not in L^2 and the product is not strictly speaking an integral. We write $\langle v, w\rangle_{\partial\Omega}$ also when $v \in H^{-\frac12}(\partial\Omega)$ and $w \in H^{\frac12}(\partial\Omega)$. To be more clear we might write $\langle v, w\rangle_{H^{\frac12}(\partial\Omega)\times H^{-\frac12}(\partial\Omega)}$ and $\langle v, w\rangle_{H^{-\frac12}(\partial\Omega)\times H^{\frac12}(\partial\Omega)}$, depending on the regularity of the arguments.

These spaces are closely related to **trace operators**. Given a smooth function $v \in C^1(\overline{\Omega})$, we define its **Dirichlet and Neumann traces**, respectively, as

$$\boxed{\gamma v := v|_{\partial\Omega}, \qquad \partial_{\mathbf{n}} v := \mathbf{n}\cdot\gamma(\nabla v)}$$

where $\mathbf{n}$ is the outward-pointing unit normal vector field on $\partial\Omega$ (which is defined almost everywhere if the domain is Lipschitz). Can we define these traces for more general functions, such as elements of Sobolev spaces?

Theorem 2.6 (Trace theorem). The Dirichlet trace γ can be extended to a surjective continuous operator mapping $\gamma : H^1(\Omega) \to H^{\frac12}(\partial\Omega)$. The kernel of γ is $H^1_0(\Omega)$.
The Neumann trace $\partial_{\mathbf{n}}$ can be extended to a surjective continuous operator $\partial_{\mathbf{n}} : H^1(\Omega;\Delta) \to H^{-\frac12}(\partial\Omega)$.

This theorem says that the spaces $H^{\frac12}(\partial\Omega)$ and $H^{-\frac12}(\partial\Omega)$ are precisely the spaces of the Dirichlet and the Neumann traces of $H^1(\Omega)$ and $H^1(\Omega;\Delta)$ functions, respectively. See [Spence14, p. 6] for more details. A consequence is that the following are equivalent norms on $H^{\pm\frac12}(\partial\Omega)$:

$$\inf\{\|U\|_{H^1(\Omega)} : \gamma U = u\} \sim \|u\|_{H^{\frac12}(\partial\Omega)}, \qquad \inf\{\|U\|_{H^1(\Omega;\Delta)} : \partial_{\mathbf{n}} U = u\} \sim \|u\|_{H^{-\frac12}(\partial\Omega)}.$$

The trace operators are local, i.e. γv and $\partial_{\mathbf{n}} v$ depend only on the value of v in an arbitrary small neighbourhood of $\partial\Omega$. Thus, if Ω is unbounded (but $\partial\Omega$ is still bounded), in the trace theorem we can substitute $H^1(\Omega)$ and $H^1(\Omega;\Delta)$ with $H^1_{\mathrm{loc}}(\overline{\Omega})$ and $H^1_{\mathrm{loc}}(\overline{\Omega};\Delta)$, respectively.

2.2.4 Green's identities

The divergence theorem says that for any $\mathbf{F} \in C^1(\overline{\Omega})^2$ we have $\int_\Omega \operatorname{div}\mathbf{F}\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \mathbf{F}\cdot\mathbf{n}\,\mathrm{d}s$, where $\mathbf{n}$ is the outward pointing unit normal vector field on $\partial\Omega$. The product rule for the divergence is $\operatorname{div}[w\mathbf{G}] = \nabla w\cdot\mathbf{G} + w\operatorname{div}\mathbf{G}$. The combination of these two ingredients gives **Green's first and second identity** for the Helmholtz equation: for $u, w \in C^2(\overline{\Omega})$, $k \in \mathbb{R}$

$$\int_\Omega (\Delta u + k^2 u) w\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} \partial_{\mathbf{n}} u\ \gamma w\,\mathrm{d}s + \int_\Omega (k^2 uw - \nabla u\cdot\nabla w)\,\mathrm{d}\mathbf{x}, \tag{18}$$

$$\int_\Omega \Big((\Delta u + k^2 u) w - u(\Delta w + k^2 w)\Big)\,\mathrm{d}\mathbf{x} = \int_{\partial\Omega} (\partial_{\mathbf{n}} u\ \gamma w - \gamma u\ \partial_{\mathbf{n}} w)\,\mathrm{d}s. \tag{19}$$

Exercise 2.7. Write in detail the proof of (18)–(19) for $u, v \in C^2(\overline{\Omega})$.

Do these identities hold for Sobolev functions?

Proposition 2.8. If Ω is a bounded Lipschitz domain, Green's first identity (18) holds for $u \in H^1(\Omega;\Delta)$ e $w \in H^1(\Omega)$. Green's second identity (19) holds for $u, v \in H^1(\Omega;\Delta)$.
The boundary integrals must be understood as the duality products $\langle \partial_{\mathbf{n}} u, \gamma\overline{w}\rangle_{\partial\Omega}$ and $\langle \gamma u, \partial_{\mathbf{n}}\overline{w}\rangle_{\partial\Omega}$.

Green's identities are the main tools in the derivation of boundary integral equations. This is the reason why the Sobolev spaces we need are $H^1(\Omega)$, $H^1(\Omega;\Delta)$ and their trace spaces $H^{\pm\frac{1}{2}}(\partial\Omega)$.

2.2.5 Variational problems, Fredholm alternative, Gårding inequality

As we do for Laplace equation, we typically write Helmholtz problems in variational form. The abstract linear **variational problem** in the (complex) Hilbert space H is

$$\text{find} \quad u \in H \qquad \text{such that} \quad \mathcal{A}(u,w) = \mathcal{F}(w) \qquad \forall w \in H, \tag{20}$$

where $\mathcal{A}$ is a sesquilinear form in H and $\mathcal{F}$ is a anti-linear functional in H. We recall that an anti-linear functional satisfies $\mathcal{F}(\lambda v + \mu w) = \overline{\lambda}\mathcal{F}(v) + \overline{\mu}\mathcal{F}(w)$ for all $\lambda, \mu \in \mathbb{C}$ and $v, w \in H$. A sesquilinear form is linear in the first argument and anti-linear in the second one.

Given a variational problem such as (20), the (conforming) **Galerkin method** consists of choosing an N-dimensional subspace $V_N \subset H$ and a basis $\varphi_1, \dots, \varphi_N$, and of looking for a solution of the problem restricted to V_N:

$$\text{find} \quad u_N \in V_N \qquad \text{such that} \quad \mathcal{A}(u_N, w_N) = \mathcal{F}(w_N) \qquad \forall w_N \in V_N. \tag{21}$$

This is achieved computationally by solving the $N \times N$ linear algebraic system $\underline{\underline{\mathbf{A}}}\mathbf{U} = \mathbf{F}$ where $A_{j,m} := \mathcal{A}(\varphi_m, \varphi_j)$, $F_j := \mathcal{F}(\varphi_j)$ and $u_N = \sum_{j=1}^N U_j \varphi_j$.

To a continuous sesquilinear form $\mathcal{A}$, we can associate a linear operator $A : H \to H^*$ by $\langle Au, w\rangle_{H^*\times H} = \mathcal{A}(u,w)$ for all $u, w \in H$, where H^* is the anti-dual of H, [Spence14, Lemma 5.4]. The operator A admits a continuous inverse if and only if, for all $\mathcal{F} \in H^*$, the variational problem (20) is well-posed.

In the case of Laplace equation, the crucial result from functional analysis is **Lax–Milgram** theorem: if $\mathcal{A}$ is **continuous** ($|\mathcal{A}(u,w)| \le C_{\mathcal{A}} \|u\|_H \|w\|_H \ \forall u, w \in H$) and **coercive**[3] ($|\mathcal{A}(w,w)| \ge \gamma_{\mathcal{A}} \|w\|_H^2 \ \forall w \in H$), and $\mathcal{F}$ is continuous ($|\mathcal{F}(w)| \le C_{\mathcal{F}} \|w\|_H \ \forall w \in H$), then the corresponding variational problem is well-posed. Moreover, several good properties of all Galerkin discretisations follow (well-posedness, stability bounds, quasi-optimality, bounds on the number of linear solver iterations, ...).

Unfortunately, for most variational formulations of the Helmholtz equation coercivity does not hold, so we cannot rely on Lax–Milgram. The branch of functional analysis that we need is called "Fredholm theory" and studies compact perturbations of operators. We recall that a linear operator $K : H_1 \to H_2$ is **compact** if the image of all bounded sequences admits a converging subsequence ($(v_j)_{j\in\mathbb{N}} \subset H_1, \|v_j\|_H \le C \ \forall j \in \mathbb{N} \Rightarrow \exists j_m \to \infty, w \in H_2$ such that $Kv_{j_m} \to w$). All compact operators are continuous. The main result is the "Fredholm alternative", which, in its simplest form, reads as follows.

Theorem 2.9 (Fredholm alternative)**.** Let $T : H_1 \to H_2$ be the sum of a continuous invertible linear operator and a compact one. Then T is injective if and only if it is surjective. In this case its inverse T^{-1} is bounded.

We say that an operator is a **Fredholm operator** if it is sum of an invertible one and a compact one (more precisely, we should say it is a Fredholm operator of index 0). Fredholm alternative says that if we want to prove that a Fredholm operator is invertible, then it suffices to prove its injectivity. A useful idea to keep in mind is that, under this respect, Fredholm operators behave like square matrices.[4]

How do we use Fredholm alternative? The general strategy is the following. We will show that some linear operator T mapping "problem solution" to "problem data" are Fredholm. When we can show that the homogeneous problem (with data equal to 0) only admits the trivial solution (i.e. T is injective), Fredholm alternative guarantees that *all* data admit a solution (i.e. T is surjective), which is unique by injectivity, and that the solution is controlled by the data (i.e. T^{-1} is bounded). **A linear problem whose "solution-to-data" operator is Fredholm is well-posed if the same operator is injective.**

[3] Here terminology can be confusing. Coercivity is sometimes called "sign-definiteness", "V-ellipticity", "strong ellipticity", or "strict coercivity", see [Spence14, §5.2]. In some of these cases, the word "coercive" is used for sesquilinear forms satisfying a Gårding inequality, which is a weaker condition. Here we follow the convention of [Spence14].

[4] Indeed, an invertible linear operator between finite-dimensional spaces corresponds to a square matrix. All finite-range operators are compact because all bounded sequences of $\mathbb{R}^n$ and $\mathbb{C}^n$ admit converging subsequences. Thus the operators between finite-dimensional spaces that are Fredholm are precisely those associated to square matrices. We know from linear algebra that a square matrix is injective if and only if it is surjective. This proves the Fredholm alternative in the case of operators between finite-dimensional spaces.

Typically, in time-harmonic problems, sesquilinear forms are not coercive but satisfy a weaker inequality, called Gårding inequality.

Definition 2.10 (Gårding inequality)**.** Let $H \subset V$ be two Hilbert spaces provided with the norms $\|\cdot\|_H$ and $\|\cdot\|_V$, and let the embedding $H \hookrightarrow V$ be continuous. A sesquilinear form satisfies a Gårding inequality if there exists two positive constants α and C_V such that

$$\mathcal{A}(v,v) \geq \alpha \|v\|_H^2 - C_V \|v\|_V^2 \qquad \forall v \in H. \tag{22}$$

Here we follow the notation of [Spence14, §5.3], where H and V are swapped with respect to the classical choice for Hilbert triples.

The main use of Gårding inequality comes from the next result, see [Spence14, Theorem 5.20].

Proposition 2.11. Assume that H and V are as in Definition 2.10, the embedding $H \hookrightarrow V$ is compact and the continuous sesquilinear form $\mathcal{A}(\cdot,\cdot)$ satisfies the Gårding inequality (22).
Then the operator $A : H \to H^*$ associated to $\mathcal{A}(\cdot,\cdot)$ is Fredholm.

Proposition 2.11 and Theorem 2.9 imply that, in order to prove well-posedness of a variational problem (20) whose sesquilinear form $\mathcal{A}(\cdot,\cdot)$ satisfies a Gårding inequality, it suffices to study the homogeneous problem (with $\mathcal{F} = 0$):

Corollary 2.12. Assume that H and V are as in Definition 2.10, the embedding $H \hookrightarrow V$ is compact and the continuous sesquilinear form $\mathcal{A}(\cdot,\cdot)$ satisfies the Gårding inequality (22).
If the homogeneous variational problem $\mathcal{A}(u,w) = 0$ for all $w \in H$ admits only the trivial solution $u = 0$ then also the non-homogeneous problem (20) is well-posed, for any $\mathcal{F} \in H^*$.

To be able to exploit Corollary 2.12, we need to know when the embedding between two function spaces is compact. A classical result, called Rellich embedding theorem, says that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, when Ω is a bounded Lipschitz domain. Similarly, one can show that $H^{s_+}(\partial\Omega) \hookrightarrow H^{s_-}(\partial\Omega)$ is compact for all $s_+ > s_-$, [Néd01, Theorem 2.5.7].

An extensive description of the relations between variational problems satisfying Lax–Milgram assumptions, Gårding inequality or the inf-sup inequality, and the consequences for the Galerkin method, can be found in [Spence14, §5.3].

2.3 Boundary value problems in bounded domains

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, Lipschitz set. The Dirichlet BVP for the Helmholtz equation is: given a source term f defined in Ω and a boundary datum g_D defined on $\partial\Omega$ find u on Ω such that

$$\Delta u + k^2 u = -f \quad \text{in } \Omega, \qquad \gamma u = g_D \quad \text{on } \partial\Omega. \tag{23}$$

We know that the Dirichlet problem for the Poisson equation (problem (23) with $k = 0$) is well-posed when the data f, g_D are sufficiently smooth. Despite the Helmholtz equation looks like an innocuous low-order perturbation of the Laplace equation, well-posedness of the Helmholtz Dirichlet problem is not guaranteed.

We start from a simple example. Let Ω be the rectangle $(0, L_1) \times (0, L_2)$. Then for all $j_1, j_2 \in \mathbb{N}$ the field $u_{j_1,j_2}(\mathbf{x}) = \sin(\frac{j_1\pi}{L_1}x_1)\sin(\frac{j_2\pi}{L_2}x_2)$ vanishes on $\partial\Omega$ and is solution of $\Delta u + k_{j_1,j_2}^2 u = 0$ with $k_{j_1,j_2}^2 = \frac{j_1^2\pi^2}{L_1^2} + \frac{j_2^2\pi^2}{L_2^2}$. So there are infinitely many values of k such that the homogeneous ($f = 0$ and $g_D = 0$) Helmholtz Dirichlet BVP admits non-trivial solutions. It follows that for these values of k the problem (23) is not well-posed: if there is a solution then it cannot be unique.

Solutions of the homogeneous Helmholtz Dirichlet problem are called **Dirichlet eigenfunctions** of the Laplacian with eigenvalue $\Lambda = k^2$, as they satisfy the eigenproblem $-\Delta u = \Lambda u$.

As a second example, if Ω is a disc of radius R, then the circular waves $J_\ell(kr)\mathrm{e}^{\pm \mathrm{i}\ell\theta}$ are Dirichlet eigenfunctions for $\Lambda = k^2$ and k such that kR is a zero of the ℓth Bessel function J_ℓ (recall the plots in the left panel of Figure 3 and the top panel of Figure 4).

In other domains we find the same situation as in the two examples described, even if we cannot compute eigenvalues and eigenfunctions explicitly.

To understand the problem in more general bounded Lipschitz domains, we study the problem from a variational point of view. From Green's first identity (18), the variational problem for the Helmholtz

Dirichlet BVP (23) with homogeneous boundary conditions $g_D = 0$ is

$$\text{find } u \in H^1_0(\Omega) \text{ such that} \quad \mathcal{A}(u,w) := \int_\Omega (\nabla u \nabla \overline{w} - k^2 u \overline{w})\,\mathrm{d}x = \int_\Omega f\overline{w}\,\mathrm{d}\mathbf{x} =: \mathcal{F}(w) \quad \forall w \in H^1_0(\Omega). \tag{24}$$

The sesquilinear form $\mathcal{A}(\cdot,\cdot)$ and the linear functional $\mathcal{F}(\cdot)$ are continuous in $H^1_0(\Omega)$. On the other hand, $\mathcal{A}(\cdot,\cdot)$ is not coercive (for k sufficiently large), as the two terms ($\nabla u \nabla \overline{w}$ and $-k^2 u\overline{w}$) have opposite signs, [Spence14, Lemma 6.2]. However, it satisfies a Gårding inequality (22) with $\alpha = 1$ and $C_V = 2k^2$:

$$\mathcal{A}(w,w) = \|w\|^2_{H^1_k(\Omega)} - 2k^2\,\|w\|^2_{L^2(\Omega)} \qquad \forall w \in H^1(\Omega).$$

Proposition 2.11, together with the compactness of $H^1_0(\Omega)$ in $L^2(\Omega)$, gives that the operator $A : H^1_0(\Omega) \to (H^1_0(\Omega))^*$, $A : u \mapsto f$, is Fredholm. Corollary 2.12 of Fredholm alternative then implies that, given Ω and k, only two situations can happen:

- If Ω and k are such that the homogeneous ($f = 0$) problem (24) admits only the trivial solution $u = 0$, then also problem (24) with any $f \in L^2(\Omega)$ (or more generally $\mathcal{F} \in (H^1_0(\Omega))^*$) admits a unique solution.
- On the other hand, if there is a $u \neq 0$ such that $\mathcal{A}(u,w) = 0\ \forall w \in H^1_0(\Omega)$, then the problem (24) is not well-posed for *any* f. It might have no solutions for some f, and many solutions for some other f.

We have proved part of the following proposition. To prove the remaining part (the existence, discreteness and divergence at infinity of the eigenvalues), one needs the spectral theory of self-adjoint compact operators, see e.g. §6 of Brezis' book. To treat non-homogeneous Dirichlet boundary conditions $g_D \neq 0$, one uses a "lifting", i.e. a $u_D \in H^1(\Omega)$ such that $\gamma u_D = g_D$ (which exists because of the surjectivity of the trace operator), and then solves for $u_0 = u - u_D \in H^1_0(\Omega)$.

Proposition 2.13. For a Lipschitz bounded domain Ω, there exist a sequence of positive numbers $k_1 < k_2 < \ldots$, with $k_j \to \infty$, such that:

- If $k = k_j$ for some j, then the Dirichlet problem (23) is not well-posed.
  In particular, the homogeneous case with $f = 0$ and $g_D = 0$ admits non-trivial solutions.
- If $0 < k \neq k_j$ for all j, then the Dirichlet problem (23) is well-posed in $H^1_0(\Omega)$ for all $f \in L^2(\Omega)$ and $g_D \in H^{\frac{1}{2}}(\partial\Omega)$.

Exercise 2.14 (Neumann Helmholtz BVP)**.**

- What are the eigenvalues and the eigenfunctions for the Laplacian with Neumann boundary conditions $\partial_{\mathbf{n}} u = 0$ on the rectangle and on the disc?
- Show that the positive Neumann eigenvalues for a rectangle coincide with the Dirichlet eigenvalues but the eigenfunctions differ. (On the other hand, using subtle properties of the Bessel functions it is possible to see that there are no Neumann eigenvalues of the disc that are also Dirichlet eigenvalues.)
- Write the variational formulation of the Neumann Helmholtz BVP with inhomogeneous conditions $\partial_{\mathbf{n}} u = g_N \in H^{-\frac{1}{2}}(\partial\Omega)$: the sesquilinear form coincides with that in (24) but the linear functional and the function space differs.

Exercise 2.15 (Absorption gives well-posedness.)**.** Show that the Dirichlet and the Neumann problems

$$\begin{aligned} \Delta u + k^2 u &= -f \quad \text{in } \Omega, \qquad \gamma u = 0 \quad \text{on } \partial\Omega, \quad \text{or} \\ \Delta u + k^2 u &= -f \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}} u = g_N \quad \text{on } \partial\Omega, \end{aligned}$$

are well-posed if $\Im k > 0$, $f \in L^2(\Omega)$ and $g_N \in H^{-\frac{1}{2}}(\partial\Omega)$.

Hint: First write the two BVPs as variational problems $\mathcal{A}(u,w) = \mathcal{F}(w)$ in $H^1_0(\Omega)$ and $H^1(\Omega)$, respectively. Then use Lax–Milgram theorem. To prove the coercivity $|\mathcal{A}(w,w)| \geq c\,\|w\|^2_{H^1_k(\Omega)}$ of the sesquilinear form obtained, first control the L^2 norm of u, then the H^1 seminorm by using the triangle inequality.

Deduce a bound on $\|u\|_{H^1_k(\Omega)}$. The bounding constant C will blow up for $\Im k \searrow 0$.

If instead of sound-soft and sound-hard conditions we have **impedance** ones we obtain a different result. Consider the impedance BVP:

$$\Delta u + k^2 u = -f \quad \text{in } \Omega, \qquad \partial_{\mathbf{n}} u - \mathrm{i}k\vartheta\,\gamma u = g_I \quad \text{on } \partial\Omega, \tag{25}$$

for $\vartheta > 0$, $f \in L^2(\Omega)$, $g_I \in H^{-\frac{1}{2}}(\partial\Omega)$. Its variational form is: find $u \in H^1(\Omega)$ such that

$$\mathcal{A}_I(u,w) := \int_\Omega (\nabla u \nabla \overline{w} - k^2 u \overline{w})\,\mathrm{d}\mathbf{x} - \mathrm{i}k\vartheta \int_{\partial\Omega} \gamma u\, \gamma\overline{w}\,\mathrm{d}s = \int_\Omega f\overline{w}\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega} g_I\, \gamma\overline{w}\,\mathrm{d}s =: \mathcal{F}(w) \quad \forall w \in H^1(\Omega). \tag{26}$$

As before, the sesquilinear form is continuous, coercive only for small k, and satisfies a Gårding inequality. So to check the well-posedness we only have to look at the homogeneous problem. If u satisfies (26) with $\mathcal{F} = 0$ (i.e. $f = 0$ and $g_I = 0$), taking the imaginary part of $\mathcal{A}_I(u,u) = 0$, we see that $\gamma u = 0$ on $\partial\Omega$, and from the boundary condition that also $\partial_{\mathbf{n}} u = 0$ on $\partial\Omega$. We will see that this implies that the impedance BVP is always well-posed.

2.4 Exterior boundary value problems

A typical problem in computational wave propagation is that of **scattering**. In a scattering problem we want to compute how a given incoming wave is perturbed by the interaction with an obstacle. Here we consider only sound-soft obstacles.

We fix some notation. Let $\boxed{\Omega_-} \subset \mathbb{R}^2$ be a bounded Lipschitz domain, denote $\boxed{\Omega_+} := \mathbb{R}^2 \setminus \overline{\Omega_-}$ and $\boxed{\Gamma} = \partial\Omega_-$. We will always assume that Ω_+ is connected, i.e. Ω_- has no holes. We choose the unit normal vector field $\boxed{\mathbf{n}}$ on $\partial\Omega$ that points out of Ω_- and into Ω_+. We will take traces of fields defined in Ω_- and in Ω_+: for clarity we will write $\boxed{\gamma^\pm}$ and $\boxed{\partial^\pm_{\mathbf{n}}}$ for the traces taken from Ω_+ and Ω_-. If $u \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ then $\gamma^+ u = \gamma^- u$ (and we might write γu); if instead $u \in H^1_{\mathrm{loc}}(\Omega_- \cup \Omega_+)$ then $\gamma^+ u$ and $\gamma^- u$ might differ. The same holds for the Neumann traces $\partial^\pm_{\mathbf{n}}$.

Let u^{Inc} be the incoming wave, or incident wave, a given Helmholtz solution which will be the datum of our scattering problem. We want to find the field u^{Scat} scattered by Ω_-, that is a Helmholtz solution in the exterior domain Ω_+ and such that $\gamma(u^{\mathrm{Tot}}) = 0$ on Γ, where $u^{\mathrm{Tot}} = u^{\mathrm{Inc}} + u^{\mathrm{Scat}}$. We see in the next section that these two conditions are not enough to determine u^{Scat}.

2.4.1 Example: scattering by a disc

Let us consider a simple example. Assume that (i) Ω_- is a disc of radius $R > 0$, centred at the origin and (ii) the trace of u^{Inc} on Γ is a circular harmonic, in polar coordinates $(\gamma^+ u^{\mathrm{Inc}})(R,\theta) = \mathrm{e}^{\mathrm{i}\ell\theta}$ for some $\ell \in \mathbb{Z}$. From §1.2.4, we know that all fields in the form[5]

$$u^{\mathrm{Scat}}_\lambda(r,\theta) = -\lambda \frac{H^{(1)}_\ell(kr)}{H^{(1)}_\ell(kR)} \mathrm{e}^{\mathrm{i}\ell\theta} - (1-\lambda)\frac{H^{(2)}_\ell(kr)}{H^{(2)}_\ell(kR)} \mathrm{e}^{\mathrm{i}\ell\theta}, \qquad \lambda \in \mathbb{C},$$

are Helmholtz solutions in Ω_+ and satisfy $\gamma^+(u^{\mathrm{Inc}} + u^{\mathrm{Scat}}_\lambda) = 0$ on the circle Γ. Which value of λ should we choose?

The scattered field is produced by the interaction of the obstacle Ω_- and the incoming field u^{Inc}. So it should look like a wave propagating away from Ω_- towards infinity.

First of all, we would like the "energy" $|u(\mathbf{x})|^2$ to decrease like r^{-1} for $r \to \infty$, so that $\int_{\{|\mathbf{x}|=R\}} |u|^2\,\mathrm{d}s$ is constant for $R \to \infty$. All Fourier–Bessel and Fourier–Hankel functions decay as $\sqrt{2/\pi kr}$ for $r \to \infty$, so this does not help choosing λ.

If we plot u^{Scat} in a position $\mathbf{x}$ very far from Ω_-, we expect it to point away from Ω_-, i.e. radially towards infinity. We would like u^{Scat} close to $\mathbf{x}$ to look like a plane wave pointing away from Ω_-, i.e. in the direction $\mathbf{d} = \frac{\mathbf{x}}{r}$:

$$u^{\mathrm{Scat}}(\mathbf{x}) \approx A\mathrm{e}^{\mathrm{i}k\mathbf{x}\cdot\frac{\mathbf{x}}{r}} = A\mathrm{e}^{\mathrm{i}kr} \qquad \mathbf{x} = (r\cos\theta, r\sin\theta).$$

Here $A \in \mathbb{C}$ includes the amplitude, proportional to $\frac{1}{\sqrt{r}}$, and the phase of the wave.

We use the fact that Bessel functions for large arguments can be approximated by the following formulas, [CK2, (3.59)]:

$$J_\ell(z) = \sqrt{\frac{2}{\pi z}} \cos\Big(z - \frac{\ell\pi}{2} - \frac{\pi}{4}\Big)\Big(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\Big), \quad Y_\ell(z) = \sqrt{\frac{2}{\pi z}} \sin\Big(z - \frac{\ell\pi}{2} - \frac{\pi}{4}\Big)\Big(1 + \mathcal{O}\Big(\frac{1}{z}\Big)\Big), \quad z \to \infty.$$

[5] Here we have chosen the Hankel functions as opposed to J_ℓ and Y_ℓ because they are different from 0 for all values of kR, so we can normalise as written.

Recalling the relations (16) between Bessel and Hankel functions, we can write the scattered field as

$$u_\lambda^{\mathrm{Scat}}(\mathbf{x}) \approx -\sqrt{\frac{2}{\pi kr}}\left(\lambda\frac{\mathrm{e}^{\mathrm{i}(kr-\frac{\ell\pi}{2}-\frac{\pi}{4}+\ell\theta)}}{H_\ell^{(1)}(kR)}+(1-\lambda)\frac{\mathrm{e}^{\mathrm{i}(-kr+\frac{\ell\pi}{2}+\frac{\pi}{4}+\ell\theta)}}{H_\ell^{(1)}(kR)}\right)\left(1+\mathcal{O}\Big(\frac{1}{kr}\Big)\right). \tag{27}$$

We see that the $H_\ell^{(1)}$ term gives a factor $\mathrm{e}^{\mathrm{i}kr}$, while $H_\ell^{(2)}$ term gives a factor $\mathrm{e}^{-\mathrm{i}kr}$. This means that only the first component is propagating outward, while the second one is directed towards the origin. So we want to keep the first term only and choose $\lambda = 1$.

Exercise 2.16. Check all computations.

Another way to see that the $\mathrm{e}^{\mathrm{i}kr}$ terms are outgoing is to recall the meaning of time-harmonic waves, as described in §1.1.2. For $u(\mathbf{x}) = r^{-1/2}\mathrm{e}^{\mathrm{i}kr}$, the time-dependent field (7) is $U(\mathbf{x},t) = r^{-1/2}\cos(kr-\omega t) = r^{-1/2}\cos(k(r-ct))$ which spreads (in time) radially from the origin. Conversely, $\tilde{u}(\mathbf{x}) = r^{-1/2}\mathrm{e}^{-\mathrm{i}kr} = \overline{u}(\mathbf{x})$ gives $\tilde{U}(\mathbf{x},t) = r^{-1/2}\cos(kr+\omega t) = U(\mathbf{x},-t)$ which moves towards the origin.

Exercise 2.17. Using the Matlab code provided, make time-harmonic animations of different combinations of Fourier–Bessel and Fourier–Hankel functions and observe in which direction they propagate.

For a general u^{Inc}, we can expand its trace on the circle Γ in circular harmonics as $u^{\mathrm{Inc}}(R,\theta) = \sum_{\ell\in\mathbb{Z}} a_\ell \mathrm{e}^{\mathrm{i}\ell\theta}$. The scattered field and the total field are then

$$u^{\mathrm{Scat}}(r,\theta) = -\sum_{\ell\in\mathbb{Z}} a_\ell \frac{H_\ell^{(1)}(kr)}{H_\ell^{(1)}(kR)}\mathrm{e}^{\mathrm{i}\ell\theta}, \qquad u^{\mathrm{Tot}}(\mathbf{x}) = u^{\mathrm{Inc}}(\mathbf{x}) + u^{\mathrm{Scat}}(\mathbf{x}).$$

This choice ensures that (i) u^{Scat} is Helmholtz solution in Ω_+, (ii) $\gamma^+ u^{\mathrm{Tot}} = 0$ on $\partial\Omega$, and (iii) u^{Scat} is made of outgoing waves only. This is an example of "Mie series".

You can see an example of scattered field computed with this formula in Figure 7 and on the webpage.

Exercise 2.18. Let u^{Inc} be a plane wave with direction $\mathbf{d}$. Compute the field scattered by a disc of radius R using Jacobi–Anger formula $\mathrm{e}^{\mathrm{i}z\cos\alpha} = \sum_{\ell\in\mathbb{Z}} \mathrm{i}^\ell J_\ell(z)\mathrm{e}^{\mathrm{i}\ell\alpha}$.

2.4.2 Sound-soft scattering problems

We have seen how to select "outgoing" waves using the expansion in polar coordinates. How to do the same without this expansion?

The radial dependence of all the outgoing terms in the circular wave approximation (27) is $\frac{1}{\sqrt{r}}\mathrm{e}^{\mathrm{i}kr}$ (ignoring high-order terms). Deriving with respect to the radial direction r we have $\partial_r(\frac{1}{\sqrt{r}}\mathrm{e}^{\mathrm{i}kr}) = \mathrm{i}k\frac{1}{\sqrt{r}}\mathrm{e}^{\mathrm{i}kr} - \frac{1}{2}r^{-3/2}\mathrm{e}^{\mathrm{i}kr}$. So if u^{Scat} is a linear combination of $H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ then it satisfies $\partial_r u^{\mathrm{Scat}} - \mathrm{i}ku^{\mathrm{Scat}} = \mathcal{O}(r^{-3/2})$. On the other hand, the bad terms $H_\ell^{(2)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ satisfy only the condition with the opposite sign $\partial_r u + \mathrm{i}ku = \mathcal{O}(r^{-3/2})$ (recall that $H_\ell^{(2)} = \overline{H_\ell^{(1)}}$). This suggests the following classical definition.

Definition 2.19. Let u be an $H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus B_R)$ solution of the Helmholtz equation in the complement of a ball. We say that u is **radiating**, or **outgoing**, if it satisfies the **Sommerfeld radiation condition**:

$$|\partial_r u - \mathrm{i}ku| = o(r^{-1/2}) \qquad r\to\infty. \tag{28}$$

Sommerfeld condition is meant to hold uniformly in all directions, namely

$$\lim_{r\to\infty}\sup_{\theta\in[0,2\pi]} \sqrt{r}\big|\partial_r u(r,\theta) - \mathrm{i}ku(r,\theta)\big| = 0.$$

Since radiating solutions can be expanded in series of Fourier–Hankel functions, Sommerfeld condition (28) is equivalent (for Helmholtz solutions only) to the apparently stronger condition

$$\exists C, R>0 \quad \text{such that} \quad \big|\partial_r u(r,\theta) - \mathrm{i}ku(r,\theta)\big| \le Cr^{-3/2} \qquad \forall r>R, \quad \theta\in[0,2\pi].$$

Sommerfeld condition also implies that $\lim_{R\to\infty}\int_{\partial B_R}|\partial_r u - \mathrm{i}ku|^2\,\mathrm{d}s = 0$, $\partial B_R = \{\mathbf{x} : |\mathbf{x}| = R\}$. The approximation (27) implies that all linear combinations of $H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ that converge outside of some ball are radiating. On the the other hand, no $H_\ell^{(2)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$ term is allowed in radiating functions.

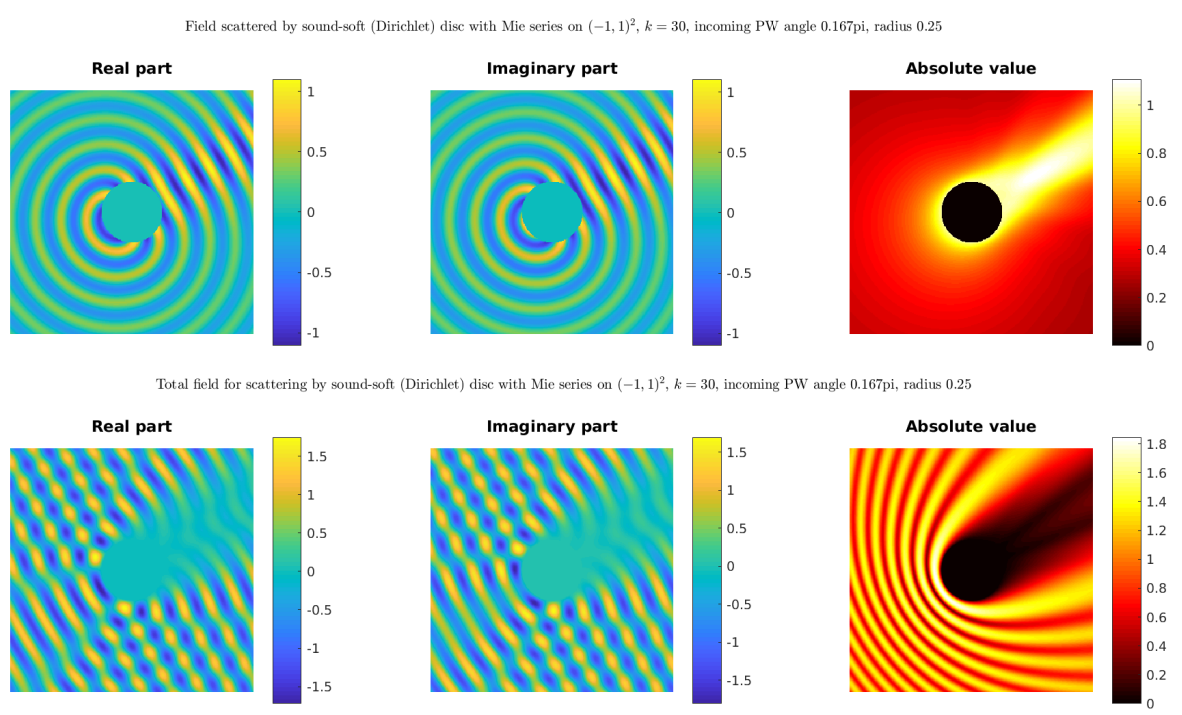


Figure 7: Scattering of a plane wave with direction $\mathbf{d} = (\frac{\sqrt{3}}{2}, \frac{1}{2})$ *by a a sound-soft disc with radius* 0.25 *at* $k = 30$*. Top: scattered field; bottom: total field. We can observe that the* u^{Scat} *field is strongest in the shadow region, and has phase opposite to* u^{Inc}*, so that* u^{Tot} *is minimal there. In* u^{Tot} *we see the complicated pattern produced by the interference between* u^{Scat} *and* u^{Inc}*.*

A more rigorous derivation of the Sommerfeld radiation condition can be done using the "limiting absorption principle": first consider the problem with absorption, i.e. $\Im k > 0$, where the $\mathrm{e}^{\mathrm{i}kr}$ behaviour corresponds to solutions decaying towards infinity, then study the limit for $\Im k \searrow 0$.

We can now define the class of exterior boundary value problems that we will consider in the following.

Definition 2.20 (Exterior Dirichlet problem—EDP)**.** Let Ω_- a bounded Lipschitz domain, $k > 0$ and $g_D \in H^{\frac{1}{2}}(\Gamma)$. We say that $u \in H^1_{\text{loc}}(\overline{\Omega_+})$ satisfies the exterior Helmholtz Dirichlet problem if

$$\begin{aligned} \Delta u + k^2 u &= 0 && \text{in } \Omega_+, \\ \gamma^+ u &= g_D && \text{on } \Gamma, \\ u &\text{ is radiating.} \end{aligned} \tag{29}$$

In the language of scattering theory:

Definition 2.21 (Sound-soft scattering problem—SSSP)**.** Let Ω_- a bounded Lipschitz domain, $k > 0$ and u^{Inc} is a Helmholtz solution in a neighbourhood of Γ. We say that $u^{\text{Scat}} \in H^1_{\text{loc}}(\overline{\Omega_+})$ satisfies the sound-soft scattering problem if

$$\begin{aligned} \Delta u^{\text{Scat}} + k^2 u^{\text{Scat}} &= 0 && \text{in } \Omega_+, \\ \gamma^+(u^{\text{Scat}} + u^{\text{Inc}}) &= 0 && \text{on } \Gamma, \\ u^{\text{Scat}} &\text{ is radiating.} \end{aligned} \tag{30}$$

The sound-soft scattering problem is an exterior Dirichlet problem with $u = u^{\text{Scat}}$ and $g_D = -\gamma^+ u^{\text{Inc}}$.

We will see that problems (29) and (30) are well-posed.

In Definition 2.21 we have assumed that u^{Inc} is defined only in a neighbourhood of the scatterer's boundary. If the incoming wave is a plane wave, then of course it is defined in the whole of $\mathbb{R}^2$, but this definition allows to include more realistic incoming waves such as "point sources", i.e. Fourier–Hankel functions centred at some point of Ω_+.

The EDP and the SSSP are defined and analysed in details in, e.g., [CK2, p. 46]. However, [CK2]

considers the 3D case (so the powers of r in Sommerfeld condition are different) and C^m spaces rather than Sobolev spaces.

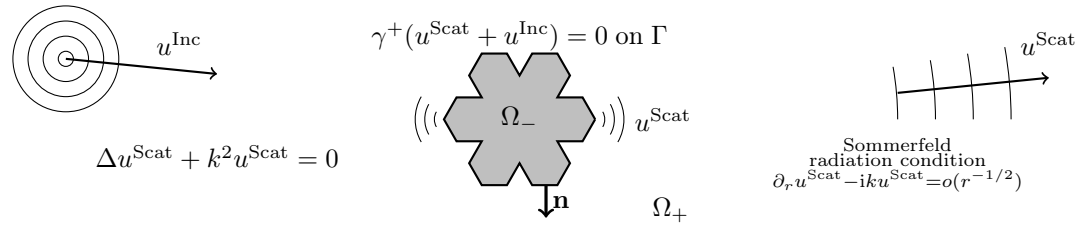


Figure 8: A diagram of the sound-soft scattering problem (30).

Remark 2.22 (Truncated problems)**.** Often one does not want to deal with BVPs posed on unbounded domains such as in Definition 2.21, for example because one wants to approximate the solution with a finite element method. A possibility to reduce this problem to one posed on a bounded domain is to choose a large ball B_R (or a different shape) with $\Omega_- \subset B_R$ and solve a Helmholtz BVP on the truncated domain $B_R \cap \Omega_+$. On Γ we impose the Dirichlet condition as above. On the artificial boundary $\Gamma_R = \partial B_R$ one has to impose some artificial boundary condition that mimics the Sommerfeld radiation condition. The simplest choice is to choose impedance conditions $\partial_r u^{\text{Scat}} - \mathrm{i}k u^{\text{Scat}} = 0$ (compare with (28)). Many more efficient and more complicated boundary conditions exist. They are called absorbing, non-reflecting, radiation, generalised-impedance boundary conditions (ABC, NRBC, GIBC...). The quality of an artificial boundary condition depends on the ability to absorb the waves coming from the domain and to not reflect them back.

Remark 2.23 (Far-field pattern)**.** It is possible to prove (e.g. [CK2, eq. (3.63)]) that if u is a radiating Helmholtz solution, then it satisfies

$$u(\mathbf{x}) = \frac{\mathrm{e}^{\mathrm{i}kr}}{\sqrt{r}}\Big(u_\infty(\theta) + \mathcal{O}(r^{-1})\Big) \qquad \text{for } r = |\mathbf{x}| \to \infty,$$

for a function $u_\infty \in C^\infty(\mathbb{S}^1)$ (recall that $\mathbb{S}^1$ is the unit circle, and that θ denotes the angular polar coordinate of $\mathbf{x}$). This means that, up to factoring out the phase factor $\mathrm{e}^{\mathrm{i}kr}$ and the decay factor $\frac{1}{\sqrt{r}}$, when we move towards infinity along a straight line in the direction θ, a radiating field converges to a given value $u_\infty(\theta)$. The function u_∞ is called *far-field pattern*. If u is defined in Ω_+ and admits Dirichlet and Neumann traces on Γ, the far-field pattern can be computed using the formula

$$u_\infty(\theta) = \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_\Gamma \Big(\gamma^+ u(\mathbf{y}) \partial_{\mathbf{n}}^+ \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} - \partial_{\mathbf{n}}^+ u(\mathbf{y}) \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}}\Big)\, \mathrm{d}s(\mathbf{y}) \qquad \mathbf{d} = (\cos\theta, \sin\theta). \tag{31}$$

The far-field pattern is one of the main quantities of interest in remote-sensing applications, for example to quantify the amount of radiation "backscattered" by an obstacle when it is hit by a wave.

Exercise 2.24. Compute (as a circular harmonic expansion) and plot with Matlab the far-field pattern of the field scattered by a disc hit by a plane wave. Use the Fourier expansion computed in Exercise 2.18. Study how the far-field pattern vary with k, R and the propagation direction of u^{Inc}. Denote $u_\infty(\theta, \xi)$ the far-field for $u^{\text{Inc}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}k(x_1 \cos\xi + x_2 \sin\xi)}$: can you find any symmetry between the two angles? See [CK2, Thm. 3.13].

Remark 2.25 (Direct and inverse scattering)**.** The SSSP is a *direct* scattering problem: we know the incoming wave, we know the obstacle, we want to compute the scattered field. In applications (such as medical imaging, oil retrieval, seismic and atmospheric remote sensing, fault detection in materials, radar and sonar...) it is very important to consider also *inverse* scattering problem: given the scattered field or the far-field (typically from measurements), one wants to compute the obstacle and/or the incoming wave. Inverse problems are ill-posed and much harder than direct ones, both theoretically and computationally. Most of the book [CK2] is devoted to inverse scattering problems.

2.5 Well-posedness of the exterior Dirichlet problem (EDP)

The most common proof of the well-posedness of the EDP (29) relies on properties of BIOs and BIEs, e.g. [CK1, Thm. 3.21] and [CK2, Thm. 3.9]. Here instead we prove well-posedness using a variational formulation on a truncated domain and the "DtN map". However, both proofs rely on the same main tools: Fredholm theory and a "Rellich lemma", which ensures uniqueness.

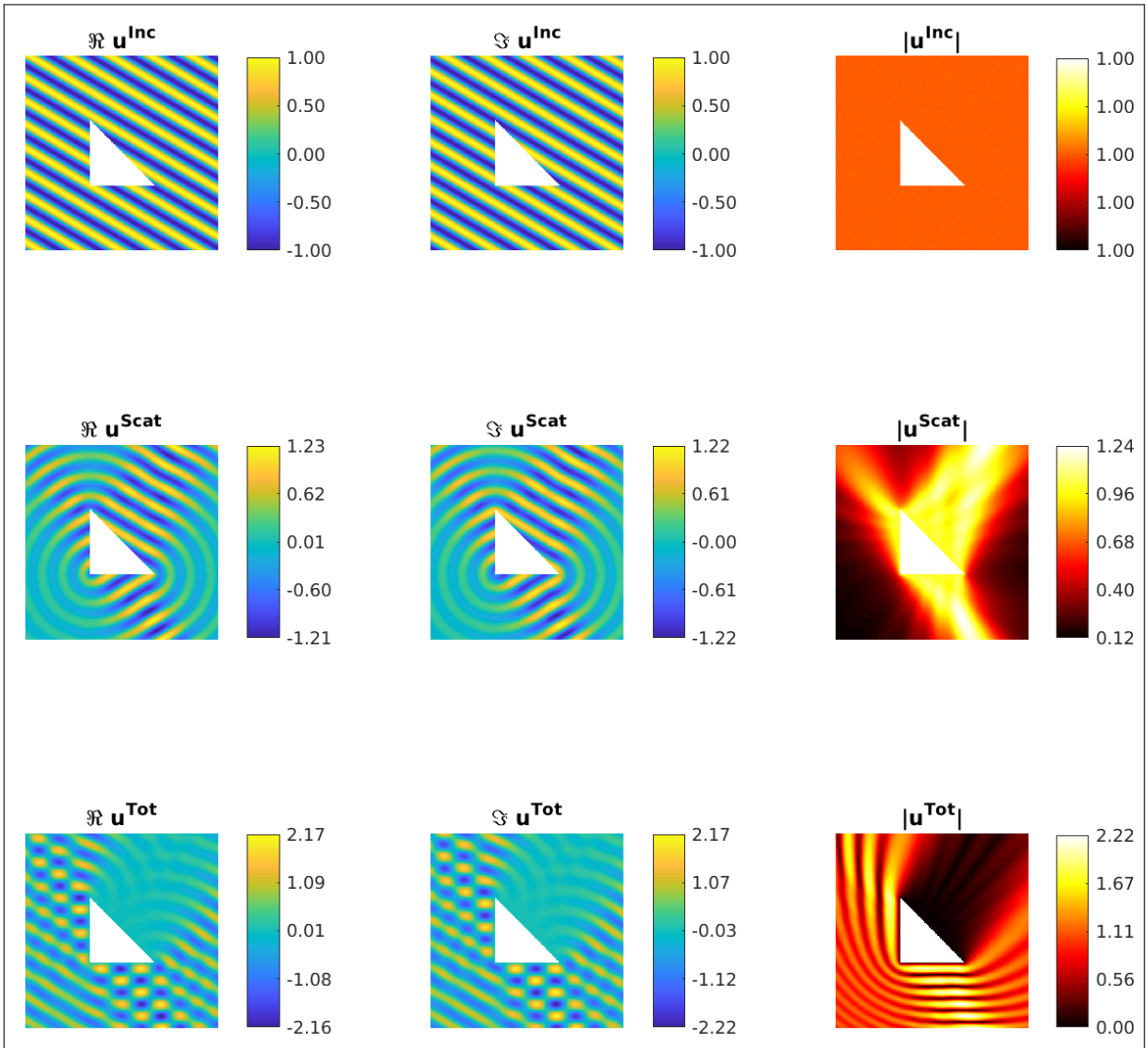


Figure 9: Scattering of a plane wave with direction $\mathbf{d} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$ *by a sound-soft triangle. Here* $k = 20$ *(so* $\lambda = \frac{\pi}{10}$*), the cathetus of the triangle has length 1 and the fields are plotted on a square of side 3.*

2.5.1 DtN map

Let $R > 0$ be the radius of an open ball B_R centred at the origin such that $\Gamma \subset B_R$. We define the operator DtN, which acts on functions defined on ∂B_R by multiplying each terms in their Fourier expansion by the ratio of the radial derivative of the corresponding Fourier–Hankel function and the value of the Fourier–Hankel function itself:

$$\mathsf{DtN}(v) = \mathsf{DtN}\Big(\sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}\Big) = \sum_{\ell\in\mathbb{Z}} T_\ell \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}, \qquad \text{for } T_\ell := \frac{kH_\ell^{(1)\prime}(kR)}{H_\ell^{(1)}(kR)}.$$

This operator is called **Dirichlet-to-Neumann (DtN) map** or **capacity operator** (see [Néd01, (2.6.92)] for the 3D version).

If u is a radiating solution in Ω_+, then in $\mathbb{R}^2 \setminus B_R$ it can[6] be expanded as $u(\mathbf{x}) = \sum_{\ell\in\mathbb{Z}} a_\ell H_\ell^{(1)}(kr)\mathrm{e}^{\mathrm{i}\ell\theta}$. Then its traces on ∂B_R are

$$\gamma^+_{\partial B_R} u = \sum_{\ell\in\mathbb{Z}} a_\ell H_\ell^{(1)}(kR)\mathrm{e}^{\mathrm{i}\ell\theta} \quad \text{and} \quad \partial^+_{\mathbf{n}\,\partial B_R}(u) = \sum_{\ell\in\mathbb{Z}} a_\ell k H_\ell^{(1)\prime}(kR)\mathrm{e}^{\mathrm{i}\ell\theta} \quad \text{thus } \mathsf{DtN}(\gamma^+_{\partial B_R} u) = \partial^+_{\mathbf{n}\,\partial B_R}(u).$$

[6] We haven't proved that *all* radiating solutions can be expanded in Fourier–Hankel series; see [CK1, Thm. 3.6] for the proof in the 3D case.

In words: **the DtN operator maps the Dirichlet trace** (on ∂B_R) **of a radiating solution to its Neumann trace** (on ∂B_R).

From the formulas (https://dlmf.nist.gov/10.6#i and https://dlmf.nist.gov/10.19#E2) for the derivative and the large-index asymptotics of the Hankel functions

$$H_\ell^{(1)\prime}(z) = \frac{H_{\ell-1}^{(1)}(z) - H_{\ell+1}^{(1)}(z)}{2}, \qquad H_\ell^{(1)}(z) \sim -\mathrm{i}\sqrt{\frac{2}{\pi}}\Big(\frac{2}{\mathrm{e}z}\Big)^\ell \ell^{\ell-\frac12} \quad \text{for } \ell \to \infty$$

we have

$$T_\ell = k\frac{H_{\ell-1}^{(1)}(kR) - H_{\ell+1}^{(1)}(kR)}{H_\ell^{(1)}(kR)} \sim k\Big(\frac{\mathrm{e}kR}{2\ell} - \frac{2\ell}{\mathrm{e}kR}\Big) = \mathcal{O}(\ell) \qquad \ell \to \infty.$$

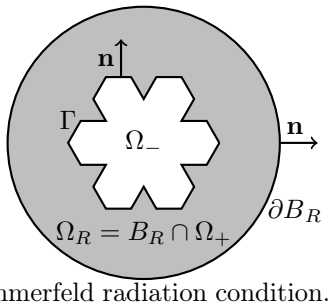
From the definition of the fractional norms (17) on the circle we have that DtN is continuous as an operator $\mathsf{DtN} : H^s(\partial B_R) \to H^{s-1}(\partial B_R)$ for any $s \in \mathbb{R}$: for some $C > 0$,

$$\|\mathsf{DtN} v\|^2_{H^{s-1}(\partial B_R)} = 2\pi \sum_{\ell\in\mathbb{Z}} |\hat v_\ell|^2 \underbrace{|T_\ell|^2}_{\sim \ell^2} (1+\ell^2)^{s-1} \le C \sum_{\ell\in\mathbb{Z}} |\hat v_\ell|^2 (1+\ell^2)^s \le C \|v\|^2_{H^s(\partial B_R)}.$$

2.5.2 Truncated problem

The EDP (29) is equivalent to the following problem on the truncated domain $\Omega_R := B_R \cap \Omega_+$:

$$\begin{aligned} \Delta u + k^2 u &= 0 && \text{in } \Omega_R, \\ \gamma u &= g_D && \text{on } \Gamma, \\ \mathsf{DtN}(\gamma u) - \partial_{\mathbf{n}} u &= 0 && \text{on } \partial B_R. \end{aligned} \tag{32}$$



The last condition on the exterior boundary ∂B_R is equivalent to the Sommerfeld radiation condition. To write this BVP as a variational problem we define the space $H^1_{0,R}(\Omega_R) := \{u \in H^1(\Omega_R) : \ \gamma u = 0 \text{ on } \Gamma\}$ (the H^1 functions whose trace vanishes on the interior boundary Γ but not necessarily on ∂B_R).

From the surjectivity of the trace operator (Theorem 2.6) there exists a lifting $u_D \in H^1(\Omega_R)$ such that $\gamma^+ u_D = g_D$; it is possible to choose $u_D \in H^1(\Omega_R; \Delta)$ (e.g. by solving an auxiliary Laplace BVP). If we can solve the problem

$$\begin{aligned} \Delta u_0 + k^2 u_0 &= -f && \text{in } \Omega_R, \\ \gamma u_0 &= 0 && \text{on } \Gamma, \\ \mathsf{DtN}(\gamma u_0) - \partial_{\mathbf{n}} u_0 &= g_R && \text{on } \partial B_R, \end{aligned} \qquad f := -\Delta u_D - k^2 u_D, \quad g_R := -\mathsf{DtN}(\gamma u_D) + \partial_{\mathbf{n}} u_D,$$

then $u = u_D + u_0$ would solve (32). (We could also choose u_D such that $g_R = 0$.) Using Green's first identity it is easy to deduce a variational problem for u_0:

$$\text{find} \quad u_0 \in H^1_{0,R}(\Omega_R) \quad \text{such that} \quad \mathcal{A}_R(u_0, w) = \mathcal{F}_R(w) \quad \forall w \in H^1_{0,R}(\Omega_R) \quad \text{where} \tag{33}$$

$$\mathcal{A}_R(u_0, w) := \int_{\Omega_R} (\nabla u_0 \nabla \overline{w} - k^2 u_0 \overline{w})\, \mathrm{d}\mathbf{x} - \int_{\partial B_R} (\mathsf{DtN}\gamma u_0)(\gamma \overline{w})\, \mathrm{d}s,$$

$$\mathcal{F}_R(w) := \int_\Omega f \overline{w}\, \mathrm{d}\mathbf{x} - \int_{\partial B_R} g_R \gamma \overline{w}\, \mathrm{d}s.$$

Using the continuity of the trace operator ($\gamma : H^1_{0,R}(\Omega_R) \to H^{\frac12}(\partial B_R)$) and the DtN map ($\mathsf{DtN} : H^{\frac12}(\partial B_R) \to H^{-\frac12}(\partial B_R)$) we deduce the continuity of $\mathcal{A}_R$ and $\mathcal{F}_R$ in $H^1_{0,R}(\Omega_R)$.

2.5.3 Gårding inequality

We first look at the real part of the boundary sesquilinear form associated to DtN: for all $v(\mathbf{x}) = \sum_{\ell\in\mathbb{Z}} \hat v_\ell \mathrm{e}^{\mathrm{i}\ell\theta} \in H^{\frac12}(\partial B_R)$ we have

$$\Re \int_{\partial B_R} (\mathsf{DtN} v)\overline{v}\, \mathrm{d}s(\mathbf{x}) = R\, \Re \int_0^{2\pi} \Big(\sum_{\ell\in\mathbb{Z}} T_\ell \hat v_\ell \mathrm{e}^{\mathrm{i}\ell\theta}\Big)\Big(\sum_{\ell\in\mathbb{Z}} \overline{\hat v_\ell} \mathrm{e}^{-\mathrm{i}\ell\theta}\Big)\, \mathrm{d}\theta = 2\pi R \sum_{\ell\in\mathbb{Z}} |\hat v_\ell|^2 \Re\{T_\ell\}.$$

For any complex-valued differentiable function f of a real variable we have

$$\Re\Big\{\frac{f'(t)}{f(t)}\Big\} = \Re\Big\{\frac{f'(t)\overline{f}(t)}{|f(t)|^2}\Big\} = \frac{f'(t)\overline{f}(t)+\overline{f'}(t)f(t)}{2|f(t)|^2} = \frac{1}{2|f(t)|^2}\frac{\partial\big(f(t)\overline{f}(t)\big)}{\partial t} = \frac{1}{2|f(t)|^2}\frac{\partial(|f(t)|^2)}{\partial t}.$$

Choosing $f(r) = H_\ell^{(1)}(kr)$ we have

$$\Re\{T_\ell\} = \Re\Big\{\frac{f'(R)}{f(R)}\Big\} = \frac{1}{2|H_\ell^{(1)}(kR)|^2}\frac{\partial(|H_\ell^{(1)}(kr)|^2)}{\partial r}\Big|_{r=R} < 0$$

since the absolute value of the Hankel function is monotonically decreasing. Combining all these ingredients we see that the sesquilinear form satisfies a Gårding inequality:

$$\begin{aligned}|\mathcal{A}_R(w,w)| \geq \Re\{\mathcal{A}_R(w,w)\} &= \int_{\Omega_R}(|\nabla w|^2 - k^2|w|^2)\,\mathrm{d}\mathbf{x} - \Re\int_{\partial B_R}(\mathsf{DtN}\gamma w)\gamma\overline{w}\,\mathrm{d}s\\ &= \|\nabla w\|^2_{L^2(\Omega_R)^2} - k^2\,\|w\|^2_{L^2(\Omega_R)} - 2\pi R\sum_{\ell\in\mathbb{Z}}|\hat{w}_\ell|^2\,\underbrace{\Re\{T_\ell\}}_{\leq 0}\\ &\geq \|\nabla w\|^2_{L^2(\Omega_R)^2} - k^2\,\|w\|^2_{L^2(\Omega_R)} = \|w\|^2_{H^1_k(\Omega_R)} - 2k^2\,\|w\|^2_{L^2(\Omega_R)},\end{aligned}$$

where we have expanded $(\gamma w)(\theta) = \sum_{\ell\in\mathbb{Z}}\hat{w}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}$ on ∂B_R. Moreover, $H^1_{0,R}(\Omega_R)$ is compactly embedded in $L^2(\Omega_R)$. From Corollary 2.12 we have that if the homogeneous version of the variational problem (33) (find $u_0 \in H^1_{0,R}(\Omega_R)$ such that $\mathcal{A}_R(u_0,w) = 0$ for all $w \in H^1_{0,R}(\Omega_R)$) admits only the trivial solution $u_0 = 0$, then (33) is well-posed for any right-hand side.

2.5.4 Uniqueness

We first prove the following important result.

Theorem 2.26 (Rellich's lemma, [CK1, Thm. 3.12])**.** If u is a radiating Helmholtz solution in Ω_+ then

$$\Im\int_{\partial B_R}\partial_{\mathbf{n}}u\;\gamma\overline{u}\,\mathrm{d}s \leq 0 \qquad \Rightarrow \qquad u = 0 \quad \text{in } \Omega_+.$$

Proof. As before, we assume that u admits a Fourier–Hankel expansion $u = \sum_{\ell\in\mathbb{Z}}\hat{u}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}H_\ell^{(1)}(kr)$, [CK1, Thm. 3.6]. With this normalisation, on ∂B_R we have

$$\begin{aligned}\Im\int_{\partial B_R}\partial_{\mathbf{n}}u\;\gamma\overline{u}\,\mathrm{d}s &= R\,\Im\int_0^{2\pi}\Big(\sum_{\ell\in\mathbb{Z}}\hat{u}_\ell\mathrm{e}^{\mathrm{i}\ell\theta}kH_\ell^{(1)\prime}(kr)\Big)\Big(\sum_{\ell\in\mathbb{Z}}\overline{\hat{u}_\ell}\mathrm{e}^{-\mathrm{i}\ell\theta}\overline{H_\ell^{(1)}(kr)}\Big)\,\mathrm{d}\theta\\ &= 2\pi R\sum_{\ell\in\mathbb{Z}}|\hat{u}_\ell|^2\Im\{kH_\ell^{(1)\prime}(kR)\overline{H_\ell^{(1)}}(kR)\}\\ &= 2\pi R\sum_{\ell\in\mathbb{Z}}|\hat{u}_\ell|^2 k\Big(Y'(kR)J(kR) - Y(kR)J'(kR)\Big) = 2\pi R\sum_{\ell\in\mathbb{Z}}|\hat{u}_\ell|^2\frac{2}{\pi R} = 4\sum_{\ell\in\mathbb{Z}}|\hat{u}_\ell|^2 \geq 0,\end{aligned}$$

from the Wronskian identity http://dlmf.nist.gov/10.5.E2 . Since all terms in the series are positive, if the series is ≤ 0 then it is 0 and $\hat{u}_\ell = 0$ for all $\ell\in\mathbb{Z}$, so $u = 0$. □

If we choose u_0 to be the solution of the homogeneous variational problem (33) with $\mathcal{F}_R = 0$, then

$$0 = \Im\{\mathcal{F}_R(u_0)\} = \Im\{\mathcal{A}_R(u_0,u_0)\} = -\Im\int_{\partial B_R}(\mathsf{DtN}\gamma u_0)\gamma\overline{u}_0\,\mathrm{d}s = -\Im\int_{\partial B_R}\partial_{\mathbf{n}}u_0\;\gamma\overline{u}_0\,\mathrm{d}s$$

and $u_0 = 0$ by Rellich's lemma 2.26.

We can now complete the proof of the well-posedness. The homogeneous variational problem ((33) with $\mathcal{F}_R = 0$) admits only the trivial solution $u_0 = 0$ (§2.5.4) and its sesquilinear form satifies a Gårding inequality (§2.5.3). By Corollary 2.12 the problem (33) is well-posed for all f and g_R. Thanks to the equivalence with (32), also the EDP (29) and the special case of the SSSP (30) are well-posed.

3 Boundary integral equations and boundary element method

3.1 Single-layer potential, operator and the first boundary integral equation

[7]We define the 2D Helmholtz fundamental solution:

$$\boxed{\Phi_k(\mathbf{x},\mathbf{y}) := \frac{\mathrm{i}}{4}H_0^{(1)}(k|\mathbf{x}-\mathbf{y}|), \qquad \mathbf{x}\neq\mathbf{y}\in\mathbb{R}^2.} \tag{34}$$

For a given point $\mathbf{y}$, this is a radiating Fourier–Hankel function of order 0 centred at $\mathbf{y}$. It is a smooth Helmholtz solution in $\mathbb{R}^2\setminus\{\mathbf{y}\}$ and has a logarithmic singularity at $\mathbf{y}$. The roles of $\mathbf{x}$ and $\mathbf{y}$ are symmetric. It represents the field produced by a point source located in $\mathbf{y}$. The value of $\Phi_k(\mathbf{x},\mathbf{y})$ only depends on the distance $|\mathbf{x}-\mathbf{y}|$ between the arguments (more precisely: it depends on the number of wavelengths contained in that distance: $\frac{|\mathbf{x}-\mathbf{y}|}{\lambda} = \frac{k|\mathbf{x}-\mathbf{y}|}{2\pi}$).

We will see that the normalisation factor $\frac{\mathrm{i}}{4}$ in (34) gives that, for all $\mathbf{y}\in\mathbb{R}^2$, $\Delta\Phi_k(\cdot,\mathbf{y})+k^2\Phi_k(\cdot,\mathbf{y}) = \delta_{\mathbf{y}}$, in the sense of distributions, where $\delta_{\mathbf{y}}$ is the Dirac delta centred at $\mathbf{y}$. Moreover this coefficient will allow to write a simple Green's integral representation.

Any linear combination $\sum_j \psi_j\Phi_k(\cdot,\mathbf{y}_j)$ of fundamental solutions centred at points $\mathbf{y}\in\overline{\Omega_-}$ satifies the Helmholtz equation in Ω_+ and is radiating.[8] We can also take a continuous linear combination of fundamental solutions, which we write as

$$\boxed{(\mathcal{S}\psi)(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \qquad \mathbf{x}\in\Omega_+,} \tag{35}$$

where ψ is a function on Γ. We can think at ψ as the density of acoustic sources generating the field $\mathcal{S}\psi$ (in the case of Laplace equation this is the density of electric charges generating an electrostatic potential). The function $\mathbf{x}\mapsto\Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})$ belongs to $C^\infty(\Omega_+)$ for any given $\mathbf{y}\in\Gamma$. Thus, by the differentiation under integral sign theorem, the function $\mathcal{S}\psi$ belongs to $C^\infty(\Omega_+)$, is radiating and is a solution of the Helmholtz equation. The operator $\mathcal{S}$ is called (acoustic) **single-layer potential** or, sometimes, **simple-layer potential**. It is possible to prove that the single layer potential is continuous as a mapping $\mathcal{S}: H^{-\frac{1}{2}}(\partial\Omega)\to H^1_{\mathrm{loc}}(\Omega_+)$.

This suggests to look for a solution of the EDP (29) in the form $u(\mathbf{x}) = (\mathcal{S}\psi)(\mathbf{x})$ for some "density" ψ. But, how can we find ψ? We need to relate $\mathcal{S}\psi$ to the boundary condition.

We first introduce the **single-layer operator** S:

$$\boxed{(S\psi)(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x},\mathbf{y})\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \qquad \mathbf{x}\in\Gamma.} \tag{36}$$

The only difference between the single-layer potential $\mathcal{S}$ and operator S is that the former is evaluated in points *off* the boundary, and the latter *on* the boundary Γ. When $\psi\in C^0(\Gamma)$, then the evaluation of $(\mathcal{S}\psi)(\mathbf{x})$ is the integral of a continuous function. On the other hand, no matter the regularity of ψ, the evaluation of $(S\psi)(\mathbf{x})$ is a singular integral, because of the singularity of $\Phi(\mathbf{x},\mathbf{y})$ at $\mathbf{x}=\mathbf{y}$. The single-layer operator is a first example of **boundary integral operator (BIO)**, in particular it is a **weakly singular integral operator**, as the singularity of Φ_k is logarithmic.

The Dirichlet trace operator relates $\mathcal{S}$ and S:

$$S\psi = \gamma^+(\mathcal{S}\psi). \tag{37}$$

To justify this (intuitively very plausible) fact when $\psi\in C^0(\Gamma)$, one can fix $\mathbf{x}\in\Gamma$ and a sequence $(\mathbf{x}_j)_{j\in\mathbb{N}}\subset\Omega_+$ with $\mathbf{x}_j\to\mathbf{x}$, define $g(\mathbf{y}) = \Phi(\mathbf{x},\mathbf{y})\psi(\mathbf{y})$ and $g_j(\mathbf{y}) = \Phi(\mathbf{x}_j,\mathbf{y})\psi(\mathbf{y})$ for $\mathbf{y}\in\Gamma$. Clearly $g_j(\mathbf{y})$ converges to $g(\mathbf{y})$ for all $\mathbf{y}\in\Gamma$ except $\mathbf{y}=\mathbf{x}$. Decomposing

$$\int_\Gamma |g_j(\mathbf{y})|\,\mathrm{d}s(\mathbf{y}) \le \int_\Gamma \big|\Phi_k(\mathbf{x}_j,\mathbf{y})-\Phi_k(\mathbf{x},\mathbf{y})\big||\psi(\mathbf{y})|\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma \big|\Phi_k(\mathbf{x},\mathbf{y})\big||\psi(\mathbf{y})|\,\mathrm{d}s(\mathbf{y})$$

[7]This section closely follows [Sayas06, §3]. However we use the notation $\mathcal{S}, S$ of [Spence14], while [Sayas06] uses $\mathcal{S}_\Gamma, V_\Gamma$.

[8]This suggests a numerical method consisting in choosing N points $\mathbf{y}_1,\dots,\mathbf{y}_N\in\Omega_-$ and in searching the coefficients $\psi_1,\dots,\psi_N$ that minimise $\left\|\sum_{j=1}^N \psi_j\Phi_k(\cdot,\mathbf{y}_j) - g_D(\cdot)\right\|_{L^2(\Gamma)}$ (or some other norm on Γ). This is a well-known scheme called the "method of fundamental solutions" (MFS). It can give extremely good accuracy but has some drawbacks: it is very sensitive with respect to the location of the $\mathbf{y}_j$s, the minimisation generally leads to ill-conditioned linear systems, it struggles to approximate solutions for scatterers with corners.

and using the properties of the Hankel function $H_0^{(1)}$ (which appears in the definition of Φ_k) obtain that $\int_\Gamma |g_j(\mathbf{y})|\,\mathrm{d}s(\mathbf{y})$ is bounded uniformly in j, if the points $\mathbf{x}_j$ are sufficiently close to $\mathbf{x}$. By Lebesgue's dominated converge theorem it follows that $\int_\Gamma g_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \to \int_\Gamma g(\mathbf{y})\,\mathrm{d}s(\mathbf{y})$, which means that $(\mathcal{S}\psi)(\mathbf{x}_j) \to (S\psi)(\mathbf{x})$.

If we are able to find ψ on Γ such that

$$\boxed{S\psi = g_D \quad \text{on } \Gamma,} \tag{38}$$

where g_D is the Dirichlet datum of the EDP (29), then

$$\boxed{u = \mathcal{S}\psi \quad \text{in } \Omega_+} \tag{39}$$

is a radiating Helmholtz solution in Ω_+ with $\gamma^+ u = \gamma^+ \mathcal{S}\psi = S\psi = g_D$ by (37), thus u itself is a **solution of the EDP** (29).

Equation (38) is the first example of **boundary integral equation** (BIE) and (39) is the corresponding **representation formula**. The unknown of the BIE is ψ, which is a distribution supported on Γ and does not need to have a physical meaning: for this reason this is called **indirect method**. If we could solve the BIE and compute ψ, then the solution u of the EDP is obtained from the representation formula, which amounts to the computation of an integral on Γ for each point $\mathbf{x} \in \Omega_+$ where we want to evaluate u.

We will see in the following that the BIE (38) is well-posed under some conditions on Γ and k.

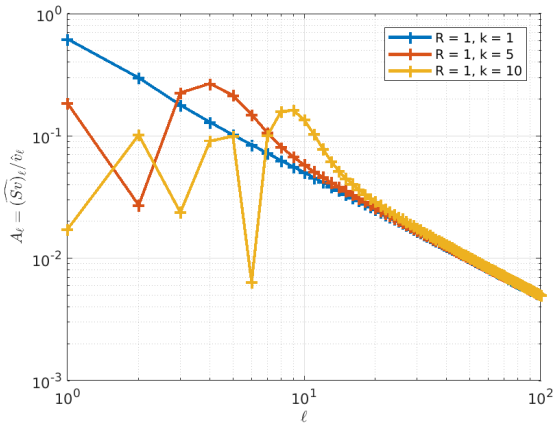


Figure 10: A log-log plot of the multipliers $A_\ell = \frac{\widehat{(Sv)}_\ell}{\hat{v}_\ell}$ *in the circular-harmonic expansion of the single-layer operator* S *on the boundary of a circle. They decay proportionally to* $\frac{1}{\ell}$*, demonstrating the continuity of* $S : H^s(\Gamma) \to H^{s+1}(\Gamma)$*. Each colour correspond to a wavenumber* k*. See Remark 3.1.*

Remark 3.1. From the continuity of the single-layer potential $\mathcal{S} : H^{-\frac{1}{2}}(\Gamma) \to H^1_{\mathrm{loc}}(\overline{\Omega_+})$, the trace formula (37) and the trace theorem 2.6, it follows that the single-layer operator is continuous as a mapping $S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$. We can try to verify this continuity for a circular boundary $\Gamma = \partial B_R = \{\mathbf{x} : |\mathbf{x}| = R\}$ for $R > 0$, where these norms can be computed from Fourier coefficients. Let $v(\theta) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}$ be a function (or distribution) defined on Γ. Then, for $\mathbf{x} = (R\cos\theta, R\sin\theta) \in \Gamma$,

$$\begin{aligned}
(Sv)(\mathbf{x}) &= \int_\Gamma \Phi(\mathbf{x},\mathbf{y}) v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = \frac{\mathrm{i}}{4} R \int_0^{2\pi} H_0^{(1)}(k|(R\cos\theta, R\sin\theta) - (R\cos\alpha, R\sin\alpha)|) \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\alpha}\,\mathrm{d}\alpha \\
&= \frac{\mathrm{i}}{4} R \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\ell\alpha} H_0^{(1)}\left(kR\sqrt{(\cos\alpha - \cos\theta)^2 + (\sin\alpha - \sin\theta)^2}\right)\mathrm{d}\alpha \\
&= \frac{\mathrm{i}}{4} R \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\ell\alpha} H_0^{(1)}\left(kR\sqrt{2(1-\cos(\alpha-\theta))}\right)\mathrm{d}\alpha \\
&= \frac{\mathrm{i}}{4} R \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\ell\alpha} H_0^{(1)}\left(kR\sqrt{2(1-\cos\alpha)}\right)\mathrm{d}\alpha \\
\Rightarrow \qquad & \widehat{(Sv)}_\ell = A_\ell \hat{v}_\ell, \qquad A_\ell := \frac{\mathrm{i}R}{4} \int_0^{2\pi} \mathrm{e}^{\mathrm{i}\ell\alpha} H_0^{(1)}\left(kR\sqrt{2(1-\cos\alpha)}\right)\mathrm{d}\alpha.
\end{aligned}$$

Each Fourier coefficient is multiplied by a factor A_ℓ computed as an integral. The integrand is singular at $\alpha = 0$. In Figure 10 we show the log-log plot of the factors A_ℓ for $0 \le \ell \le 100$, $R = 1$ and different values of k, computed by approximating the integral with a quadrature formula. The coefficients decay as $A_\ell \sim \ell^{-1}$. Recallying the definition of the fractional Sobolev norms (17) on the circle, this gives that $\|Sv\|_{H^{s+1}(\Gamma)} \le C\,\|v\|_{H^s(\Gamma)}$ for all $s \in \mathbb{R}$ and all $v \in H^s(\Gamma)$. This kind of bounds can be proved rigorously for all boundaries and for a range of s that depends on the boundary regularity.

3.2 Piecewise-constant BEM for the single-layer BIE (38)

We have seen that if we were able to find a solution ψ to the BIE (38) then we would have a solution $u = \mathcal{S}\psi$ of the EDP. In general we cannot solve the BIE analytically, thus we resort to a numerical method.

The **boundary element method** (BEM) consists of choosing an N-dimensional space $V_N \subset H^{-\frac{1}{2}}(\Gamma)$ and looking for a $\psi_N \in V_N$ that approximately solves the BIE (38). There are two ways of imposing the BIE.

- **Collocation-BEM**. We choose N points $\mathbf{x}_1, \dots, \mathbf{x}_N$ on Γ and look for
$$\psi_N \in V_N \text{ such that } (S\psi_N)(\mathbf{x}_j) = g_D(\mathbf{x}_j), \quad j = 1, \dots, N.$$
- **Galerkin-BEM**. The Galerkin method requires a sesquilinear form. When we deal with BVPs for PDEs we obtain sesquilinear forms from integration by parts. With BIEs we simply multiply the equation by a test function and integrate on Γ. So the Galerkin-BEM reads: find $\psi_N \in V_N$ such that[9]
$$\mathcal{A}(\psi_N, \xi_N) := \int_\Gamma (S\psi_N)\overline{\xi_N}\,\mathrm{d}s = \int_\Gamma g_D\overline{\xi_N}\,\mathrm{d}s =: \mathcal{F}(\xi_N) \qquad \forall \xi_N \in V_N$$

We recall that $H^{-\frac{1}{2}}(\Gamma)$ is a space larger than $L^2(\Gamma)$, so it accommodates discontinuous functions. This makes the construction of the discrete space V_N simpler. The simplest choice of V_N is the following: we partition the curve Γ in a mesh $\mathcal{T}_N(\Gamma)$ of N (possibly curvilinear) segments $K_1, \dots, K_N \subset \Gamma$ (with $\bigcup_{j=1}^N \overline{K_j} = \Gamma$ and $K_j \cap K_{j'} = \emptyset$ for $j \ne j'$) and choose V_N to be the space of **piecewise constant** functions on $\mathcal{T}_N(\Gamma)$.

The obvious **basis** $\{\varphi_j\}_{j=1}^N$ of V_N is defined by $\varphi_j(\mathbf{x}) = 1$ if $\mathbf{x} \in K_j$ and $\varphi_j(\mathbf{x}) = 0$ if $\mathbf{x} \in \Gamma \setminus K_j$.

We expand the BEM solution in coordinates as $\boxed{\psi_N = \sum_{j=1}^N \Psi_j \varphi_j}$, where $\Psi_j = (\boldsymbol{\Psi})_j$ is the jth element of the vector $\boldsymbol{\Psi} \in \mathbb{C}^n$. So $\psi_N(\mathbf{x}) = \Psi_j$ if $\mathbf{x} \in K_j$.

With this discrete space and basis, in the collocation-BEM we choose the **collocation nodes** $\mathbf{x}_j$ such that $\mathbf{x}_j \in K_j$ for all $j = 1, \dots, N$ and obtain the linear system $\boxed{\underline{\underline{\mathbf{A}}}^C \boldsymbol{\Psi} = \mathbf{F}^C}$, where

$$\boxed{A^C_{j,m} := (S\varphi_m)(\mathbf{x}_j) = \int_\Gamma \Phi_k(\mathbf{x}_j, \mathbf{y})\varphi_m(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = \int_{K_m} \Phi_k(\mathbf{x}_j, \mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad F^C_j := g_D(\mathbf{x}_j).} \tag{40}$$

Indeed, by the linearity of S, if ψ_N is the collocation-BEM solution then its coefficient vector $\boldsymbol{\Psi}$ solves $\underline{\underline{\mathbf{A}}}^C \boldsymbol{\Psi} = \mathbf{F}^C$:

$$F^C_j = g_D(\mathbf{x}_j) = (S\psi_N)(\mathbf{x}_j) = \Big(S \sum_{m=1}^N \Psi_m \varphi_m\Big)(\mathbf{x}_j) = \sum_{m=1}^N \Psi_m (S\varphi_m)(\mathbf{x}_j) = (\underline{\underline{\mathbf{A}}}^C \boldsymbol{\Psi})_j.$$

To be able to evaluate the collocation-BEM right-hand side vector $\mathbf{F}^C$ we need $g_D \in C^0(\Gamma)$, which is typically satisfied if the EDP comes from a scattering problem such as (30).

Similarly, if ψ_N is the Galerkin-BEM solution then its coefficient vector $\boldsymbol{\Psi}$ solves $\boxed{\underline{\underline{\mathbf{A}}}^G \boldsymbol{\Psi} = \mathbf{F}^G}$ with

$$\begin{aligned} A^G_{j,m} :&= \mathcal{A}(\varphi_m, \varphi_j) = \langle S\varphi_m, \varphi_j\rangle_\Gamma = \int_\Gamma (S\varphi_m)(\mathbf{x})\overline{\varphi_j}(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_\Gamma \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y})\varphi_m(\mathbf{y})\overline{\varphi_j}(\mathbf{x})\,\mathrm{d}s(\mathbf{y})\,\mathrm{d}s(\mathbf{x}) \\ &= \int_{K_j}\int_{K_m} \Phi_k(\mathbf{x}, \mathbf{y})\,\mathrm{d}s(\mathbf{y})\,\mathrm{d}s(\mathbf{x}), \\ F^G_j :&= \mathcal{F}(\varphi_j) = \langle g_D, \varphi_j\rangle_\Gamma = \int_\Gamma g_D(\mathbf{x})\overline{\varphi_j}(\mathbf{x})\,\mathrm{d}s(\mathbf{x}) = \int_{K_j} g_D(\mathbf{x})\,\mathrm{d}s(\mathbf{x}). \end{aligned} \tag{41}$$

[9] To be more precise, in general we should use duality products in place of integrals: $\mathcal{A}(\psi_N, \xi_N) := \langle S\psi_N, \xi_N\rangle_\Gamma = \langle g_D, \xi_N\rangle_\Gamma =: \mathcal{F}(\xi_N)\ \forall \xi_N \in V_N$. The duality $\langle S\psi_N, \xi_N\rangle_\Gamma$ makes sense because $\psi_N, \xi_N \in V_N \subset H^{-\frac{1}{2}}(\Gamma)$ so $S\psi_N \in H^{\frac{1}{2}}(\Gamma)$. However, since in all concrete cases we choose $V_N \subset L^2(\Gamma)$, the expression with the boundary integrals is correct.

Multiplying each row of $\underline{\underline{\mathbf{A}}}^C$ and $\mathbf{F}^C$ by the length of the corresponding element, we see that the collocation-BEM is a Galerkin-BEM with a simple 1-point quadrature. Formally, one can also think at the collocation method as a Petrov–Galerkin method with delta functions as test functions, but this is not mathematically correct in the $H^{\pm\frac{1}{2}}(\Gamma)$ framework because delta functions do not belong to $H^{-\frac{1}{2}}(\Gamma)$.

Exercise 3.2. Show that the Galerkin-BEM with piecewise-constant functions is equivalent to impose that the *integral averages* of $S\psi_N$ and g_D coincide on each element.

Both matrices $\underline{\underline{\mathbf{A}}}^C$ and $\underline{\underline{\mathbf{A}}}^G$ are **dense**: this is a major difference between the BEM and the finite element method (FEM). The Galerkin matrix is also complex-symmetric, but not Hermitian.

This shortcoming of the BEM with respect to the FEM is compensated by a dimensional reduction: to solve a 2D problem we only need to mesh a 1D object, the boundary Γ. Thus typically BEM requires much fewer degrees of freedom (DOFs) than FEM for comparable problems and accuracies.

Another advantage of BEM is that it deals with a BVP posed on the unbounded domain Ω_+ by discretising only a bounded object, Γ. To treat the EDP with FEM one has to truncate Ω_+ as in Remark 2.22, introducing additional errors.

The collocation-BEM is clearly simpler to implement than the Galerkin-BEM. However, in many situations the choice of the collocation nodes adversely affects the performance of the method. Moreover, the stability and convergence theory for the Galerkin-BEM is much more complete.

3.2.1 BEM and quadrature

From (40) and (41) we see that to compute each entry of the system matrix we need to compute an **integral of the fundamental solution**: it is a single integral on a mesh element for the collocation-BEM and a double integral on the Cartesian product of two elements for the Galerkin-BEM. To compute each entry of the right-hand side vectors, in the collocation-BEM we only need to evaluate the boundary datum g_D while for the Galerkin-BEM we need an integral over an element.

All these integrals require accurate **quadrature** formulas: these are among the main difficulties in a BEM implementation. In particular, for both matrices, the **diagonal entries require the approximation of singular integrals**, because of the (logarithmic) singularity of $\Phi_k(\mathbf{x},\mathbf{y})$ at $\mathbf{x}=\mathbf{y}$.

Let us assume that Ω_- is a connected **polygon** and each mesh element is a **straight segment**. For $j=1,\dots,N$, the element K_j has endpoints $\mathbf{p}_j$ and $\mathbf{p}_{j+1}$ and length $h_j := |\mathbf{p}_{j+1}-\mathbf{p}_j|$ (of course $\mathbf{p}_{N+1}=\mathbf{p}_1$). The element is parametrised by $\mathbf{X}_j : (0,h_j) \to K_j$, $\mathbf{X}_j(s) := \mathbf{p}_j + s\boldsymbol{\tau}_j$, where $\boldsymbol{\tau}_j := \frac{\mathbf{p}_{j+1}-\mathbf{p}_j}{|\mathbf{p}_{j+1}-\mathbf{p}_j|}$ is the unit tangent vector to K_j. Recalling the definition of the fundamental solution (34), the entries of the BEM matrices and vectors are then computed as integrals over intervals and rectangles:

$$A^C_{j,m} = \frac{\mathrm{i}}{4}\int_0^{h_m} H_0^{(1)}(k|\mathbf{p}_m + s\boldsymbol{\tau}_m - \mathbf{x}_j|)\,\mathrm{d}s, \qquad F^C_j = g_D(\mathbf{x}_j), \qquad F^G_j = \int_0^{h_j} g_D(\mathbf{p}_j + s\boldsymbol{\tau}_j)\,\mathrm{d}s,$$

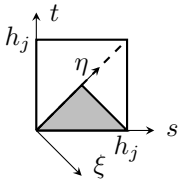
$$A^G_{j,m} = \int_0^{h_j}\left(\int_0^{h_m} \Phi_k(\mathbf{p}_j + t\boldsymbol{\tau}_j, \mathbf{p}_m + s\boldsymbol{\tau}_m)\,\mathrm{d}s\right)\mathrm{d}t = \frac{\mathrm{i}}{4}\int_0^{h_j}\left(\int_0^{h_m} H_0^{(1)}\big(k|\mathbf{p}_j - \mathbf{p}_m + t\boldsymbol{\tau}_j - s\boldsymbol{\tau}_m|\big)\,\mathrm{d}s\right)\mathrm{d}t.$$

The jth diagonal entry of the collocation-BEM matrix is a singular integral, as $\mathbf{x}_j \in K_j$. A simple recipe to compute $A^C_{j,j}$ is to split the element in the two components of $K_j \setminus \{\mathbf{x}_j\}$ and apply Gauss quadrature on each side. The obvious choice for the collocation nodes is to take the element midpoints: $\mathbf{x}_j = \frac{1}{2}(\mathbf{p}_j + \mathbf{p}_{j+1})$.

The jth diagonal entry of the Galerkin-BEM matrix is a double integral on the square $\{0<s,t<h_j\}$, whose integrand has a singularity along the diagonal $s=t$. One can use a quadrature formula for triangles (e.g. based on Duffy transform) on each half of the square. Alternatively, splitting the square in four triangle, exploiting the symmetries, and using the isometric change of variables $\xi = \frac{s-t}{\sqrt{2}}$, $\eta = \frac{s+t}{\sqrt{2}}$, we obtain

$$A^G_{j,j} = \int_0^{h_j}\int_0^{h_j} \frac{\mathrm{i}}{4}H_0^{(1)}\big(k|s-t|\big)\,\mathrm{d}s\,\mathrm{d}t = 4\int_0^{\frac{h_j}{\sqrt{2}}}\left(\int_\xi^{\frac{h_j}{\sqrt{2}}} \frac{\mathrm{i}}{4}H_0^{(1)}(\sqrt{2}k\xi)\,\mathrm{d}\eta\right)\mathrm{d}\xi$$

$$= \int_0^{\frac{h_j}{\sqrt{2}}} \mathrm{i}\Big(\frac{h_j}{\sqrt{2}} - \xi\Big)H_0^{(1)}(\sqrt{2}k\xi)\,\mathrm{d}\xi.$$



This is a one-dimensional integral with a weak singularity at one extreme.

If all elements are identical, e.g. straight segments of the same length, then also the diagonal terms are identical and they need to be computed only once.

In the Galerkin-BEM, also when two element share an endpoint we have a singularity. For example, if the segments K_j and K_{j+1} are aligned ($\boldsymbol{\tau}_j = \boldsymbol{\tau}_{j+1}$), we have

$$A^G_{j,j+1} = \int_0^{h_j} \left(\int_0^{h_{j+1}} \frac{\mathrm{i}}{4} H_0^{(1)}\big(k|h_j - s + t|\big)\,\mathrm{d}t \right) \mathrm{d}s.$$

This is a double integral on the rectangle $\{0 < s < h_j, 0 < t < h_{j+1}\}$ with a singularity at the vertex $s = h_j$, $t = 0$.

Exercise 3.3. Write the entries of the collocation- and Galerkin-BEM when Ω_- is not a polygon but a general Lipschitz domain and its boundary is defined by a parametrisation $\mathbf{X} : [0, L] \to \Gamma$.

Remark 3.4 (Singularity extraction quadrature)**.** A typical technique to compute the singular integrals is the "singularity extraction". The small-argument asymptotics of the Hankel function ($H_0^{(1)}(z) \sim \frac{2\mathrm{i}}{\pi} \log z$ for $z \searrow 0$) give $\Phi_k(\mathbf{x}, \mathbf{y}) = -\frac{1}{2\pi} \log|\mathbf{x} - \mathbf{y}| + \mathcal{R}(\mathbf{x}, \mathbf{y})$ for a reminder $\mathcal{R}$ of class C^1. Inserting this in the expression of $A^{C/G}_{j,j}$ one can compute analytically the terms coming from the log and use a standard quadrature for the remainder $\mathcal{R}$.

Once we have assembled and solved the BEM linear system we have obtained an approximation $\psi_N \in V_N$ of the solution $\psi \in H^{-\frac{1}{2}}(\Gamma)$ of the BIE (38). However to approximate the solution of the EDP (29)/SSSP (30), we need to approximate u in the unbounded domain Ω_+. Recalling the representation formula $u = \mathcal{S}\psi$ (39), the BEM approximation of u is

$$u_N(\mathbf{x}) = (\mathcal{S}\psi_N)(\mathbf{x}) = \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y}) \psi_N(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = \sum_{j=1}^N \Psi_j \int_{K_j} \Phi_k(\mathbf{x}, \mathbf{y})\,\mathrm{d}s(\mathbf{y}) \qquad \mathbf{x} \in \Omega_+.$$

Again, each term in this sum is an integral that needs to be approximated with a quadrature formula. For all $\mathbf{x} \in \Omega_+$ the integrand is C^∞, however if $\mathbf{x}$ lies very close to Γ the accurate evaluation of $u_N(\mathbf{x})$ requires a careful use of the quadrature as the integral is near-singular.

Remark 3.5. Another difficulty is the **oscillatory behaviour** of both Φ_k and the solution for large values of the wavenumber k.

A first issue is that to approximate the solution one needs to use more DOFs for larger values of k. A typical recipe for "engineering accuracy" (a few percent relative errors) is to use at least 10 DOFs per wavelength λ. This means that the length of each element should not exceed $\frac{\lambda}{10} = \frac{\pi}{5k}$. (Sometimes the rule of thumb is to use 6 DOFs per wavelength, corresponding roughly to $kh_j \le 1$, for 10%/15% error.) This implies that the number of DOFs must grow like $N = \mathcal{O}(k)$ for increasing k: high-frequency problem are computationally very expensive. On the other hand, a 2D FEM needs at least $\mathcal{O}(k^2)$ DOFs as the wavelength has to be resolved in two dimensions.

The fundamental solution oscillates with wavelength close to $\lambda = \frac{2\pi}{k}$. Thus, even for elements K_j, K_m far from each other, in the assembly of the matrices $\underline{\underline{\mathbf{A}}}^{C/G}$ one has to use a sufficiently accurate quadrature formula to take into account the oscillations.

Obvious improvements of the piecewise-constant BEM use piecewise-polynomial discrete spaces of higher order. They can be discontinuous, or $C^0(\Gamma)$ or of higher continuity (at least away from corners of Γ) such as splines. The choice of basis functions, collocation points and quadrature rules is in general non-trivial. When Γ is smooth one can use also global functions such as mapped trigonometric functions, in the spirit of spectral methods. Close to the corners of Γ, if it is not smooth, the solution has a singularity: to approximate it efficiently one can use a graded mesh, i.e. whose elements are smaller the closer they are to a corner.

3.2.2 BEM coding project

Implement the collocation-BEM method for the scattering of a plane wave by a polygon Ω_-. Use a discrete space V_N of piecewise-constant functions. Choose a mesh such that the elements on a given side of Γ have equal length. Plot the scattered field u_N and the total field on a portion of Ω_+. You can use the quadrature routine provided.

A possible suggestion for the main steps in the code:

1. Prepare the geometric data structure. Given the vertices of the polygon, decide the number of elements on each side, and generate (for each element K_j) the endpoint $\mathbf{p}_j$, the length h_j, the tangent vector

$\boldsymbol{\tau}_j$, and the collocation point $\mathbf{x}_j$, chosen as the element midpoint. (In Matlab it might be convenient to treat points and vectors in the plane as complex numbers with the usual identification $\mathbb{C} \sim \mathbb{R}^2$; then the `abs` function allows for immediate computation of distances.)

2. Assemble the matrix $\underline{\underline{\mathbf{A}}}^C$ (being careful with the quadrature) and the right-hand side $\mathbf{F}^C$.
3. Solve the linear system.
4. Evaluate the near-field u_N on a grid of points in 2D using the representation formula and plot it.

   To generate the grid of points for the plot use `meshgrid`. To plot the field you can use one of the Matlab commands `pcolor`, `surf`, `mesh` or `contour`. To hide the grid points that lie inside Ω_- you can locate them with `inpolygon` and set them to 0 or `NaN` (or use the command `patch`).

Plenty of interesting extensions are possible:

- Choose as incident wave u^{Inc} a fundamental solution centred in some $\mathbf{x}_0 \in \Omega_+$. (The total field cannot be plotted close to $\mathbf{x}_0$.) You can also try more exotic incident waves, such as Herglotz functions.
- Implement the problem of scattering by multiple polygons, i.e. Ω_- made of several components.
- Implement the Galerkin-BEM for the same problem, on the same mesh. The only difference is in the assembly of the matrix and the right-hand side. Compare the solutions obtained with the two versions of the scheme.
- Generate a time-harmonic animation of the scattered and total fields (use the file provided).
- Implement the singularity extraction quadrature of Remark 3.4.
- Use meshes refined locally towards the corners of Γ.
- Study the dependence of the results on the parameters. You can see how the plots and the norms of the solution vary when you change the number N of DOFs, the wavenumber k, the quality of the quadrature, the shape of Ω_-...

  You can plot the convergence of the error committed by the scheme against N. You can measure the error e.g. as $\|\psi_N - \psi_{ref}\|_{L^2(\Gamma)}$ or $\|u_N - u_{ref}\|_{L^2(\Omega_*)}$ where Ω_* is the portion of Ω_+ where you plot the near-field and ψ_{ref}/u_{ref} are the reference solutions obtained with the finest mesh.

  You can also test the code (for a square scatterer) by computing the near-field error against the reference solution provided. This was computed with MPSpack `https://github.com/ahbarnett/mpspack`.

  How does the condition number depends on N?
- Extend the code to curvilinear polygons and/or smooth scatterers.
- Implement a spectral BEM on a smooth curvilinear scatterer (hard!).

  Choose a smooth scatterer defined by a parametrisation $\mathbf{X} : [0, 2\pi] \to \Gamma$. E.g. a shape often used as example for scattering problems is the kite $\mathbf{X}(t) = (\cos t + 0.65(\cos 2t - 1),\ 1.5 \sin t)$, [CK2, p. 71]. For $L \in \mathbb{N}$ and $N = 2L + 1$, choose as basis functions the mapped complex exponentials $\varphi_\ell(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\ell\mathbf{X}^{-1}(\mathbf{x})}$, $-L \leq \ell \leq L$, or the corresponding trigonometric functions.
- ...

3.3 Green's integral representation

The Green's integral representation, or Green's third identity, is an important tool to derive new BIEs and BEMs, and to understand the properties of the corresponding BVPs. We write it and prove it for bounded and unbounded Lipschitz domains.

We keep using the notation introduced in §2.4. In particular, we recall that the unit normal $\mathbf{n}$ on the boundary of the bounded domain Ω_- is defined to point outwards, into the complement Ω_+; this enters the definition of the Neumann trace $\partial_{\mathbf{n}}$. Since the fundamental solution Φ_k depends on two variables, we write $\frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}$ and $\frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}$ to make clear which is the variable with respect to which we derive and take the trace. We recall that when we write $\int_\Gamma \varphi\overline{\psi}\,\mathrm{d}s$ for $\varphi \in H^{-\frac{1}{2}}(\partial\Omega)$ and $\psi \in H^{\frac{1}{2}}(\partial\Omega)$ (or vice versa) we mean the duality product $\langle \varphi, \psi \rangle_\Gamma$.

Theorem 3.6 (Green's representation in Ω_-)**.** Let Ω_- be a bounded Lipschitz domain and $u \in H^1(\Omega_-; \Delta) \cap C^2(\Omega_-)$ be a Helmholtz solution in Ω_-. Then:

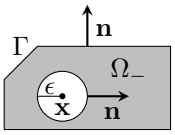
$$\int_\Gamma \left(\partial_{\mathbf{n}}^- u(\mathbf{y}) \Phi_k(\mathbf{x}, \mathbf{y}) - \gamma^- u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}) = \begin{cases} u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_-, \\ 0 & \text{if } \mathbf{x} \in \Omega_+. \end{cases} \tag{42}$$

Proof. If $\mathbf{x} \in \Omega_+$, then both u and Φ_k are Helmholtz solution in Ω_- thus the volume integral in Green's second identity (19) vanishes and what is left is (42).

If $\mathbf{x} \in \Omega_-$ we take $\epsilon > 0$ such that the ball $B_\epsilon(\mathbf{x}) := \{\mathbf{z} \in \mathbb{R}^2 : |\mathbf{z} - \mathbf{x}| < \epsilon\} \subset \Omega_-$. Applying again Green's second identity in $\Omega_- \setminus \overline{B_\epsilon(\mathbf{x})}$ we have

$$\int_\Gamma \left(\partial_\mathbf{n}^- u(\mathbf{y}) \Phi_k(\mathbf{x},\mathbf{y}) - \gamma^- u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}) = \int_{\partial B_\epsilon(\mathbf{x})} \left(\partial_\mathbf{n} u(\mathbf{y}) \Phi_k(\mathbf{x},\mathbf{y}) - \gamma u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}),$$

where $\mathbf{n}$ points out of the ball (this is why we do not have a minus sign in front). We now want to take the limit for $\epsilon \searrow 0$ of the right-hand side, and see that the first term vanish, while the second converges to $u(\mathbf{x})$. The advantage of using a ball centred at $\mathbf{x}$, is that on its boundary the value of $\Phi_k(\mathbf{x},\cdot)$ and its normal derivative are constant.



To this purpose, we need some properties of Hankel functions: the formula for the first derivative of $H_0^{(1)}$ and the asymptotics for small arguments (from https://dlmf.nist.gov/10.7):

$$\frac{\partial}{\partial z} H_0^{(1)}(z) = -H_1^{(1)}(z), \qquad H_0^{(1)}(z) \sim \frac{2\mathrm{i}}{\pi} \log z, \qquad H_\ell^{(1)}(z) \sim -\frac{\mathrm{i}}{\pi} (\ell - 1)! \frac{2^\ell}{z^\ell}, \qquad \ell \in \mathbb{N}, \qquad z \to 0. \tag{43}$$

(Here $a(z) \sim b(z)$ for $z \to 0$ means that $\lim_{z \searrow 0} \frac{a(z)}{b(z)} = 1$.)

We take the limit $\lim_{\epsilon \to 0}$ of the first term using the expression of the fundamental solution, the divergence theorem, $\operatorname{div} \nabla = \Delta$, $\Delta u = -k^2 u$, the asymptotics (43), the mean value theorem (in the form $\lim_{\epsilon \to 0} \frac{1}{\pi \epsilon^2} \int_{B_\epsilon(\mathbf{x})} f(\mathbf{y}) \, \mathrm{d}\mathbf{y} = f(\mathbf{x})$), the boundedness of u in $\mathbf{x}$:

$$\begin{aligned}
\int_{\partial B_\epsilon(\mathbf{x})} \partial_\mathbf{n} u(\mathbf{y}) \Phi_k(\mathbf{x},\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) &= \frac{\mathrm{i}}{4} H_0^{(1)}(k\epsilon) \int_{\partial B_\epsilon(\mathbf{x})} \partial_\mathbf{n} u(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \\
&= \frac{\mathrm{i}}{4} H_0^{(1)}(k\epsilon) \int_{B_\epsilon(\mathbf{x})} \Delta u(\mathbf{y}) \, \mathrm{d}\mathbf{y} \\
&= -\frac{\mathrm{i}}{4} H_0^{(1)}(k\epsilon) k^2 \int_{B_\epsilon(\mathbf{x})} u(\mathbf{y}) \, \mathrm{d}\mathbf{y} \sim -\frac{\mathrm{i}}{4} \Big(\frac{2\mathrm{i}}{\pi} \log k\epsilon \Big) k^2 \pi \epsilon^2 u(\mathbf{x}) \xrightarrow{\epsilon \searrow 0} 0.
\end{aligned}$$

The symbol "$\sim$" stands for "have the same limit for $\epsilon \to 0$". Using the expression of the radial derivative of Φ_k, the asymptotics (43), the mean value theorem ($\lim_{\epsilon \to 0} \frac{1}{2\pi\epsilon} \int_{\partial B_\epsilon(\mathbf{x})} f(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) = f(\mathbf{x})$) we get

$$\begin{aligned}
-\int_{\partial B_\epsilon(\mathbf{x})} u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})} \, \mathrm{d}s(\mathbf{y}) &= k \frac{\mathrm{i}}{4} \, H_1^{(1)}(k\epsilon) \int_{\partial B_\epsilon(\mathbf{x})} u(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \\
&\sim k \frac{\mathrm{i}}{4} \Big(\frac{-2\mathrm{i}}{\pi k \epsilon} \Big) \int_{\partial B_\epsilon(\mathbf{x})} u(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \sim \Big(\frac{1}{2\pi\epsilon} \Big) 2\pi\epsilon u(\mathbf{x}) = u(\mathbf{x}).
\end{aligned}$$

(This also explains why we chose the coefficient $\frac{\mathrm{i}}{4}$ in the definition (34) of Φ_k: it allows to write a simple Green's representation.) □

From Green's representation it follows immediately that:

Corollary 3.7. If u is a Helmholtz solution in Ω_- and $\gamma^- u = \partial_\mathbf{n}^- u = 0$ then $u = 0$.
In particular, $u \neq 0$ is not simultaneously Dirichlet and Neumann eigenfunction in Ω_- for the Laplacian.

We have seen at the end of §2.3 that the solution of the impedance BVP (25) in Ω_- satifies a Fredholm-type variational problem. We have also seen that the solution u_0 of the homogeneous impedance BVP has zero traces: $\gamma^- u_0 = \partial_\mathbf{n}^- u_0 = 0$. From Green's representation (42), $u_0 = 0$, so the operator associated to the sesquilinear form $\mathcal{A}_I(\cdot,\cdot)$ is injective. Since the operator is Fredholm, by Theorem 2.9 it is also invertible and the following corollary holds.

Corollary 3.8 (Well-posedness of interior impedance BVP)**.** The impedance BVP (25) (equivalently, (26)) in Ω_- is well-posed.

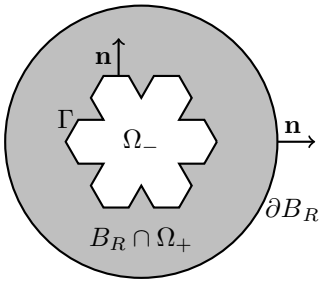
The Fredholm alternative implies that the solution of the impedance BVP (25) exists and is unique, and also that its $H_k^1(\Omega_-)$ norm is controlled by the norm of the data (f and g_I). However, the bounding constant is not explicit as in the situations where we can use Lax–Milgram theorem.

Exercise 3.9. Let $\Omega_R = B_R \setminus \overline{\Omega_-}$, where B_R is an open ball containing $\overline{\Omega_-}$. Write a truncation of the EDP (29) to Ω_R, as described in Remark 2.22 imposing impedance boundary conditions on ∂B_R and sound-soft conditions on Γ. Show that the BVP obtained is well-posed.

Remark 3.10. If u in (42) were not Helmholtz solution we would need to add the left-hand side of Green's representation the volume integral term $-\int_{\Omega_-}\big(\Delta u(\mathbf{y})+k^2u(\mathbf{y})\big)\Phi_k(\mathbf{x},\mathbf{y})\,\mathrm{d}\mathbf{y}$. When $\mathbf{x}\in\Omega_-$ the integrand is weakly singular, see (43), and the integral is well-defined.

Remark 3.11. In the 3-dimensional case all the arguments are similar. The fundamental solution has the simpler expression $\Phi_k(\mathbf{x},\mathbf{y})=\frac{\mathrm{e}^{\mathrm{i}k|\mathbf{x}-\mathbf{y}|}}{4\pi|\mathbf{x}-\mathbf{y}|}$, which does not involve Bessel and Hankel function, so checking the limits for $\epsilon\searrow 0$ is simpler.

To write a Green's representation formula for unbounded domains, we denote by B_R the ball of radius $R>0$ centred at $\mathbf{0}$ and fix $\mathbf{n}=\frac{\mathbf{x}}{r}$ on ∂B_R. We will work in the bounded region $B_R\cap\Omega_+$, for sufficiently large R and use Sommerfeld condition to take the limit $R\to\infty$. The unit normal $\mathbf{n}$ points into $B_R\cap\Omega_+$ on the inner boundary Γ and out of $B_R\cap\Omega_+$ on the outer boundary ∂B_R.



Lemma 3.12. Let Ω_- be a bounded Lipschitz domain, $\Gamma=\partial\Omega_-$, $\Omega_+=\mathbb{R}^2\setminus\overline{\Omega_-}$, and $u,w\in H^1_{\mathrm{loc}}(\overline{\Omega_+};\Delta)\cap C^2(\Omega_+)$ be two *radiating* Helmholtz solutions in Ω_+. Then:

$$\lim_{R\to\infty}\int_{\partial B_R}|u|^2\,\mathrm{d}s<\infty,\qquad \Im\int_\Gamma\partial_{\mathbf{n}}u\,\gamma\overline{u}\,\mathrm{d}s\ge 0\qquad \lim_{R\to\infty}\int_{\partial B_R}(\partial_{\mathbf{n}}u\,w-u\partial_{\mathbf{n}}w)\,\mathrm{d}s=0. \quad (44)$$

Proof. We first prove the boundedness of the limit of $\|u\|_{L^2(\partial B_R)}$. The imaginary part of Green's first identity (18) with $w=\overline{u}$ in $B_R\cap\Omega_+$ gives

$$\begin{aligned}\Im\int_\Gamma\partial_{\mathbf{n}}u\,\gamma\overline{u}\,\mathrm{d}s&=\Im\int_{\partial B_R}\partial_{\mathbf{n}}u\,\gamma\overline{u}\,\mathrm{d}s+\underbrace{\Im\int_{B_R\cap\Omega_+}(k^2|u|^2-|\nabla u|^2)\,\mathrm{d}\mathbf{x}}_{=0,\text{ imaginary part of real value}}\\&=\frac{1}{2k}\int_{\partial B_R}\big(k^2|u|^2+|\partial_{\mathbf{n}}u|^2-|\partial_{\mathbf{n}}u-\mathrm{i}ku|^2\big)\,\mathrm{d}s,\end{aligned}$$

where we have used the identity $|a-\mathrm{i}b|^2=|a|^2+|b|^2-2\Re\{a\,\overline{\mathrm{i}b}\}=|a|^2+|b|^2-2\Im\{a\bar{b}\}$, which holds for all $a,b\in\mathbb{C}$, applied to $a=\partial_{\mathbf{n}}u$, $b=ku$. Taking the limit for $R\to\infty$, the term $\int_{\partial B_R}|\partial_{\mathbf{n}}u-\mathrm{i}ku|^2\,\mathrm{d}s\to 0$, by the Sommerfeld condition (28). The left-hand side is independent of R, thus

$$\lim_{R\to\infty}\frac{1}{2k}\Big(k^2\,\|u\|^2_{L^2(\partial B_R)}+\|\partial_{\mathbf{n}}u\|^2_{L^2(\partial B_R)}\Big)=\Im\int_\Gamma\partial_{\mathbf{n}}u\,\gamma\overline{u}\,\mathrm{d}s<\infty.$$

Since the norms are non-negative and the limit is finite, each of them is bounded, which is the desired inequality. In particular $u=\mathcal{O}(r^{-1/2})$ for $r\to\infty$. Moreover the left-hand side is non-negative, so also the second inequality is proved.

Sommerfeld condition, together with $u,w=\mathcal{O}(r^{-1/2})$, gives the identity involving w:

$$\begin{aligned}\int_{\partial B_R}(\partial_{\mathbf{n}}u\,w-w\partial_{\mathbf{n}}u)\,\mathrm{d}s&=\int_{\partial B_R}\Big(\big(\mathrm{i}ku+o(R^{-1/2})\big)\,w-u\big(\mathrm{i}kw+o(R^{-1/2})\big)\Big)\,\mathrm{d}s\\&=\int_{\partial B_R}\Big(o(R^{-1/2})\mathcal{O}(R^{-1/2})-\mathcal{O}(R^{-1/2})o(R^{-1/2})\Big)\,\mathrm{d}s=\int_{\partial B_R}o(R^{-1})\,\mathrm{d}s\xrightarrow{R\to\infty}0.\end{aligned}$$

□

Recall that we have already seen in Theorem 2.26 a stronger version of the second inequality in (44): if this integral is 0 then $u=0$ (see the first formula in the proof to relate the integrals on Γ and on ∂B_R).

The Sommerfeld radiation condition can be extended to problems with complex wavenumbers with $\Im k\ge 0$, Lemma 3.12 holds also in this case, see [CK1, Thm. 3.3].

Theorem 3.13 (Green's representation in Ω_+)**.** Let Ω_- be a bounded Lipschitz domain, $\Gamma = \partial\Omega_-$, $\Omega_+ = \mathbb{R}^2 \setminus \overline{\Omega_-}$, and $u \in H^1_{\mathrm{loc}}(\overline{\Omega_+}; \Delta) \cap C^2(\Omega_+)$ be a *radiating* Helmholtz solution in Ω_+. Then:

$$-\int_\Gamma \left(\partial_{\mathbf{n}}^+ u(\mathbf{y}) \Phi_k(\mathbf{x}, \mathbf{y}) - \gamma^+ u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_-, \\ u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_+. \end{cases} \tag{45}$$

Proof. Let $R > 0$ be the radius of a ball such that $\Gamma \subset B_R$ and, if $\mathbf{x} \in \Omega_+$ also $\mathbf{x} \in B_R$. Then Green's representation (42) applied in $\Omega_+ \cap B_R$ gives

$$\left(\int_{\partial B_R} - \int_\Gamma \right) \left(\partial_{\mathbf{n}}^+ u(\mathbf{y}) \Phi_k(\mathbf{x}, \mathbf{y}) - \gamma^+ u(\mathbf{y}) \frac{\partial \Phi_k(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \right) \mathrm{d}s(\mathbf{y}) = \begin{cases} 0 & \text{if } \mathbf{x} \in \Omega_-, \\ u(\mathbf{x}) & \text{if } \mathbf{x} \in \Omega_+. \end{cases}$$

Taking the limit for $R \to \infty$, the integral over ∂B_R vanishes by the identity in (44) and we conclude. □

Exercise 3.14. Show that the fundamental solution deserves its name: for any $\mathbf{y} \in \mathbb{R}^2$, it satisfies $\Delta \Phi_k(\cdot, \mathbf{y}) + k^2 \Phi_k(\cdot, \mathbf{y}) = \delta_{\mathbf{y}}$ in distributional sense, where $\delta_{\mathbf{y}}$ is the Dirac delta at $\mathbf{y}$.

This means that $\int_{\mathbb{R}^2} \Phi_k(\mathbf{x}, \mathbf{y}) \big(\Delta \rho(\mathbf{x}) + k^2 \rho(\mathbf{x}) \big) \, \mathrm{d}\mathbf{x} = \rho(\mathbf{y})$ for all $\rho \in \mathscr{D}(\mathbb{R}^2)$ and $\mathbf{y} \in \mathbb{R}^2$.

Hint: integrate by parts in $\mathbb{R}^2 \setminus B_\epsilon(\mathbf{y})$ and take the limit for $\epsilon \to 0$ using the technique of the proof of Theorem 3.6.

3.4 Double-layer potential and operator

Green's representation formulas (42) and (45) mean that all Helmholtz solutions in Ω_- and all radiating Helmholtz solutions in Ω_+ can be written as boundary integrals over Γ. The integral of $\partial_{\mathbf{n}} u \; \Phi_k$ is the single layer potential $\mathcal{S}\psi$ for $\psi = \partial_{\mathbf{n}}^\pm u$ we already know from (35). Now we are evaluating $(\mathcal{S}\psi)(\mathbf{x})$ for both $\mathbf{x} \in \Omega_+$ and $\mathbf{x} \in \Omega_-$, so we are slightly extending the definition of the single layer operator to all points in the complement of Γ:

$$(\mathcal{S}\psi)(\mathbf{x}) := \int_\Gamma \Phi_k(\mathbf{x}, \mathbf{y}) \psi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \qquad \mathbf{x} \in \Omega_+ \cup \Omega_-.$$

In particular, for any $\psi \in H^{-\frac{1}{2}}(\Gamma)$, $\mathcal{S}\psi$ is a Helmholtz solution both in Ω_- and Ω_+ and is radiating. With the same reasoning as in (37), the Dirichlet traces from both sides of Γ coincide:

$$\gamma^+(\mathcal{S}\psi) = \gamma^-(\mathcal{S}\psi) = S\psi.$$

In particular, if $\psi \in C^0(\Gamma)$, then $\mathcal{S}\psi \in C^0(\mathbb{R}^2)$ but $\mathcal{S}\psi \notin C^1(\mathbb{R}^2)$ because, as we will see, the normal derivative of $\mathcal{S}\psi$ jumps on Γ.

The second term in (42) and (45) is a new potential:

$$\boxed{(\mathcal{D}\psi)(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_k(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} \psi(\mathbf{y}) \, \mathrm{d}s(\mathbf{y}) \qquad \mathbf{x} \in \Omega_- \cup \Omega_+.} \tag{46}$$

This is called acoustic **double-layer potential**.[10] For a function ψ on Γ, sufficiently smooth, $\mathcal{S}\psi$ is a smooth Helmholtz solution in both Ω_- and Ω_+, and satisfies the radiation condition. It can be proved that it is continuous as mapping $\mathcal{D} : H^{\frac{1}{2}}(\Gamma) \to H^1_{\mathrm{loc}}(\Omega_- \cup \Omega_+)$ (here we need the density to be in $H^{\frac{1}{2}}(\Gamma)$, while for the single-layer potential $H^{-\frac{1}{2}}(\Gamma)$ was enough, this is because the singularity of $\nabla \Phi_k$ is stronger than that of Φ_k).

Then Green's representation can be written as:

$$\boxed{\begin{aligned} \text{if } u \text{ is Helmholtz solution in } \Omega_- : \quad & u = \mathcal{S}\partial_{\mathbf{n}}^- u - \mathcal{D}\gamma^- u \quad && \text{in } \Omega_-, \\ \text{if } u \text{ is radiating Helmholtz solution in } \Omega_+ : \quad & u = -\mathcal{S}\partial_{\mathbf{n}}^+ u + \mathcal{D}\gamma^+ u \quad && \text{in } \Omega_+. \end{aligned}} \tag{47}$$

This means that any (radiating) Helmholtz solution is known once we know the "Cauchy data" γu and $\partial_{\mathbf{n}} u$, i.e. its Dirichlet and Neumann traces.

[10] The double layer potential has this name because it can be thought as the acoustic potential generated by two "sheets" of charges with opposite signs, parallel to Γ, in the limit when the distance between the sheets decreases to 0 (from $\frac{\partial \Phi_k(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} = \lim_{\delta \to 0} \frac{1}{2\delta} (\Phi_k(\mathbf{x}, \mathbf{y} + \delta \mathbf{n}) - \Phi_k(\mathbf{x}, \mathbf{y} - \delta \mathbf{n}))$.

As we did for the single layer, we can define the **double-layer operator**:

$$\boxed{(D\psi)(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \qquad \mathbf{x}\in\Gamma.} \tag{48}$$

If Γ is of class C^2 and $\psi \in C^0(\Gamma)$, then $D\psi$ is well-defined as a standard (weakly singular) integral and $S\psi \in C^0(\Gamma)$. On the other hand, if Γ is only Lipschitz and $\psi \in L^2(\Gamma)$, then $D\psi$ must be understood as Cauchy principal value:

$$(D\psi)(\mathbf{x}) = \lim_{\epsilon\to 0}\int_{\Gamma\cap\{\mathbf{y}:|\mathbf{y}-\mathbf{x}|>\epsilon\}} \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \qquad \text{a.e. } \mathbf{x}\in\Gamma.$$

Then $D\psi \in L^2(\Gamma)$.

Exercise 3.15. Assume that $\Gamma_* \subset \Gamma$ is a straight segment and that $\psi \in C^0(\Gamma)$ is supported in Γ_*. Show that $(D\psi)(\mathbf{x}) = 0$ for all for all $\mathbf{x}\in\Gamma_*$.

Hint: you do not need the precise value of Φ_k but only the fact that it only depends on $|\mathbf{x}-\mathbf{y}|$.

We have seen in (37) that $S = \gamma\mathcal{S}$. One might expect that $D = \gamma\mathcal{D}$, but this is not the case. To see this fact, we first extend Green's representation (42)/(45)/(47) to the case $\mathbf{x}\in\Gamma$.

For $\mathbf{x}\in\Gamma$ define

$$\sigma(\mathbf{x}) := \lim_{\epsilon\to 0}\frac{1}{2\pi\epsilon}\int_{\mathbf{y}\in\Omega_-,|\mathbf{y}-\mathbf{x}|=\epsilon} \mathrm{d}s.$$

If Γ is C^1 in $\mathbf{x}$ then $\sigma = \frac{1}{2}$; if Γ forms an angle with opening α at $\mathbf{x}$ then $\sigma(\mathbf{x}) = \frac{\alpha}{2\pi}$. By Rademacher theorem (Lipschitz functions are differentiable a.e.), for a Lipschitz Γ, $\sigma = \frac{1}{2}$ almost everywhere on Γ.

Lemma 3.16. Let $u \in H^1(\Omega_-;\Delta)\cap C^0(\overline{\Omega_-})$ be a Helmholtz solution. Then

$$\int_\Gamma \left(\partial_{\mathbf{n}}^- u(\mathbf{y})\Phi_k(\mathbf{x},\mathbf{y}) - \gamma^- u(\mathbf{y})\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\right)\mathrm{d}s(\mathbf{y}) = \sigma(\mathbf{x})u(\mathbf{x}) \qquad \mathbf{x}\in\Gamma. \tag{49}$$

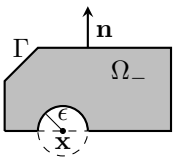
If $u \in H^1_{\mathrm{loc}}(\Omega_+;\Delta)\cap C^0(\overline{\Omega_+})$ is a radiating Helmholtz solution then

$$-\int_\Gamma \left(\partial_{\mathbf{n}}^+ u(\mathbf{y})\Phi_k(\mathbf{x},\mathbf{y}) - \gamma^+ u(\mathbf{y})\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\right)\mathrm{d}s(\mathbf{y}) = (1-\sigma)(\mathbf{x})u(\mathbf{x}) \qquad \mathbf{x}\in\Gamma. \tag{50}$$

Proof. We prove the first identity, the second one is analogous.

For $\mathbf{x}\in\Gamma$, we apply Green's second identity on $\Omega_- \setminus \overline{B_\epsilon(\mathbf{x})}$. Its boundary is decomposed in $\Gamma\setminus B_\epsilon(\mathbf{x})$ and $\partial B_\epsilon(\mathbf{x})\cap\Omega_-$; on both parts we choose $\mathbf{n}$ pointing outwards.

$$\begin{aligned} 0 &= \int_{\partial(\Omega_-\setminus\overline{B_\epsilon(\mathbf{x})})}\left(\partial_{\mathbf{n}}^- u(\mathbf{y})\Phi_k(\mathbf{x},\mathbf{y}) - \gamma^- u(\mathbf{y})\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\right)\mathrm{d}s(\mathbf{y}) \\ &= \left(\int_{\Gamma\setminus B_\epsilon(\mathbf{x})} + \int_{\partial B_\epsilon(\mathbf{x})\cap\Omega_-}\right)\left(\partial_{\mathbf{n}}^- u(\mathbf{y})\Phi_k(\mathbf{x},\mathbf{y}) - \gamma^- u(\mathbf{y})\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\right)\mathrm{d}s(\mathbf{y}). \end{aligned}$$



The limit for $\epsilon\to 0$ of the integral over $\Gamma\setminus B_\epsilon(\mathbf{x})$ is exactly $(S\partial_{\mathbf{n}}^- u)(\mathbf{x}) - (D\gamma^- u)(\mathbf{x})$, the left-hand side of the assertion. Proceeding as in the proof of Theorem 3.6 we see that $\lim_{\epsilon\to 0}\int_{\partial B_\epsilon(\mathbf{x})\cap\Omega_-}\partial_{\mathbf{n}}^- u(\mathbf{y})\Phi_k(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y}) = 0$, while the last term gives:

$$\begin{aligned} -\int_{\partial B_\epsilon(\mathbf{x})\cap\Omega_-}\gamma^- u(\mathbf{y})\frac{\partial\Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\,\mathrm{d}s(\mathbf{y}) &= k\frac{\mathrm{i}}{4}H_1^{(1)}(k\epsilon)\int_{\partial B_\epsilon(\mathbf{x})\cap\Omega_-} u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \\ &\sim k\frac{\mathrm{i}}{4}\left(\frac{-2\mathrm{i}}{\pi k\epsilon}\right)\int_{\partial B_\epsilon(\mathbf{x})\cap\Omega_-} u(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \sim \left(\frac{1}{2\pi\epsilon}\right)2\pi\sigma(\mathbf{x})\epsilon u(\mathbf{x}) = \sigma(\mathbf{x})u(\mathbf{x}). \end{aligned}$$

□

The assertions of Lemma 3.16 can be written in terms of layer operators as (compare against (47))

$$\boxed{S\partial_{\mathbf{n}}^- u - D\gamma^- u = \sigma\gamma^- u, \qquad D\gamma^+ u - S\partial_{\mathbf{n}}^+ u = (1-\sigma)\gamma^+ u.} \tag{51}$$

Now let u be a Helmholtz solution in Ω_- and denote $\psi := \gamma^- u \in H^{\frac{1}{2}}(\Gamma)$. We have

$$\psi = \gamma^- u \overset{\text{Green repr. (42)}}{=} \gamma^- \mathcal{S}\partial_{\mathbf{n}}^- u - \gamma^- \mathcal{D}\psi \overset{\text{single-layer trace (37)}}{=} S\partial_{\mathbf{n}}^- u - \gamma^- \mathcal{D}\psi \qquad \text{and} \qquad \sigma\psi \overset{(51)}{=} S\partial_{\mathbf{n}}^- u - D\psi$$

Taking the difference between these two equations we have the Dirichlet **trace formula for the double layer potential**:

$$\boxed{\gamma^- \mathcal{D}\psi = D\psi - (1-\sigma)\psi, \qquad \gamma^+ \mathcal{D}\psi = D\psi + \sigma\psi.} \tag{52}$$

The second of these equations is obtained similarly using (45) in place of (42) and (50) in place of (49). Recall that $\sigma = \frac{1}{2}$ in all smooth points, so (52) reads almost everywhere as

$$\gamma^{\pm}\mathcal{D}\psi = D\psi \pm \frac{1}{2}\psi, \quad \text{or, in operator form, } \gamma^{\pm}\mathcal{D} = D \pm \frac{1}{2}I,$$

where I is the identity operator. All these formulas tell us that the Dirichlet trace of $\mathcal{D}$ is not simply D but a correction term is needed, due to the singular behaviour of $\frac{\partial \Phi_k}{\partial n}$. Taking the difference between the two equations in (52), the correction terms $\pm\frac{1}{2}\psi$ give the **jump relation**:

$$[\![\mathcal{D}\psi]\!] := \gamma^+ \mathcal{D}\psi - \gamma^- \mathcal{D}\psi = \psi.$$

Given a ψ on the boundary Γ, the double layer potential $\mathcal{D}\psi$ is a radiating Helmholtz solution in the complement of Γ, whose jump on Γ is ψ itself.

For simplicity, in the following we will write $\frac{1}{2}$ instead of σ and $1-\sigma$, with the implicit convention that equalities on Γ hold almost everywhere (everywhere except possibly at corners).

3.5 Neumann traces of the potentials: two more BIOs

In the previous section we have learned how to construct fields in $\Omega_- \cup \Omega_+$ from distributions defined on Γ:

$$\forall \psi \in H^{-\frac{1}{2}}(\Gamma), \quad \forall \varphi \in H^{\frac{1}{2}}(\Gamma), \qquad \mathcal{S}\psi,\ \mathcal{D}\varphi \in H^1(\Omega_-;\Delta) \times H^1_{\text{loc}}(\Omega_+;\Delta) \tag{53}$$

are radiating Helmholtz solution in the complement of the boundary Γ. Moreover, the Dirichlet traces of $\mathcal{S}\psi$ coincide: $\gamma^+\mathcal{S}\psi = \gamma^-\mathcal{S}\psi$, so $\mathcal{S}\psi \in H^1_{\text{loc}}(\mathbb{R}^2;\Delta)$, while $\mathcal{D}\varphi$ is discontinuous on Γ, (52).

We now want to look at the Neumann traces of $\mathcal{S}\psi$, $\mathcal{D}\varphi$. To this purpose, we need to introduce two more BIOs (the last ones!): the **adjoint double-layer operator** D' and the **hypersingular operator** H:

$$\boxed{(D'\varphi)(\mathbf{x}) := \int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{x})}\varphi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad (H\varphi)(\mathbf{x}) := \frac{\partial}{\partial n(\mathbf{x})}\int_\Gamma \frac{\partial \Phi_k(\mathbf{x},\mathbf{y})}{\partial n(\mathbf{y})}\varphi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \quad \mathbf{x}\in\Gamma.} \tag{54}$$

Note that D' differs from D only in that the normal derivation is taken with respect to a different variable of Φ_k. If Γ is not C^2 or $\varphi \notin C^0(\Gamma)$, the adjoint double layer operator has to be understood as a principal value integral, in the same way as D. Its name and notation are due to the fact that it is possible to prove, using Fubini theorem and with some complications due to the singularity of the integrand, that $\int_\Gamma (D\varphi)\psi\,\mathrm{d}s = \int_\Gamma \varphi(D'\psi)\,\mathrm{d}s$ for all $\varphi,\psi \in L^2(\Gamma)$.

The hypersingular operator is more complicated: it has to be understood as a limit $(H\varphi)(\mathbf{x}) = \lim_{\mathbf{z}\to\mathbf{x}} \mathbf{n}(\mathbf{x})\cdot\nabla(\mathcal{D}\varphi)(\mathbf{x})$, for a suitable choice of the points $\mathbf{z}$. In its definition we are not allowed move $\frac{\partial}{\partial n(\mathbf{x})}$ inside the integral because the second derivatives of Φ_k are not integrable.

The main properties of the BIOs are their relations with the traces of the two layer potentials:

$$\boxed{\begin{aligned} &\gamma^{\pm}\mathcal{S} = S, && \gamma^{\pm}\mathcal{D} = D \pm \frac{1}{2}I, \\ &\partial_{\mathbf{n}}^{\pm}\mathcal{S} = D' \mp \frac{1}{2}I, && \partial_{\mathbf{n}}^{\pm}\mathcal{D} = H. \end{aligned}} \tag{55}$$

Here I is the identity operator. We have already derived the formulas of the Dirichlet traces, those for the Neumann ones are proved in a similar way, [CK1, §2.4–2.5]. Taking the difference between outer and inner traces we find the **jump relations** (we have already encountered those for the Dirichlet traces)

$$\boxed{\begin{aligned} [\![\gamma\mathcal{S}\psi]\!] &= \gamma^+\mathcal{S}\psi - \gamma^-\mathcal{S}\psi = 0, && [\![\gamma\mathcal{D}\psi]\!] = \gamma^+\mathcal{D}\psi - \gamma^-\mathcal{D}\psi = \psi, \\ [\![\partial_{\mathbf{n}}\mathcal{S}\psi]\!] &= \partial_{\mathbf{n}}^+\mathcal{S}\psi - \partial_{\mathbf{n}}^-\mathcal{S}\psi = -\psi, && [\![\partial_{\mathbf{n}}\mathcal{D}\psi]\!] = \partial_{\mathbf{n}}^+\mathcal{D}\psi - \partial_{\mathbf{n}}^-\mathcal{D}\psi = 0. \end{aligned}} \tag{56}$$

From (55), using (53) and the trace theorem 2.6, the mapping (continuity) properties of the BIOs follow:

$$S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad D : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma), \quad D' : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma), \quad H : H^{\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma).$$

From (55) we also see that all four operators are averages of traces of the potentials:

$$S = \{\!\{\gamma\mathcal{S}\}\!\} = \frac{\gamma^+\mathcal{S} + \gamma^-\mathcal{S}}{2}, \qquad D = \{\!\{\gamma\mathcal{D}\}\!\} = \frac{\gamma^+\mathcal{D} + \gamma^-\mathcal{D}}{2},$$
$$D' = \{\!\{\partial_{\mathbf{n}}\mathcal{S}\}\!\} = \frac{\partial_{\mathbf{n}}^+\mathcal{S} + \partial_{\mathbf{n}}^-\mathcal{S}}{2}, \qquad H = \{\!\{\partial_{\mathbf{n}}\mathcal{D}\}\!\} = \frac{\partial_{\mathbf{n}}^+\mathcal{D} + \partial_{\mathbf{n}}^-\mathcal{D}}{2}.$$

These formulas can be taken as alternative rigorous definitions of the four BIOs, given those of the two layer potentials.

Remark 3.17. There is no universal notation for boundary integral operators and sometimes the same symbol is used by different authors to mean different BIOS: comparing references can be a nightmare. To help navigating the literature, we list here the notation used in several good references on BIEs for Helmholtz (some of these only consider the 3D case).

	Φ_k	$\mathcal{S}$	$\mathcal{D}$	S	D	D'	H
[Spence14, p. 36], [Chandler-Wilde et al, Acta Num. 2012, pp. 108–113]	Φ_k	$\mathcal{S}_k$	$\mathcal{D}_k$	S_k	D_k	D'_k	H_k
[Sayas06, §3,§11.1]	ϕ	$\mathcal{S}_\Gamma$	$\mathcal{D}_\Gamma$	V_Γ	K_Γ	K_Γ^t	W_Γ
[CK1, §2.7], [CK2, (3.8–11)]	Φ			S	K	K'	T
[Néd01, p. 116]	E			S	D	D^*	N
[Antoine, notes 2012, §3.3]	G	L	S	$\mathcal{L}$	$\mathcal{N}$	$\mathcal{D}$	$\mathcal{S}$
[Hsiao and Wendland 2008, §2.1]	E_k	V_k	W_k	V_k	K_k	K'_k	D_k
[Martin 2006, §5.1–5.3]	G	S	D	S	K	$\overline{K}^*$	N
[McLean 2000, pp. 217–218]	G	SL	DL	S	T	$\widetilde{T}$	R
[Sauter and Schwab 2011, §3.9]	G_k	S_k	D_k	V_k	K_k	K'_k	W_k
[Steinbach 2008, §6.9] (W used for Laplace d.l.p.)	U_k^*	$\widetilde{V}_k$		V_k	K_k	K'_k	D_k

Moreover, in some cases the hypersingular operator is defined with the opposite sign, e.g. [Sayas06, §11.1]. In other cases all four BIOs include a factor 2 [CK2, eq. (3.8)–(3.11)] (to avoid the factor $\frac{1}{2}$ in the trace relations). [Martin, (5.1)] defines the fundamental solution as (-2) times our (standard) definition.

3.5.1 Consequences for the single-layer BIE applied to the SSSP

Consider the SSSP (30) and the corresponding single-layer BIE $S\psi = g_D$ (38). The representation formula $u^{\text{Scat}} = (\mathcal{S}\psi)|_{\Omega_+}$ (39) gives the value of the scattered field in Ω_+ as a single-layer potential. Denote by u^- the same potential evaluated in Ω_-, i.e. $u^- = (\mathcal{S}\psi)|_{\Omega_-}$. Then u^- is a Helmholtz solution in Ω_- with trace $\gamma^- u^- = \gamma^-\mathcal{S}\psi = S\psi = g_D = -\gamma u^{\text{Inc}}$. We now assume that: (i) k^2 is not a Dirichlet eigenvalue and (ii) u^{Inc} is an incoming wave that is Helmholtz solution also in Ω_-, e.g. a plane wave. Then

$$(\mathcal{S}\psi)|_{\Omega_-} = u^- = -u^{\text{Inc}}.$$

This equality has a few useful consequences.

From one of the jump relations (56) we can relate the BIE density ψ to a "physical" quantity, the Neumann trace of the total field:

$$\psi = -[\![\partial_{\mathbf{n}}\mathcal{S}\psi]\!] = \partial_{\mathbf{n}}^-\mathcal{S}\psi - \partial_{\mathbf{n}}^+\mathcal{S}\psi = \partial_{\mathbf{n}}(-u^{\text{Inc}}) - \partial_{\mathbf{n}}^+ u^{\text{Scat}} = -\partial_{\mathbf{n}}^+ u^{\text{Tot}}.$$

This allow to compute the Neumann trace of the scattered field from the data and the BIE solution as

$$\partial_{\mathbf{n}}^+ u^{\text{Scat}} = -\psi - \partial_{\mathbf{n}} u^{\text{Inc}}.$$

We can use this formula to compute the far-field pattern u_∞ (31) of the SSSP (30) solution u from the BIE (38) solution ψ:

$$\begin{aligned} u_\infty(\theta) &= \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_\Gamma \Big(\gamma^+ u^{\text{Scat}}(\mathbf{y}) \partial_{\mathbf{n}} \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} - \partial_{\mathbf{n}}^+ u^{\text{Scat}}(\mathbf{y}) \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} \Big) \,\mathrm{d}s(\mathbf{y}) \\ &= \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_\Gamma \Big(-\gamma u^{\text{Inc}}(\mathbf{y}) \partial_{\mathbf{n}} \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} + \big(\psi(\mathbf{y}) + \partial_{\mathbf{n}} u^{\text{Inc}}(\mathbf{y})\big) \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} \Big) \,\mathrm{d}s(\mathbf{y}) \\ &= \frac{\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}}{\sqrt{8\pi k}} \int_\Gamma \psi(\mathbf{y}) \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}} \,\mathrm{d}s(\mathbf{y}) \qquad\qquad \mathbf{d} = (\cos\theta, \sin\theta) \end{aligned}$$

where the last equality comes from Green's representation (42) applied to u^{Inc} and the plane wave $\mathbf{y} \mapsto \mathrm{e}^{-\mathrm{i}k\mathbf{y}\cdot\mathbf{d}}$.

Exercise 3.18. Use your BEM code to approximate the far-field pattern of the field scattered by a polygon; see an example in Figure 11.

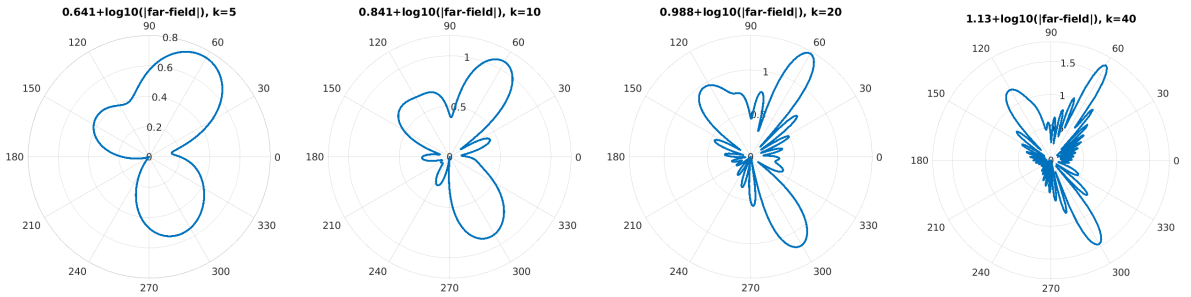


Figure 11: Some polar logarithmic plots (Matlab's `polarplot` *command) of the magnitudes of far-field patterns* $\log_{10}|u_\infty|$ *computed with BEM. In this scattering problem, a plane wave with direction* $\frac{\pi}{3}$ *hits a sound-soft triangular scatterer with vertices* $(0,0)$, $(1,0)$ *and* $(0,1)$, *as in Figure 9. Each plot corresponds to a different wavenumber* ($k = 5, 10, 20, 40$)*: for increasing frequencies the far field becomes more complex and focused in few directions. The far-field pattern has maximal intensity in the direction* $\frac{\pi}{3}$ *of the incoming wave (up right), where the triangle projects its shadow. Two other peaks are in directions* $-\frac{\pi}{3}$ *and* $\frac{2\pi}{3}$*, corresponding to the wave reflected by the two illuminated sides. The field in all other directions is due to the diffraction by the corners.*

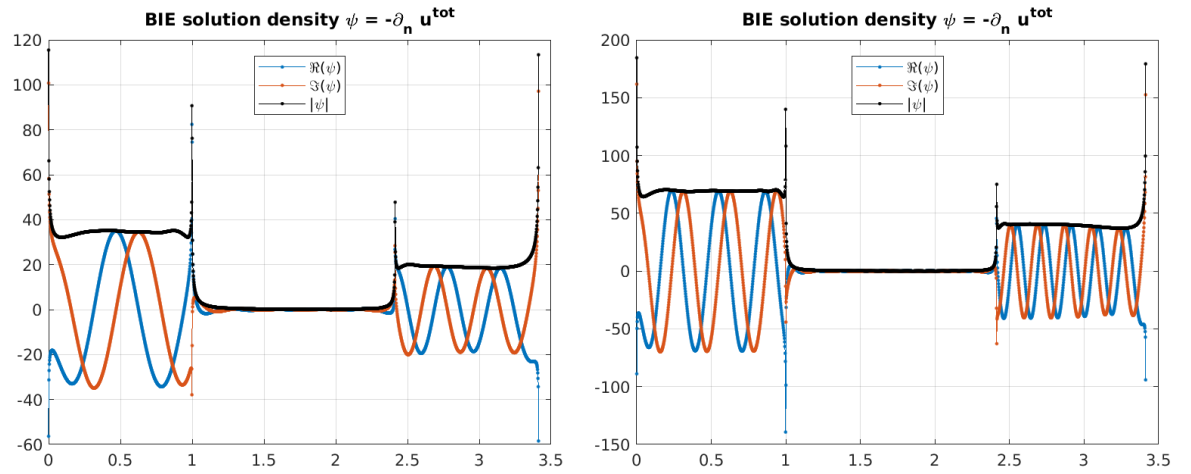


Figure 12: Left: the density $\psi = -\partial^+_{\mathbf{n}} u^{\mathrm{Tot}}$ *for the problem of Figure 9 with* $k = 20$*, computed with the BEM of §3.2. The* x*-axis represent the curvilinear abscissa along the boundary of the triangle* Γ*, starting from the lower-left vertex and proceeding anticlockwise. We observe that* ψ *oscillates on the two illuminated sides of* Γ*, is small (but non-zero) on the shadow side, and has singularities at the three vertices. Right: the same for* $k = 40$*.*

Remark 3.19 (Checking BEM accuracy)**.** The formula $(\mathcal{S}\psi + u^{\mathrm{Inc}})|_{\Omega_-} = 0$ is useful to check a BEM implementation of the BIE (38). The routine used to evaluate the numerical near-field $u_N = \mathcal{S}\psi_N$ in a portion of Ω_+ can be used to approximate $\mathcal{S}\psi$ in Ω_-. The value $|\mathcal{S}\psi_N + u^{\mathrm{Inc}}|$ in Ω_- must be small for an accurate BEM implementation and must decrease to 0 when the BEM mesh is refined.

Test your BEM code by computing e.g. $\|\mathcal{S}\psi + u^{\mathrm{Inc}}\|_{L^2(\Omega_-)} / \|u^{\mathrm{Inc}}\|_{L^2(\Omega_-)}$ and see how this ratio depend on the problem parameters and on the numerical ones (k, Ω_-, N, quadrature,...).

3.6 Well-posedness of the single-layer BIE

We want to study the well-posedness of the single-layer BIE $S\psi = g_D$ (38). To this purpose, we want to verify that the single-layer operator $S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is (i) injective and (ii) Fredholm. When both conditions are satisfied, then Fredholm alternative (Theorem 2.9) implies that S is invertible and the BIE is well-posed. However, injectivity is not always true: the EDP (29) is always well-posed (§2.5) but its BIE (38) might fail.

3.6.1 Injectivity of the single-layer operator

As in §2.3, two cases may happen:

- If $\Lambda = k^2$ is a Laplace–Dirichlet eigenvalue in Ω_-, then there exists an eigenfunction $w \neq 0$ such that $\Delta w + k^2 w = 0$ and $\gamma^- w = 0$. Define $\psi = \partial_{\mathbf{n}}^- w$. By Green's representation (42), $w = \mathcal{S}\partial_{\mathbf{n}}^- w - \mathcal{D}\gamma^- w = \mathcal{S}\partial_{\mathbf{n}}^- w = \mathcal{S}\psi$. Then $\psi \neq 0$ because a non-trivial Dirichlet eigenfunction has non-trivial Neumann trace (Corollary 3.7) and $S\psi = \gamma^-\mathcal{S}\psi = \gamma^- w = 0$. In this case the single-layer operator is not injective: $0 \neq \psi \in \ker S$. The BIE (38) is not well-posed.
- If $\Lambda = k^2$ is not a Laplace–Dirichlet eigenvalue in Ω_-, then assume that $S\psi = 0$ and define $u = \mathcal{S}\psi$. We have that $u^- = u|_{\Omega_-}$ is Helmholtz solution in Ω_- and $\gamma^- u^- = \gamma^-\mathcal{S}\psi = S\psi = 0$. But the interior homogeneous Helmholtz Dirichlet BVP is well-posed by Proposition 2.13 and admits only the solution $u^- = 0$. Similarly $u^+ = u|_{\Omega_+}$ is a radiating Helmholtz solution in Ω_+ with $\gamma^+ u^+ = \gamma^+\mathcal{S}\psi = S\psi = 0$. By the well-posedness of the EDP of §2.5 also $u^+ = 0$. The jump relation (56) gives $\psi = [\![\partial_{\mathbf{n}}\mathcal{S}\psi]\!] = \partial_{\mathbf{n}}^+ u^+ - \partial_{\mathbf{n}}^- u^- = 0$, so the single-layer operator is injective.

  Combining with Proposition 2.13 we obtain the following fact.

For each Ω_- there exist a sequence of positive number $k_1 < k_2 < \ldots$, $\lim_{j\to\infty} k_j = \infty$, such that S is injective if and only if $k \neq k_j$ for all $j \in \mathbb{N}$.

These values are called **spurious resonances** or **spurious frequencies**. Even if the BIE (38) is not solvable in this case, the EDP (29) is well-posed: the interior eigenvalues perturb the BIE formulation of the exterior problem. This can be understood as follows: the same BIE solves both an exterior and an interior Helmholtz Dirichlet problem (with solutions $(\mathcal{S}\psi)|_{\Omega_+}$ and $(\mathcal{S}\psi)|_{\Omega_-}$), when the latter is not well-posed then the BIE cannot be well-posed either. We will see other (slightly more complicated) BIEs that always admit a solution.

Exercise 3.20. Spurious resonances affect numerical computations. Plot the condition number of the BEM matrix $\underline{\underline{\mathbf{A}}}^{C/G}$ and the accuracy test of Remark 3.19 (e.g. $\left\|\mathcal{S}\psi_N + u^{\mathrm{Inc}}\right\|_{L^2(\Omega_-)}$) for several values of k close to a resonance to see how they blow up.

Hint: choose Ω_- as a square, so that the values of k_j are easily computed by hand as in §2.3.

3.6.2 The single-layer operator is Fredholm

We now want to show that the single layer operator $S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is Fredholm, i.e. it is sum of an invertible and a compact operator.

We define the single-layer operator for Laplace equation as

$$(S_0\psi)(\mathbf{x}) := -\frac{1}{2\pi}\int_\Gamma \log\frac{|\mathbf{x}-\mathbf{y}|}{d}\psi(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \qquad \mathbf{x}\in\Gamma \tag{57}$$

where d is a parameter[11] satisfying $d > \mathrm{diam}(\Gamma) = \sup_{\mathbf{x},\mathbf{y}\in\Gamma}|\mathbf{x}-\mathbf{y}|$.

The following two facts hold:

- $S - S_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is compact.
- $S_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is coercive, i.e. $\langle S_0\psi, \psi\rangle_\Gamma \geq c\,\|\psi\|^2_{H^{-\frac{1}{2}}(\Gamma)}$ for all $\psi \in H^{-\frac{1}{2}}(\Gamma)$.

Then, by Lax–Milgram, $S_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is invertible.

By Fredholm alternative 2.9, $S = S_0 + (S - S_0)$ is invertible if and only if it is injective.

By §3.6.1, the single-layer BIE $S\psi = g_D$ is well-posed (i.e. S is invertible) if and only if $-k^2$ is not a Laplace–Dirichlet eigenvalue for Ω_-. Combining with what we already know about the eigenvalues, we obtain the following fact.

For each Ω_- there exist a sequence of positive number $k_1 < k_2 < \ldots$, $\lim_{j\to\infty} k_j = \infty$, such that the BIE $S\psi = g_D$ is well-posed for all $g_D \in H^{\frac{1}{2}}(\Gamma)$ if and only if $k \neq k_j$ for all $j \in \mathbb{N}$.

We study the properties of compactness and coercivity in the following, starting from the case of a circular scatterer.

[11] Different values of d give different "versions" of the single layer. This corresponds to adding a constant to S_0: recall the difference between Laplace and Helmholtz solutions $\Delta u = 0 \Rightarrow \Delta(u+c) = 0$ but $(\Delta + k^2)u = 0 \Rightarrow (\Delta + k^2)(u+c) \neq 0$ for all constants $c \neq 0$. This is related to the fact that the 2D Laplace fundamental solution does not decay to 0 at infinity. We will see in §3.6.5 that the precise value of d only matters to ensure the coercivity of S_0. This is not true in 3D, where there is no need for the parameter d.

Remark 3.21. This decomposition of the Helmholtz operator in a "Laplace part" and "whatever is left" should remind you the technique used in §2.3 for Helmholtz problems on bounded domains. In that case, using the Gårding inequality, we have decomposed the Helmholtz sesquilinear form (either $\mathcal{A}$ of (24) or $\mathcal{A}_I$ of (26)) in a coercive part corresponding to an elliptic equation and a compact perturbation term multiple of $k^2 \int_\Omega u\overline{w}\,\mathrm{d}\mathbf{x}$.

3.6.3 Continuity, compactness and coercivity of single-layer BIOs on a circle

Let K be a BIO on the circle ∂B_R with kernel $\kappa : \mathbb{R} \to \mathbb{C}$, i.e.

$$(Kv)(\mathbf{x}) = \int_{\partial B_R} \kappa(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}), \tag{58}$$

for v defined on Γ. The distance between two points on the circle in polar coordinates reads

$$|\mathbf{x}-\mathbf{y}| = R|\mathrm{e}^{\mathrm{i}\theta_\mathbf{x}} - \mathrm{e}^{\mathrm{i}\theta_\mathbf{y}}| = R|1-\mathrm{e}^{\mathrm{i}(\theta_\mathbf{x}-\theta_\mathbf{y})}| = R\sqrt{2-2\cos(\theta_\mathbf{x}-\theta_\mathbf{y})}.$$

The action of the operator K on a function v can be written as the multiplication of the Fourier coefficients of the argument $v(\mathbf{x}) = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}$ by some coefficients K_ℓ:

$$\begin{aligned}(Kv)(\mathbf{x}) = \int_{\partial B_R} \kappa(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) &= R\int_0^{2\pi} \kappa(R\sqrt{2-2\cos(\theta_\mathbf{x}-\theta)}) \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}\,\mathrm{d}\theta \qquad (\alpha = \theta - \theta_\mathbf{x}) \\ &= \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta_\mathbf{x}} \underbrace{R\int_0^{2\pi} \kappa(R\sqrt{2-2\cos\alpha})\mathrm{e}^{\mathrm{i}\ell\alpha}\,\mathrm{d}\alpha}_{=:K_\ell} = \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell K_\ell\, \mathrm{e}^{\mathrm{i}\ell\theta_\mathbf{x}}.\end{aligned}$$

If $K_\ell = \mathcal{O}(\ell^a)$ for some $a \in \mathbb{R}$ then, from the definition (17) of the Sobolev spaces on the circular boundary, $K : H^s(\partial B_R) \to H^{s-a}(\partial B_R)$ as a bounded operator. But, how to estimate the K_ℓ?

The values K_ℓ are the Fourier coefficients of the function $\alpha \mapsto R\kappa(R\sqrt{2-2\cos\alpha})$ on $(0, 2\pi)$. Parseval's theorem ($\int_0^{2\pi} |f(\theta)|^2\,\mathrm{d}\theta = 2\pi\sum_{\ell\in\mathbb{Z}} |\hat{f}_\ell|^2$) implies that the Fourier coefficients of an $L^2(0,2\pi)$ function decay as $o(\ell^{-1/2})$. As we have seen in §2.2.3 using that $(\mathrm{e}^{\mathrm{i}\ell\theta})' = \mathrm{i}\ell\mathrm{e}^{\mathrm{i}\ell\theta}$, if $f' \in L^2(0,2\pi)$ then $\hat{f}_\ell = o(\ell^{-3/2})$.

The function $\alpha \mapsto R\sqrt{2-2\cos\alpha}$ is Lipschitz (verify that its derivative is $\pm\sqrt{\frac{1+\cos\alpha}{2}}$). Thus, if $\kappa \in L^2(0,2R)$ then $K_\ell = o(\ell^{-1/2})$, if moreover $\kappa' \in L^2(0,2R)$ then $K_\ell = o(\ell^{-3/2})$. So, useful relations between the properties of the kernel κ and the continuity of the operator K in the form (58) are

$$\kappa \in L^2(0,2R) \;\Rightarrow\; K : H^s(\partial B_R) \to H^{s+\frac{1}{2}}(\partial B_R), \qquad \kappa \in H^1(0,2R) \;\Rightarrow\; K : H^s(\partial B_R) \to H^{s+\frac{3}{2}}(\partial B_R).$$

(From the formulas above, the properties of $\kappa(t)$ for $t > 2R$, the diameter of the circle, are irrelevant.)

What are the kernels of the Helmholtz and Laplace single layer operators? We have

$$\begin{aligned} K = S \quad &\Rightarrow \quad \kappa(t) = \frac{\mathrm{i}}{4}H_0^{(1)}(kt) && \in L^2(0,2R), \\ K = S_0 \quad &\Rightarrow \quad \kappa(t) = -\frac{1}{2\pi}\log\frac{|t|}{d} && \in L^2(0,2R), \\ K = S - S_0 \quad &\Rightarrow \quad \kappa(t) = \frac{\mathrm{i}}{4}H_0^{(1)}(kt) + \frac{1}{2\pi}\log\frac{|t|}{d} && \in H^1(0,2R). \end{aligned}$$

The first line gives $S : H^s(\partial B_R) \to H^{s+\frac{1}{2}}(\partial B_R)$. This is not new: we already mentioned that $S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ for all Lipschitz boundaries, which, for $s = -\frac{1}{2}$, is a stronger result. We have also seen from the numerical computations in Figure 10 and Remark 3.1 that $K_\ell \sim \ell^{-1}$, so we cannot expect any stronger continuity property than this.

We now look at the difference between Helmholtz and Laplace single-layer operators $S - S_0$. From the asymptotic formula $H_0^{(1)}(z) = \mathrm{i}\frac{2}{\pi}\log z + 1 + \mathrm{i}\frac{2}{\pi}(\gamma-2) + \mathcal{O}(z^2)$ by, e.g., https://dlmf.nist.gov/10.8.E2 with the Euler's constant $\gamma \approx 0.57721$ we have that κ is bounded and κ' is bounded (with a jump at 0 because of the $\sqrt{2-2\cos\alpha}$ term). So $S - S_0 : H^{-\frac{1}{2}}(\partial B_R) \to H^1(\partial B_R)$. (In Figure 13 we demonstrate numerically a stronger continuity property, i.e. that $S - S_0 : H^{-\frac{1}{2}}(\partial B_R) \to H^{\frac{5}{2}}(\partial B_R)$.) Since the inclusion $\iota : H^1(\partial B_R) \to H^{\frac{1}{2}}(\partial B_R)$ is compact, then **the difference between the two single layers $S - S_0 : H^{-\frac{1}{2}}(\partial B_R) \to H^{\frac{1}{2}}(\partial B_R)$ is compact**.[12]

[12] Recall that the property of compactness of an operator depends heavily on the norms of the function spaces chosen as

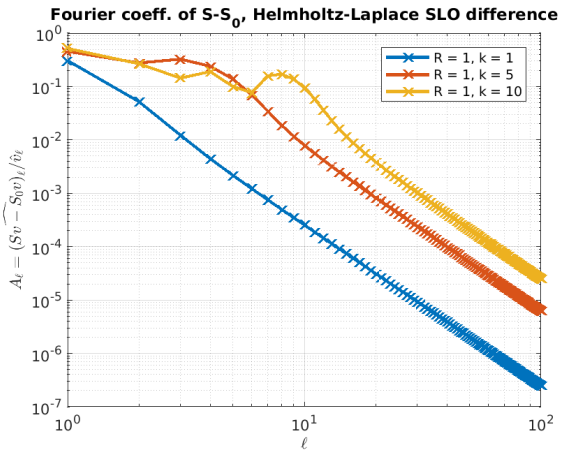


Figure 13: A log-log plot of the Fourier coefficients K_ℓ for the operator $S - S_0$, difference between Helmholtz and Laplace single layer operators, on a circle. The coefficients decay as $K_\ell \sim \ell^{-3}$, so the operator is continuous $H^s(\partial B_R) \to H^{s+3}(\partial B_R)$. In particular it is compact $H^{-\frac{1}{2}}(\partial B_R) \to H^{\frac{1}{2}}(\partial B_R)$. Compare with the coefficients of S in Figure 10.

We can use the expansion in circular harmonics also to check the coercivity of an integral operator. The sesquilinear form associated to K diagonalises in the Fourier basis (recall §2.2.3):

$$\langle Kv, w\rangle_{\partial B_R} = \left\langle \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell K_\ell \mathrm{e}^{\mathrm{i}\ell\theta_{\mathbf{x}}}, \sum_{m\in\mathbb{Z}} \hat{w}_\ell \mathrm{e}^{\mathrm{i}m\theta_{\mathbf{x}}} \right\rangle_{\partial B_R} = 2\pi R \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell K_\ell \overline{\hat{w}_\ell}.$$

If $K_\ell \in \mathbb{R}$ and $K_\ell \geq c(1+\ell^2)^s$ for all ℓ and some $c > 0$, $s \in \mathbb{R}$ [13], then $|\langle Kv, v\rangle_{\partial B_R}| \geq 2\pi R c \sum_{\ell\in\mathbb{Z}} |\hat{v}_\ell|^2 (1+\ell^2)^s = c\,\|v\|^2_{H^s(\partial B_R)}$, i.e. K is coercive in $H^s(\partial B_R)$.

Now look at the Laplace single layer, i.e. K with $\kappa(t) = -\frac{1}{2\pi}\log\frac{|t|}{d}$. Since $\kappa(R\sqrt{2-2\cos(t)})$ is real and even-symmetric, its Fourier coefficients K_ℓ are real.

Exercise 3.22. Compute with Matlab the coefficients K_ℓ for the Laplace single-layer S_0 and show that they satisfy $K_\ell(1+\ell^2)^{\frac{1}{2}} > c$. (For $R = 1$, $c \approx \frac{1}{2}$.)

From this exercise it follows that the **Laplace single-layer operator S_0 is coercive in $H^{-\frac{1}{2}}(\partial B_R)$.**

3.6.4 Compactness of $S - S_0$

We sketch the main ideas used to prove the compactness of $(S - S_0) : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$. Making them precise and rigorous is not trivial.

The key result to prove compactness of BIOs is the following: an operator $K : L^2(\Gamma) \to L^2(\Gamma)$ in the form $(Kv)(\mathbf{x}) = \int_\Gamma \kappa(|\mathbf{x}-\mathbf{y}|)v(\mathbf{y})\,\mathrm{d}s(\mathbf{y})$ is compact if the kernel $\kappa : \mathbb{R} \to \mathbb{C}$ is a *bounded* function (L^∞).

From the asymptotic expansion of the Hankel function at the origin, as in §3.6.3, we see that the kernel of $S - S_0$ is bounded (and continuous). The operator T defined by $Tv = (Sv - S_0 v)'$, where the derivative is the tangential derivative along Γ, also has a bounded (but discontinuous) kernel, for the same reason. From this it follows that $S - S_0 : L^2(\Gamma) \to H^1(\Gamma)$ is a compact operator. However we want to lower the Sobolev exponents of both spaces by $\frac{1}{2}$.

From functional analysis we know that if an operator $K : H_1 \to H_2$ is compact, then its adjoint $K^* : H_2^* \to H_1^*$ (defined by $(K^*\varphi)(\psi) = \varphi(K\psi)$ for $\varphi \in H_2^*$ and $\psi \in H_1$) is also compact.

Fubini theorem implies that the single layer is self-adjoint in $L^2(\Gamma)$: $\int_\Gamma (S\varphi)\psi\,\mathrm{d}s = \int_\Gamma \varphi(S\psi)\,\mathrm{d}s$ for all $\varphi, \psi \in L^2(\Gamma)$ (you can prove this). The same holds for S_0. Thus the adjoint of $S - S_0 : L^2(\Gamma) \to H^1(\Gamma)$ is the extension of $S - S_0$ itself to $S - S_0 : H^{-1}(\Gamma) \to L^2(\Gamma)$, where $H^{-1}(\Gamma)$ is the dual of $H^1(\Gamma)$ as in §2.2.3,

domain and codomain. E.g. the identity operator $I : H^1(\Omega) \to H^1(\Omega)$ is not compact for a bounded Lipschitz Ω, but when we view it as $I : H^1(\Omega) \to L^2(\Omega)$ (and we call it embedding) then it is compact (Rellich theorem). The technique used in this section is very standard: we show that an operator K maps in a space that is slightly smoother (here $H^1(\partial B_R)$) than the desired domain (here $H^{\frac{1}{2}}(\partial B_R)$), then we compose the operator with the embedding (here $\iota : H^1(\partial B_R) \to H^{\frac{1}{2}}(\partial B_R)$) and if this embedding is compact the same holds for $\iota \circ K$. Recall that the composition of a continuous operator and a compact one is compact; you can prove this from the definition.

[13] We could also take $\Re\{\alpha K_\ell\} > c(1+\ell^2)^s$ for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$.

and is compact. A technique called "operator interpolation" allows to deduce from the compactness of $S - S_0$ in $L^2(\Gamma) \to H^1(\Gamma)$ and $H^{-1}(\Gamma) \to L^2(\Gamma)$ the compactness in all intermediate spaces, in particular the compactness of $S - S_0 : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$.

3.6.5 Coercivity of S_0

We sketch the proof of the coercivity of the Laplace single-layer operator S_0 following [Steinbach 2008, Thm. 6.22–23] and [McLean 2000, Thm. 8.12–16]. In this section we use real-valued function spaces.

Most of the results derived and stated in the previous sections (traces, jumps, ...) for the Helmholtz equation hold also for the Laplace equation, using $\Phi_0(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi}\log\frac{|\mathbf{x}-\mathbf{y}|}{d}$ as fundamental solution.

Fix $\psi \in H^{-\frac{1}{2}}(\Gamma)$ and denote $u = \mathcal{S}_0\psi \in H^1(\Omega_-,\Delta) \times H^1_{\mathrm{loc}}(\overline{\Omega_+},\Delta)$, $\mathcal{S}_0$ being the Laplace single-layer potential. Then $\Delta u = 0$ in $\Omega_- \cup \Omega_+$, $[\![\gamma u]\!] = 0$ and $[\![\partial_{\mathbf{n}} u]\!] = -\psi$, in analogy to (56).

Integration by parts (Green's first identity (18) with $k = 0$) gives that $\|\nabla u\|^2_{L^2(\Omega_-)} = \int_{\Omega_-} \nabla u \cdot \nabla \overline{u}\,\mathrm{d}\mathbf{x} = \int_\Gamma \partial^-_{\mathbf{n}} u \gamma^- \overline{u}\,\mathrm{d}s$. The radiation condition for Laplace equation (which we do not discuss here) gives that a similar identity holds in Ω_+: if the 0-average condition $\langle\psi, 1\rangle_\Gamma = 0$ holds, then $\|\nabla u\|^2_{L^2(\Omega_+)} = \int_{\Omega_+} \nabla u \cdot \nabla\overline{u}\,\mathrm{d}\mathbf{x} = -\int_\Gamma \partial^+_{\mathbf{n}} u \gamma^+ \overline{u}\,\mathrm{d}s$. Note that (1) this integral/norm on Ω_+ would not be bounded in the Helmholtz case, (2) in 2D this holds only for potentials u whose density ψ satisfies the 0-average condition, (3) in 3D this holds for all $\psi \in H^{-\frac{1}{2}}(\Gamma)$.

The continuity of the normal trace operator

$$H(\mathrm{div};\Omega_\pm) = \{\mathbf{v} \in L^2(\Omega_\pm)^2; \mathrm{div}\,\mathbf{v} \in L^2(\Omega_\pm)\} \ni \mathbf{v} \mapsto \gamma^\pm\mathbf{v}\cdot\mathbf{n} \in H^{-\frac{1}{2}}(\Gamma)$$

applied to ∇u, which is divergence-free because $\Delta u = 0$, allows to control the Neumann traces with the L^2 traces of the gradient. Combining all this we have that, for all $\psi \in H^{-\frac{1}{2}}(\Gamma)$ with $\langle\psi,1\rangle_\Gamma = 0$,

$$\begin{aligned}
\|\psi\|^2_{H^{-\frac{1}{2}}(\Gamma)} &= \|[\![\partial_{\mathbf{n}} u]\!]\|^2_{H^{-\frac{1}{2}}(\Gamma)} && \psi = -[\![\partial_{\mathbf{n}} u]\!] \\
&\le 2\left\|\partial^-_{\mathbf{n}} u\right\|^2_{H^{-\frac{1}{2}}(\Gamma)} + \left\|\partial^+_{\mathbf{n}} u\right\|^2_{H^{-\frac{1}{2}}(\Gamma)} && \text{triangle inequality} \\
&\le C(\|\nabla u\|^2_{L^2(\Omega_-)^2} + \|\nabla u\|^2_{L^2(\Omega_+)^2}) && \text{normal trace continuity in } H(\mathrm{div};\Omega_\pm) \\
&= C\int_{\Omega_-\cup\Omega_+} \nabla u\cdot\nabla\overline{u}\,\mathrm{d}\mathbf{x} \\
&= C\int_\Gamma (\partial^-_{\mathbf{n}} u\gamma^-\overline{u} - \partial^+_{\mathbf{n}} u\gamma^+\overline{u})\,\mathrm{d}s && \text{Green's first identity, in 2D it requires } \langle\psi,1\rangle_\Gamma = 0 \\
&= -C\int_\Gamma [\![\partial_{\mathbf{n}} u]\!]\gamma\overline{u}\,\mathrm{d}s && \gamma^+ u = \gamma^- u \\
&= C\int_\Gamma \psi\overline{S_0\psi}\,\mathrm{d}s && [\![\partial_{\mathbf{n}} u]\!] = -\psi, \quad \gamma u = \gamma\mathcal{S}_0\psi = S_0\psi.
\end{aligned}$$

This is precisely the coercivity of S_0 in $H^{-\frac{1}{2}}_*(\Gamma) := \{\psi \in H^{-\frac{1}{2}}(\Gamma) : \langle\psi,1\rangle_\Gamma = 0\}$. (In 3D this would hold in the whole of $H^{-\frac{1}{2}}(\Gamma)$, so the proof would be complete.)

To deal with the general case ($\psi \in H^{-\frac{1}{2}}(\Gamma)$ instead of $\psi \in H^{-\frac{1}{2}}_*(\Gamma)$) we have to work a bit more. The coercivity for 0-average densities ensures that there exists a unique (non zero)

$$\beta_* \in H^{-\frac{1}{2}}_*(\Gamma) \qquad \langle S_0\beta_*, \xi_*\rangle_\Gamma = \langle S_0 1, \xi_*\rangle_\Gamma \qquad \forall \xi_* \in H^{-\frac{1}{2}}_*(\Gamma).$$

Define $\beta_{\mathrm{eq}} := \frac{1}{|\Gamma|}(1 - \beta_*) \in H^{-\frac{1}{2}}(\Gamma)$. Then β_{eq} is real, $\langle\beta_{\mathrm{eq}},1\rangle_\Gamma = 1$, and $\langle S_0\beta_{\mathrm{eq}},\xi_*\rangle_\Gamma = 0$ for all $\xi_* \in H^{-\frac{1}{2}}_*(\Gamma)$, so $S_0\beta_{\mathrm{eq}}$ is a constant. Recalling the definition (57) of S_0,

$$\begin{aligned}
(S_0\beta_{\mathrm{eq}})(\mathbf{x}) &= \frac{1}{2\pi}\int_\Gamma \beta_{\mathrm{eq}}(\mathbf{y})\big(\log d - \log|\mathbf{x}-\mathbf{y}|\big)\,\mathrm{d}s(\mathbf{y}) \\
&= \frac{1}{2\pi}\log d\int_\Gamma \beta_{\mathrm{eq}}(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \frac{1}{2\pi}\int_\Gamma \beta_{\mathrm{eq}}(\mathbf{y})\log|\mathbf{x}-\mathbf{y}|\,\mathrm{d}s(\mathbf{y}) \\
&= \frac{1}{2\pi}\log d - \frac{1}{2\pi}\int_\Gamma \beta_{\mathrm{eq}}(\mathbf{y})\log|\mathbf{x}-\mathbf{y}|\,\mathrm{d}s(\mathbf{y}) && \forall \mathbf{x} \in \Gamma.
\end{aligned}$$

This is where we need the (so far unused) parameter $d > 0$: if d is sufficiently large then $S_0\beta_{\rm eq} > 0$. It is possible to show that $d > \operatorname{diam}\Gamma$ is enough to guarantee that $S_0\beta_{\rm eq} > 0$.[14] Then also

$$\langle S_0\beta_{\rm eq}, \beta_{\rm eq}\rangle_\Gamma = S_0\beta_{\rm eq}\langle 1, \beta_{\rm eq}\rangle_\Gamma = S_0\beta_{\rm eq} > 0.$$

We want to decompose a general $\psi \in H^{-\frac{1}{2}}(\Gamma)$ in a $H_*^{-\frac{1}{2}}(\Gamma)$ component and a remainder: instead of taking a constant remainder as one might expect, we take a remainder whose image under S_0 is constant, i.e. a multiple of $\beta_{\rm eq}$. For all $\psi \in H^{-\frac{1}{2}}(\Gamma)$ define

$$\psi_* := \psi - \langle\psi, 1\rangle_\Gamma\beta_{\rm eq} \quad \Rightarrow \quad \langle\psi_*, 1\rangle_\Gamma = \langle\psi, 1\rangle_\Gamma\Big(1 - \langle\beta_{\rm eq}, 1\rangle_\Gamma\Big) = 0.$$

Now we use the decomposition $\psi = \psi_* + \langle\psi, 1\rangle_\Gamma\beta_{\rm eq}$ to show the positivity of the single-layer potential:

$$\begin{aligned}\langle S_0\psi, \psi\rangle_\Gamma &= \langle S_0(\psi_* + \langle\psi, 1\rangle_\Gamma\beta_{\rm eq}), \psi_* + \langle\psi, 1\rangle_\Gamma\beta_{\rm eq}\rangle_\Gamma \\ &= \langle S_0\psi_*, \psi_*\rangle_\Gamma + 2\langle\psi, 1\rangle_\Gamma \underbrace{\langle S_0\beta_{\rm eq}, \psi_*\rangle_\Gamma}_{=0} + |\langle\psi, 1\rangle_\Gamma|^2 \underbrace{\langle S_0\beta_{\rm eq}, \beta_{\rm eq}\rangle_\Gamma}_{>0} \geq c\Big(\|\psi_*\|^2_{H^{-\frac{1}{2}}(\Gamma)} + |\langle\psi, 1\rangle_\Gamma|^2\Big)\end{aligned}$$

which gives coercivity when combined with the triangle inequality

$$\|\psi\|_{H^{-\frac{1}{2}}(\Gamma)} \leq \|\psi_*\|_{H^{-\frac{1}{2}}(\Gamma)} + |\langle\psi, 1\rangle_\Gamma|\,\|\beta_{\rm eq}\|_{H^{-\frac{1}{2}}(\Gamma)} \leq C\Big(\|\psi_*\|_{H^{-\frac{1}{2}}(\Gamma)} + |\langle\psi, 1\rangle_\Gamma|\Big).$$

Exercise 3.23. Using the BIO expansion in §3.6.3 show that for a circle $\Gamma = \partial B_R$, we have $\beta_* = 0$, $\beta_{\rm eq} = \frac{1}{2\pi R}$, $S_0\beta_{\rm eq} = \frac{1}{2\pi R}S_0 1 = \frac{1}{2\pi}\log\frac{d}{R}$. Thus $d > R$ is enough to prove the coercivity of S_0 on $H^{-\frac{1}{2}}(\partial B_R)$.

Hint: use that $\int_0^{2\pi}\log\sqrt{2-2\cos\alpha}\,\mathrm{d}\alpha = 0$ and the properties of the logarithm.

Remark 3.24. The coercivity of S_0 implies that the single-layer BIE for the Laplace equation $S_0\psi = g_D$ is always well-posed. Moreover, as we are in a Lax–Milgram setting, every Galerkin-BEM discretisation of $\langle S_0\psi, \xi\rangle_\Gamma = \langle g_D, \xi\rangle_\Gamma$ is well-posed, quasi-optimal and gives symmetric positive-definite matrices. This is not true in the Helmholtz case but some conditions on the Galerkin discrete space are needed to ensure well-posedness, see [Spence14, Thm. 5.21].

3.7 The BIE zoo

We have seen that the single-layer BIE fails for some values of k. We want to derive some other BIEs that allow to compute the solution of the EDP/SSSP also for these values of k. We will write a total of six BIEs; their properties are summarised in Table 1.

First of all, it is instructive to recall how we found the BIE (38). We wrote the solution u of the EDP (29) as a single layer $u = \mathcal{S}\psi$, then we took the Dirichlet trace γ^+ of this representation, and from one of the trace formulas (55) obtained the BIE $S\psi = g_D$ (recall that we need to impose the boundary condition $\gamma^+u = g_D$). Also for the other BIEs the key steps will be the same: (i) choose a potential representation, (ii) take a trace using (55).

3.7.1 Indirect double-layer BIE

If instead of a single-layer we assume that the EDP solution is a double-layer potential

$$u = \mathcal{D}\psi, \qquad \psi \in H^{\frac{1}{2}}(\Gamma),$$

taking the Dirichlet trace γ^+ (55) we obtain

$$\boxed{\Big(\frac{1}{2} + D\Big)\psi = g_D} \quad \text{in } H^{\frac{1}{2}}(\Gamma), \qquad \psi \in H^{\frac{1}{2}}(\Gamma). \tag{59}$$

This is another BIE for the same BVP. Here and in the following, $\frac{1}{2}$ stands for the identity operator multiplied by the number $\frac{1}{2}$, i.e. the equation is to be read $\frac{1}{2}\psi + D\psi = g_D$.

This can be discretised with collocation-BEM or Galerkin-BEM in the same way as §3.2. We encounter a couple of difficulties. A first difference is that the singularity of D is stronger than that of S, so the quadrature requires more care.

[14] The value $\mathrm{e}^{\int_\Gamma \beta_{\rm eq}(\mathbf{y})\log|\mathbf{x}-\mathbf{y}|\,\mathrm{d}s(\mathbf{y})}$, which is independent of $\mathbf{x}$, is called "logarithmic capacity of Γ", while $\beta_{\rm eq}$ is the "equilibrium density". In 2D electrostatic, the electric charge on an isolated conductor Ω_- distributes on the boundary Γ proportionally to $\beta_{\rm eq}$, in such a way that the electrostatic potential $S_0\beta_{\rm eq}$ is constant on Γ and takes value $\mathcal{S}_0\beta_{\rm eq}$ in Ω_+.

A second difference is that this BIE is posed in $H^{\frac{1}{2}}(\Gamma)$ instead of $H^{-\frac{1}{2}}(\Gamma)$. The functions of $H^{\frac{1}{2}}(\Gamma)$ are in general not necessarily continuous, but if they are piecewise-polynomial then they must also be continuous.[15] This implies that the BEM discrete space V_N cannot be made of piecewise-constant functions. The simplest choice is to take V_N as the space of **piecewise-linear functions** on a mesh.

Is the BIE (59) well-posed? I.e. is the operator $\frac{1}{2} + D$ invertible? It is possible to prove that the this operator is Fredholm. To study injectivity, once again we have to look at some interior problem.

Exercise 3.25. Show the following.

- If w is a Neumann–Laplace eigenfunction in Ω_- for k, then its trace $\psi = \gamma^- w$ satisfies $\frac{1}{2}\psi + D\psi = 0$.

  Hint: use Green's representation.
- If k^2 is not a Neumann eigenvalue in Ω_-, then $\frac{1}{2} + D$ is injective.

  Hint: take $u = \mathcal{D}\psi$ for $\psi \in \ker(\frac{1}{2} + D)$. Use the well-posedness of the EDP and both jump relations.
- Deduce that the BIE (59) is injective if and only if k^2 is not a Neumann eigenfunction.

From this exercise it follows that the BIE (59) is well-posed except for a discrete set of frequencies.

Remark 3.26 (What is ψ?)**.** If ψ is solution of (59), then $u^- = (\mathcal{D}\psi)|_{\Omega_-}$ is Helmholtz solution in Ω_- with Neumann trace $\partial_{\mathbf{n}}^- u^- = \partial_{\mathbf{n}}^+ u$. Differently from §3.5.1, this is not immediately related to the incoming field u^{Inc}. However, if k^2 is not a Neumann eigenvalue, u^- is well-defined as a solution of an interior Neumann problem. From the jump relation (56), $\psi = [\![\gamma \mathcal{D}\psi]\!] = \gamma^+ u - \gamma^- u^- = g_D - \gamma^- u^-$. The solution of the BIE (59) is the difference between the datum g_D and the trace of the solution of an auxiliary interior Neumann problem, whose boundary datum is a trace of u itself.

Remark 3.27. Using $\frac{\partial}{\partial z} H_0^{(1)}(z) = -H_1^{(1)}(z)$, we can write more explicit formulas for the double-layer and the adjoint double-layer operators:

$$(D\psi)(\mathbf{x}) = \frac{\mathrm{i}k}{4} \int_\Gamma H_1^{(1)}(k|\mathbf{x}-\mathbf{y}|) \frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \psi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}),$$
$$(D'\psi)(\mathbf{x}) = \frac{\mathrm{i}k}{4} \int_\Gamma H_1^{(1)}(k|\mathbf{x}-\mathbf{y}|) \frac{(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}(\mathbf{x})}{|\mathbf{x}-\mathbf{y}|} \psi(\mathbf{y})\, \mathrm{d}s(\mathbf{y}).$$

From this formulas, we see that if Γ is a polygon and E is one of its edges, the points $\mathbf{y} \in E$ do not contribute to the computation of $(D\psi)(\mathbf{x})$ for $\mathbf{x} \in E$, because $(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y}) = 0$. So $H_1^{(1)}$ is evaluated only for $|\mathbf{x}-\mathbf{y}| > \mathrm{dist}(\mathbf{x}, \partial E)$, i.e. away from the singularity. The singularity in the integrand has to be treated carefully when $\mathbf{x}$ is very close to a corner, while it is harmless otherwise. If Γ is smooth then $\frac{(\mathbf{x}-\mathbf{y})\cdot\mathbf{n}(\mathbf{y})}{|\mathbf{x}-\mathbf{y}|} \to 0$ for $\mathbf{y} \to \mathbf{x}$, partially compensating the strong singularity of $H_1^{(1)}$.

3.7.2 Direct BIE

We have constructed two BIEs (38) and (59) by searching for EDP solutions in the form $u = \mathcal{S}\psi$ and $u = \mathcal{D}\psi$, respectively. Green's representation formula (47) allows to represent any radiating solution u in Ω_+ as linear combination of both potentials applied to the traces of u itself:

$$u = -\mathcal{S}\partial_{\mathbf{n}}^+ u + \mathcal{D}\gamma^+ u.$$

When u is solution of the EDP, one of the traces is given: $\gamma^+ u = g_D$. So we can choose as unknown the other one: $\psi = \partial_{\mathbf{n}}^+ u$. How to obtain a BIE from this?

Taking the Dirichlet and the Neumann traces γ^+ and $\partial_{\mathbf{n}}^+$ of Green's representation and using the trace formulas (55), we obtain

$$\boxed{S\psi = \Big(D - \frac{1}{2}\Big) g_D} \qquad \text{in } H^{\frac{1}{2}}(\Gamma), \tag{60}$$

$$\boxed{\Big(\frac{1}{2} + D'\Big)\psi = H g_D} \qquad \text{in } H^{-\frac{1}{2}}(\Gamma). \tag{61}$$

[15] Exercise: compute the Fourier series of $\chi(t) = \begin{cases} 1 & 0 < t < \pi, \\ 0 & \pi < t < 2\pi. \end{cases}$ Show that $\chi \in H^s(\mathbb{S}^1)$ if and only if $s < \frac{1}{2}$.

Can you find a $v \in H^{\frac{1}{2}}(\mathbb{S}^1) \setminus C^0(\mathbb{S}^1)$?

We reiterate that here the unknown stands for the Neumann datum and the underlying representation formula is that coming from Green's formula:

$$\psi = \partial_{\mathbf{n}}^{+} u \quad \in H^{-\frac{1}{2}}(\Gamma), \qquad u = -\mathcal{S}\psi + \mathcal{D}g_D \quad \text{in } \Omega_+.$$

Some terminology. BIEs such as (60)–(61) whose unknown is the missing Cauchy datum are called **direct** BIEs; BIEs such as (38) and (59) where the unknown is not directly linked to the EDP are called **indirect**. BIEs (38) and (60) are called BIEs of the **first kind**, as the unknown ψ only appears as argument of a BIO, while (59) and (61) are called of the **second kind** as the unknown ψ also appears outside the integral operator (typically a linear BIE $\alpha\psi + T\psi = f$ for a BIO T and data f is called of the first kind if $\alpha = 0$ and of the the second kind if α is a non-zero coefficient).

The first-kind direct BIE (60) has at the left-hand side the same operator S as the indirect BIE (38) we know well. The right-hand side instead is slightly more complicated, as it involves the double-layer operator. So (60) is well-posed exactly when (38) is, i.e. away from Dirichlet eigenvalues. The matrix $\underline{\underline{\mathbf{A}}}^{C/G}$ of a BEM implementation for this formulation is identical to the matrix for the same method applied to (38). The right-hand side vector $\mathbf{F}^{C/G}$ is slightly more complicated to code and more expensive to compute as it requires the implementation of the double-layer operator.

The second-kind direct BIE (61) has at the left-hand side the *adjoint* of the operator present in the indirect BIE (59). Theorem 1.28 of [CK1] implies that $(\frac{1}{2} + D')$ is injective if and only if $(\frac{1}{2} + D)$ is injective (this requires the Fredholm property of the operators). So (61) is well-posed away from Neumann eigenvalues, exactly as (59). The implementation of a BEM discretisation of (61) also requires an approximation of the hypersingular operator H for the right-hand side.

3.7.3 Indirect combined field integral equation: Brakhage–Werner equation

We have seen four different BIEs for the same EDP, and none of them is invertible for *all* positive values of k, which is quite disappointing. However all is not lost: the formulations considered were deduced from some special representations of u in terms of layer potentials, we need to choose some other such representation.

We now choose, arbitrarily, to search for some u in the form

$$u = (\mathcal{D} - \mathrm{i}\eta\mathcal{S})\psi, \qquad \psi \in H^{\frac{1}{2}}(\Gamma) \tag{62}$$

where $\eta > 0$ is a parameter. Taking the Dirichlet trace, this is solution of the EDP if ψ is solution of

$$\boxed{\left(\frac{1}{2} + D - \mathrm{i}\eta S\right)\psi = g_D} \quad \text{in } H^{\frac{1}{2}}(\Gamma). \tag{63}$$

The operator $A := (\frac{1}{2} + D - \mathrm{i}\eta S) : H^{\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$ is Fredholm. Is it injective?

Let $A\psi = 0$ for some $\psi \in H^{\frac{1}{2}}(\Gamma)$. Define u as in (62). Then $u|_{\Omega_+}$ is solution of the EDP with $g_D = 0$, so $u = 0$ in Ω_+. The jump relations (56) give

$$-\gamma^{-}u = [\![\gamma u]\!] = [\![\mathcal{D}\psi]\!] = \psi, \qquad -\partial_{\mathbf{n}}^{-}u = [\![\partial_{\mathbf{n}} u]\!] = [\![-\mathrm{i}\eta\partial_{\mathbf{n}}\mathcal{S}\psi]\!] = \mathrm{i}\eta\psi \quad \Rightarrow \quad \partial_{\mathbf{n}}^{-}u - \mathrm{i}\eta u = 0.$$

So $u|_{\Omega_-}$ is solution of a homogeneous impedance BVP (25) in Ω_- with $\vartheta = \frac{\eta}{k}$. From the well-posedness of the impedance BVP (§2.3 and Corollary 3.8) $u = 0$ and from the jump relation again $\psi = -\gamma^{-}u = 0$. We conclude that the operator A is injective.

The BIE (63) is well-posed for all Γ, $k > 0$, $\eta > 0$ and $g_D \in H^{\frac{1}{2}}(\Gamma)$.

We have finally found a BIE that is invertible for all wavenumbers! The BIE (63) is often called **Brakhage–Werner equation** (even if it was introduce independently in three papers by Brakhage and Werner, by Leis and by Panič, all in 1965).

Exercise 3.28. Let ψ be the solution of the Brakhage–Werner BIE (63) and $u = (\mathcal{D} - \mathrm{i}\eta\mathcal{S})\psi$ in $\Omega_+ \cup \Omega_-$. Use the trace formulas (55) to show that $\partial_{\mathbf{n}}^{-}u - \mathrm{i}\eta\gamma^{-}u = \partial_{\mathbf{n}}^{+}u - \mathrm{i}\eta\gamma^{+}u$ (careful with the signs!).

Deduce, using the jump relations, that the solution ψ of the BIE is the jump between the Dirichlet traces of the EDP solution $u|_{\Omega_+}$ and the solution $u|_{\Omega_-}$ of an impedance BVP in Ω_- with data $\partial_{\mathbf{n}}^{+}u - \mathrm{i}\eta\gamma^{+}u$ and impedance parameter $\vartheta = \frac{\eta}{k}$.

3.7.4 Direct combined field integral equation: Burton–Miller equation

Can we find a *direct* method that is well-posed for all values of k? We know that $\psi = \partial_{\mathbf{n}}^+ u$ solves both direct equations (60)–(61). We take a linear combination of the two equations:

$$\boxed{\Big(\frac{1}{2} + D' - \mathrm{i}\eta S\Big)\psi = \Big[H - \mathrm{i}\eta\Big(D - \frac{1}{2}\Big)\Big] g_D} \quad \text{in } H^{-\frac{1}{2}}(\Gamma). \tag{64}$$

Again, here $\eta > 0$ is a parameter. This is called **Burton–Miller** or (direct) **combined field** integral equation (CFIE). This is a second-kind direct equation, so, as in §3.7.2, the density and the representation formula are

$$\psi = \partial_{\mathbf{n}}^+ u \quad \in H^{-\frac{1}{2}}(\Gamma) \qquad \text{and} \qquad u = -\mathcal{S}\partial_{\mathbf{n}}^+ u + \mathcal{D}\gamma^+ u = -\mathcal{S}\psi + \mathcal{D} g_D \quad \text{in } \Omega_+.$$

The operator $A' := (\frac{1}{2} + D' - \mathrm{i}\eta S) : H^{-\frac{1}{2}}(\Gamma) \to H^{-\frac{1}{2}}(\Gamma)$ at the left-hand side differs from the operator A of the Brakhage–Werner equation only in that D is substituted by D'.

To study the injectivity of A', let $A'\psi = 0$ and $u = -\mathcal{S}\psi$. Then $\partial_{\mathbf{n}}^- u - \mathrm{i}\eta\gamma^- u = -A'\psi = 0$, so $u = 0$ in Ω_- by the well-posedness of the homogeneous interior impedance BVP. By the jump formula $\gamma^+ u = \gamma^- u$, $u|_{\Omega_+}$ is solution of the homogeneous EDP, so it also vanish, and $\psi = -[\![\partial_{\mathbf{n}} \mathcal{S}\psi]\!] = [\![u]\!] = 0$.

Similarly to the previous section, A' is also Fredholm, thus the BIE (64) is well-posed.

Remark 3.29. How to choose the parameter $\eta > 0$ in (63) or (64)? From the expression of A and A', we can guess that η has the dimension of the inverse of a length: η multiplies the operator S which acts as the inverse of a derivation ($S : H^{-\frac{1}{2}}(\Gamma) \to H^{\frac{1}{2}}(\Gamma)$) and is added to the identity. So a plausible choice is $\eta \sim k$. It turns out that $\eta = k$ is also a good choice to reduce the condition number of a BEM discretisation of either (63) or (64) for large values of k.

Remark 3.30 (Variational formulations of II kind BIEs)**.** In the first-kind equations (38) and (60) the operator to be inverted is S, which maps $H^{-\frac{1}{2}}(\Gamma)$ (the space where we look for the unknown) to its dual $H^{\frac{1}{2}}(\Gamma)$. So testing the BIEs against elements of the same space is simple: the sesquilinear form $\mathcal{A}_S(\psi, \xi) = \langle S\psi, \xi\rangle_\Gamma$ is well-defined for $\psi, \xi \in H^{-\frac{1}{2}}(\Gamma)$ and involves the extension $\langle\cdot,\cdot\rangle_\Gamma$ of the $L^2(\Gamma)$ scalar product. This is why in the implementation of the Galerkin-BEM matrix $\underline{\underline{\mathbf{A}}}^G$ we are allowed to use integrals over Γ (recall that we also chose basis functions in $L^2(\Gamma)$).

For the second-kind integral equations we have to be more careful. E.g., in (64), the operator A' maps $H^{-\frac{1}{2}}(\Gamma)$ to itself. So, we cannot write $\langle A'\psi, \xi\rangle_\Gamma$ for $\psi, \xi \in H^{-\frac{1}{2}}(\Gamma)$ because it is not defined, but should use the sesquilinear form $(A'\psi, \xi)_{H^{-\frac{1}{2}}(\Gamma)}$, where $(\cdot,\cdot)_{H^{-\frac{1}{2}}(\Gamma)}$ is the scalar product in $H^{-\frac{1}{2}}(\Gamma)$. Implementing a BEM discretisation of this variational problem is hard, as it requires to evaluate the non-local $H^{-\frac{1}{2}}(\Gamma)$ scalar product.[16]

On the other hand, if the EDP datum g_D is at least in $H^1(\Gamma)$, which is the case for smooth incoming waves, then the right-hand side $[H - \mathrm{i}\eta(D - \frac{1}{2})]g_D \in L^2(\Gamma)$ so we can use the variational formulation

$$(A'\psi, \xi)_{L^2(\Gamma)} = \int_\Gamma \Big(\frac{1}{2} + D' - \mathrm{i}\eta S\Big)\psi\, \overline{\xi}\,\mathrm{d}s = \int_\Gamma \Big[H - \mathrm{i}\eta\Big(D - \frac{1}{2}\Big)\Big] g_D \overline{\xi}\,\mathrm{d}s \qquad \forall \xi \in L^2(\Gamma).$$

This is well-posed because A' is Fredholm also as a mapping $A' : L^2(\Gamma) \to L^2(\Gamma)$. The Galerkin-BEM is then implemented with the same techniques of §3.2.

An alternative would be to use a Petrov–Galerkin approach: take test functions in a space different from the trial space. For example one can seek $\psi \in H^{-\frac{1}{2}}(\Gamma)$ such that $\langle A'\psi, \xi\rangle_\Gamma = \langle [H - \mathrm{i}\eta(D - \frac{1}{2})]g_D, \xi\rangle_\Gamma$ for all $\xi \in H^{\frac{1}{2}}(\Gamma)$. The Galerkin-BEM discretisation of this problem can use piecewise-constant trial functions ψ_N and continuous piecewise-linear test functions ξ_N. The two discrete spaces have to be defined on different compatible "dual" meshes.

Remark 3.31 (Advantages of direct formulation)**.** The BEM approximation of Burton–Miller equation (64) is slightly more complicated and expensive than Brakhage–Werner (63), as the right-hand side involves two BIOs. Similarly, the direct equations (60)&(61) require a more complicated right-hand side and representation formula than (38)&(59). What is the advantage of a direct formulation against an indirect one?

[16] On a circular boundary $\Gamma = \mathbb{S}^1$, this would be easier as the scalar product is computed from the circular harmonics expansion: $(\sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \mathrm{e}^{\mathrm{i}\ell\theta}, \sum_{\ell\in\mathbb{Z}} \hat{w}_\ell \mathrm{e}^{\mathrm{i}\ell\theta})_{H^{-\frac{1}{2}}(\mathbb{S}^1)} = 2\pi \sum_{\ell\in\mathbb{Z}} \hat{v}_\ell \overline{\hat{w}_\ell} (1+\ell^2)^{\frac{1}{2}}$ (recall definition (17)). Still, unless the Fourier coefficients of all the functions involved are already known, this is more complicated than the simple integral appearing in the $\langle\cdot,\cdot\rangle_\Gamma$ duality.

In a direct formulation, if we have some information on the properties of the EDP solution u we can include it in the design of the approximating space V_N to improve its accuracy and efficiency. For instance, in some situations, PDE theory and high-frequency asymptotics permit to estimate the location and the strength of the singularities of $\psi = \partial_{\mathbf{n}}^+ u^{\mathrm{Scat}}$, its oscillations, the different behaviour in the shadow and the illuminated parts of Γ; see, e.g., Figure 12[17]. This knowledge allows to construct discrete spaces V_N that ensure high accuracy with small numbers of DOFs.

In an indirect method, the BIE solution ψ depends also on the trace of some eigenvalue problem (see Rem 3.26 and Ex. 3.28), so its efficient approximation would require also the knowledge of the corresponding eigenfunction, which is not directly related to the physical scattering problem and might contain expensive-to-approximate "unphysical" singularities.

Moreover, often the quantity of interest is not u^{Scat} or u^{Tot} in Ω_+, but something dependent on $\partial_{\mathbf{n}}^+ u^{\mathrm{Scat}}$, such as the far-field pattern u_∞ (31). This is easily and accurately computed with a direct method or with (38), but not with (59) and (63).

	BIE	representation formula	density $\psi =$	unknown ψ in	direct/ indirect	kind	fails for
(38)	$S\psi = g_D$	$u = \mathcal{S}\psi$	$-\partial_{\mathbf{n}}^+ u^{\mathrm{Tot}}$	$H^{-\frac{1}{2}}(\Gamma)$	indirect	I	Dir. eig.
(59)	$(\frac{1}{2} + D)\psi = g_D$	$u = \mathcal{D}\psi$	Rem.3.26	$H^{\frac{1}{2}}(\Gamma)$	indirect	II	Neum. eig.
(60)	$S\psi = (D - \frac{1}{2})g_D$	$u = \mathcal{D}g_D - \mathcal{S}\psi$	$\partial_{\mathbf{n}}^+ u$	$H^{-\frac{1}{2}}(\Gamma)$	direct	I	Dir. eig.
(61)	$(\frac{1}{2} + D')\psi = Hg_D$	$u = \mathcal{D}g_D - \mathcal{S}\psi$	$\partial_{\mathbf{n}}^+ u$	$H^{-\frac{1}{2}}(\Gamma)$	direct	II	Neum. eig.
(63)	$(\frac{1}{2} + D - \mathrm{i}\eta S)\psi = g_D$	$u = (\mathcal{D} - \mathrm{i}\eta\mathcal{S})\psi$	Ex. 3.28	$H^{\frac{1}{2}}(\Gamma)$	indirect	II	never!
(64)	$(\frac{1}{2}+D'-\mathrm{i}\eta S)\psi$ $=[H-\mathrm{i}\eta(D-\frac{1}{2})]g_D$	$u = \mathcal{D}g_D - \mathcal{S}\psi$	$\partial_{\mathbf{n}}^+ u$	$H^{-\frac{1}{2}}(\Gamma)$	direct	II	never!

Table 1: Six BIEs for the EDP (29).
They are all described in [CK1]: (38) (3.44), (59) (3.26), (60) (3.83), (61) (3.81), (63) (3.51), (64) (3.84).

Remark 3.32. In these notes we have considered exterior Dirichlet BVPs for the 2D Helmholtz equation. However BIEs and BEM have a much broader range of applicability. They can be used to model, analyse and approximate Helmholtz BVPs posed on bounded domains, on domains with unbounded boundaries, with other boundary conditions (Neumann, impedance, mixed), transmission problems (i.e. multiple Helmholtz equations with different wavenumbers on different domains, coupled by Dirichlet and Neumann conditions),....

BIEs can be used for much more general linear PDEs (of all kinds: elliptic, parabolic and hyperbolic), including systems of PDEs such as those of elasticity and electromagnetism (recall §1.1.3–1.1.4). The main requirement for implementing a BEM is that the fundamental solution of the problem is known, either in exact or approximate form.

A BEM can be coupled with a FEM (or another volume-based method) for approximating problems with different physical models in different subdomains. Typically, FEMs are used in small regions of high geometric complexity, variable coefficients or nonlinearities, and BEMs are used to deal with unbounded regions where coefficients are constant.

Another important numerical method for the discretisation of BIEs is Nyström method, which can converge extremely fast for smooth scatterers; see [CK2, §3.5] for the application to 2D Brakhage–Werner equation or [Sayas15, pp. 33 and 36] for the Laplace case.

Plenty of information can be found in the references in the bibliography and in the books mentioned in Remark 3.17.

[17] Figure 12 shows the density ψ for the indirect BIE (38). However we have seen in §3.5.1 that the solution of this BIE has the physical interpretation $\psi = -\partial_{\mathbf{n}}^+ u^{\mathrm{Tot}}$, while this is not true for the solutions of the other indirect BIEs (59) and (63).

A Useful calculus formulas and notation

$$B_R(\mathbf{x}) := \{\mathbf{y} \in \mathbb{R}^n : \ |\mathbf{y}-\mathbf{x}| < R\}, \qquad B_R := B_R(\mathbf{0}),$$

$$\mathrm{e}^{\mathrm{i}z} = \cos z + \mathrm{i}\sin z, \qquad \cos z = \frac{\mathrm{e}^{\mathrm{i}z}+\mathrm{e}^{-\mathrm{i}z}}{2}, \qquad \sin z = \frac{\mathrm{e}^{\mathrm{i}z}-\mathrm{e}^{-\mathrm{i}z}}{2\mathrm{i}},$$

$$\mathbf{v}\times\mathbf{w} := (v_2w_3 - v_3w_2,\ v_3w_1 - v_1w_3,\ v_1w_2 - v_2w_1), \qquad \mathbf{u}\times(\mathbf{v}\times\mathbf{w}) = \mathbf{v}(\mathbf{u}\cdot\mathbf{w}) - \mathbf{w}(\mathbf{u}\cdot\mathbf{v}),$$

$$\nabla u := \Big(\frac{\partial u}{\partial x_1},\ \ldots,\ \frac{\partial u}{\partial x_n}\Big),$$

$$\operatorname{div}\mathbf{v} := \nabla\cdot\mathbf{v} := \frac{\partial v_1}{\partial x_1} + \cdots + \frac{\partial v_n}{\partial x_n},$$

$$\Delta u := \nabla^2 u := \operatorname{div}(\nabla u) \quad = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2} \quad \overset{\text{if } n=2}{=} \quad \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} \quad = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2},$$

$$\operatorname{curl}\mathbf{v} := \nabla\times\mathbf{v} := \Big(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3},\ \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1},\ \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\Big),$$

$$\operatorname{curl}\nabla u = \mathbf{0}, \qquad \operatorname{div}\operatorname{curl}\mathbf{v} = 0,$$

$$\operatorname{curl}\operatorname{curl}\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \boldsymbol{\Delta}\mathbf{v}$$

$$= \Big(\tfrac{\partial^2 v_2}{\partial x_1\partial x_2} + \tfrac{\partial^2 v_3}{\partial x_1\partial x_3} - \tfrac{\partial^2 v_1}{\partial x_2^2} - \tfrac{\partial^2 v_1}{\partial x_3^2},\quad \tfrac{\partial^2 v_1}{\partial x_1\partial x_2} + \tfrac{\partial^2 v_3}{\partial x_2\partial x_3} - \tfrac{\partial^2 v_2}{\partial x_1^2} - \tfrac{\partial^2 v_2}{\partial x_3^2},\quad \tfrac{\partial^2 v_1}{\partial x_1\partial x_3} + \tfrac{\partial^2 v_2}{\partial x_2\partial x_3} - \tfrac{\partial^2 v_3}{\partial x_1^2} - \tfrac{\partial^2 v_3}{\partial x_2^2}\Big).$$

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