On a conservative isogeometric scheme for the wave equation

PhD Spring Workshop 2022

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Introduction

Model problem

Let $\Omega = \mathcal{F}(\hat{\Omega})$, where $\mathcal{F}$ is an isogeometric map, and $\hat{\Omega} = [0,1]^d$, with $d \in \mathbb{N}$. Given $T \in \mathbb{R}^+$, we consider:

$$u_{tt} - \mathrm{div}(c^2 \nabla u) = 0, \text{ in } \Omega \times [0,T].$$

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Introducing $\mathbf{v} := c\nabla u$ and $\phi := u_t$, this leads to the equations:

$$\begin{cases} \mathbf{v}_t = c\nabla\phi & \text{in } \Omega \times [0,T], \\ \phi_t = \operatorname{div}(c\mathbf{v}) & \text{in } \Omega \times [0,T]. \end{cases}$$

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The problem will be find $\mathbf{v} \in V = \mathbf{H}(c, \operatorname{div}; \Omega)$ and $\phi \in Q = L^2(\Omega)$ such that:

$$\int_\Omega \mathbf{v}_t \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} = -\int_\Omega \operatorname{div}(c\mathbf{w}) \, \phi \, \mathrm{d}\mathbf{x} + \underbrace{\int_{\partial\Omega} c\phi\mathbf{w} \cdot \mathbf{n} \mathrm{d}\Gamma}_{=0} \qquad \forall \mathbf{w} \in V,$$

$$\int_\Omega \phi_t \psi \, \mathrm{d}\mathbf{x} = \int_\Omega \operatorname{div}(c\mathbf{v}) \, \psi \, \mathrm{d}\mathbf{x}, \qquad \forall \psi \in Q.$$

Isogeometric framework

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$$\mathcal{F} := \sum_{\mathrm{i} \in \mathrm{J}} \mathrm{c}_{\mathrm{i}} \hat{B}_{\mathrm{i},\mathrm{p}}.$$

a linear combination of B-splines (or NURBS) of degree p.

Example

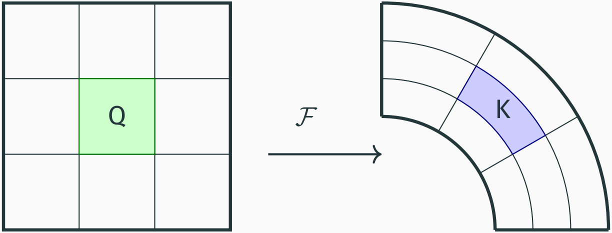

Figure 1: Mesh $\widehat{\mathcal{M}}$ in the parametric domain, and its image $\mathcal{M}$ on the physical domain.

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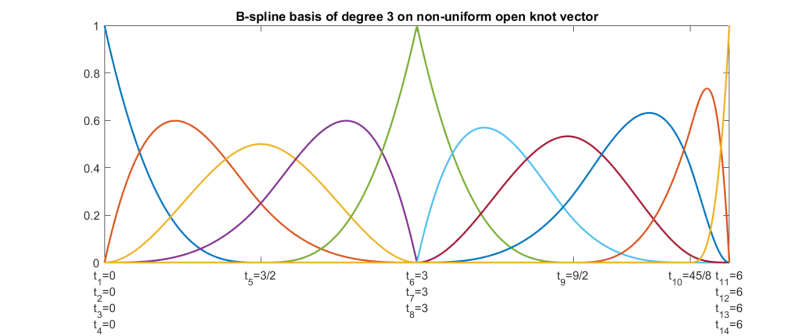

Figure 2: An example of univariate B-spline basis functions.

Project the equations

Modified problem

We introduce $\Pi^1 : V \to V_h$ and $\Pi^2 : Q \to Q_h$, and consider:

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$$\begin{aligned} \int_\Omega \mathbf{v}_t \cdot \mathbf{w} \, \mathrm{d}\mathbf{x} &= -\int_\Omega \Pi^2(\mathrm{div}(c\mathbf{w}))\phi \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{w} \in V, \\ \phi_t &= \Pi^2(\mathrm{div}(c\mathbf{v})). \end{aligned} \tag{1}$$

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We retrieve the conservation of total Energy, that is:

$$E(t) := \frac{1}{2} \int_\Omega |\mathbf{v}|^2 + \phi^2 \mathrm{d}\mathbf{x} = E_0.$$

How to choose Π^i?

We ask Π^1 and Π^2 to commute with the following diagram:

$$\begin{array}{ccc} \mathbf{H}(c, \mathrm{div}; \Omega) & \xrightarrow{\mathrm{div}} & L^2(\Omega) \\ \Big\downarrow \Pi^1 & & \Big\downarrow \Pi^2 \\ V_h & \xrightarrow{\mathrm{div}} & Q_h, \end{array} \tag{2}$$

that is

$$\Pi^2(\mathrm{div}(c\mathbf{v})) = \mathrm{div}(\Pi^1(c\mathbf{v})).$$

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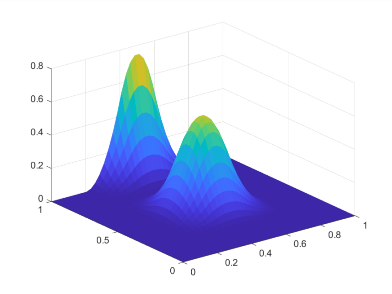
Isogeometric De Rham complex

- V_h and Q_h are suitable pushforwards of spline spaces over $\hat{\Omega}$;
- Π^i are suitable pullbacks of Tensor-Product univariate quasi-interpolants $\hat{\pi}_p$; [Beirão da Veiga et al., 2014]
- We took the quasi-interpolants described in [Lee et al., 2000]

Push-forward:



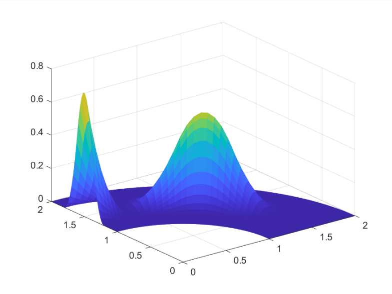


Figure 3: *Left:* two multivariate B-splines over the parametric domain $\hat{\Omega}$. *Right:* push-forward of the two B-splines with the Piola transformations.

Univariate quasi-interpolants from [Lee et al., 2000]:

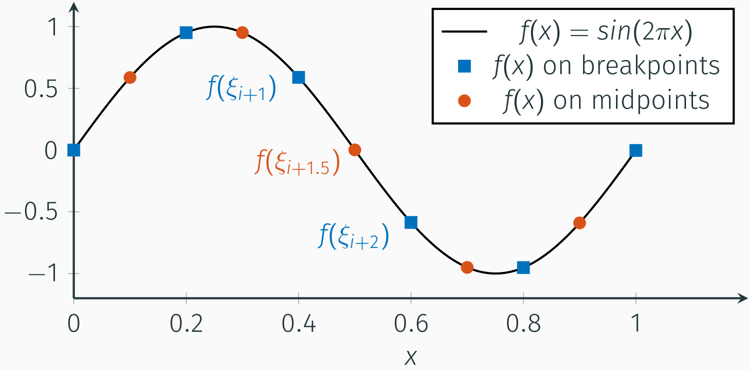


Figure 4: Example of point-wise evaluation of $f(x) = sin(2\pi x)$ for projection with $\hat{\pi}_2$. The explicit formula is $\lambda_{i,2}(f) = -\frac{1}{2}f(\xi_{i+1}) + 2f(\xi_{i+1.5}) - \frac{1}{2}f(\xi_{i+2})$.

Semi-discrete mixed formulation

The space semi-discretization of the modified problem (1) is: find $\mathbf{v} \in V_h$ and $\phi \in Q_h$, such that

$$\begin{aligned} \int_\Omega \mathbf{v}_t \cdot \mathbf{w_h} \, \mathrm{d}\mathbf{x} &= -\int_\Omega \mathrm{div}(\Pi^1(c\mathbf{w}_h))\phi \, \mathrm{d}\mathbf{x}, \quad \forall \mathbf{w}_h \in V_h, \\ \phi_t &= \Pi^2(\mathrm{div}(c\mathbf{v})). \end{aligned} \tag{3}$$

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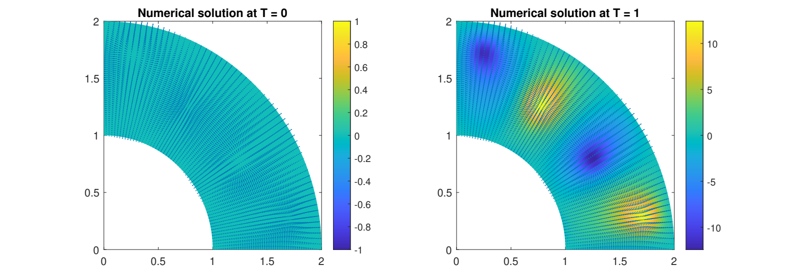
Note:

- This semi-discretization + Crank-Nicolson in time is Energy preserving;
- We are changing the test functions !

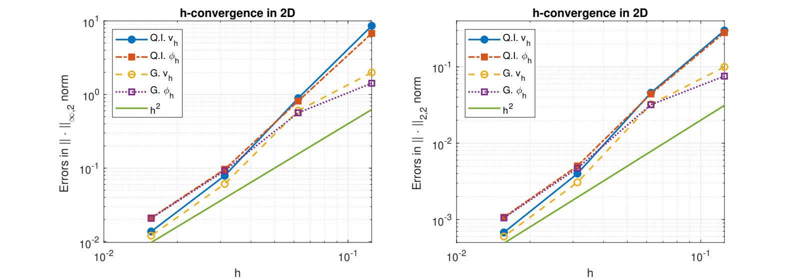
Numerical results

Solutions:



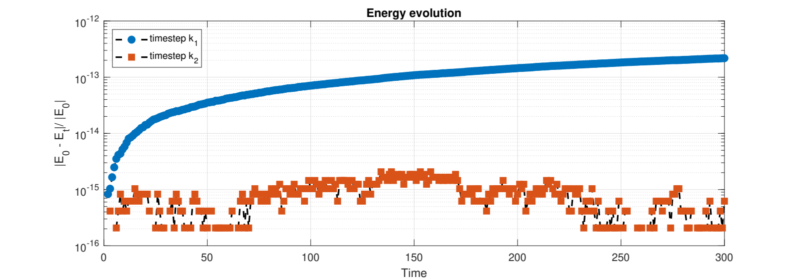
Numerical solution for Dirichlet homogeneous boundary conditions at $T = 1$. Mesh width $h = 0.0156$ and $k = 5e - 04$. Coefficient $c = sin(2\pi x_1)sin(2\pi x_2) + 2$. Solutions computed with GeoPDEs [Vázquez, 2016].

Convergence rates:



Left: errors in $\|\cdot\|_{\infty,2}$ norm for solutions with (Q.I.) and (G) methods with homogeneous Dirichlet boundary conditions. *Right:* errors with $\|\cdot\|_{2,2}$ norm for the same problems.

Energy conservation:



Energy conservation plots for Dirichlet homogeneous boundary conditions. Mesh width $h = 0.0312$, while $T = 300$ and two different partitions with $k_1 = 0.2$ and $k_2 = 0.01$.

Conclusive remarks

Relevant remarks

- We have a new discretization scheme that hides the (non-constant) coefficients into the test functions;
- We achieve optimal convergence rates as in a standard Galerking discretization;
- We preserved the total Energy of the system for long time simulations;
- To assemble the matrices, we have to compute the projections $\Pi^1(c\mathsf{w}_h)$ letting vary w_h in the set of basis functions of V_h. This projections are local.

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Open problems:

- Analytical error estimates?
- Minimal hypothesis on c?
- Parallel computation?

References

- Beirão da Veiga, L., Buffa, A., Sangalli, G., and Vázquez, R. (2014).
  Mathematical analysis of variational isogeometric methods.
  Acta Numerica, 23:157.
- Lee, B.-G., Lyche, T., and Mørken, K. (2000).
  Some examples of quasi-interpolants constructed from local spline projectors.
  Mathematical methods for curves and surfaces: Oslo, pages 243–252.
- Vázquez, R. (2016).
  A new design for the implementation of isogeometric analysis in Octave and Matlab: GeoPDEs 3.0.
  Computers and Mathematics with Applications, 72(3):523–554.


Thank
You!