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Mathematical and Numerical Challenges in Optical Tomography

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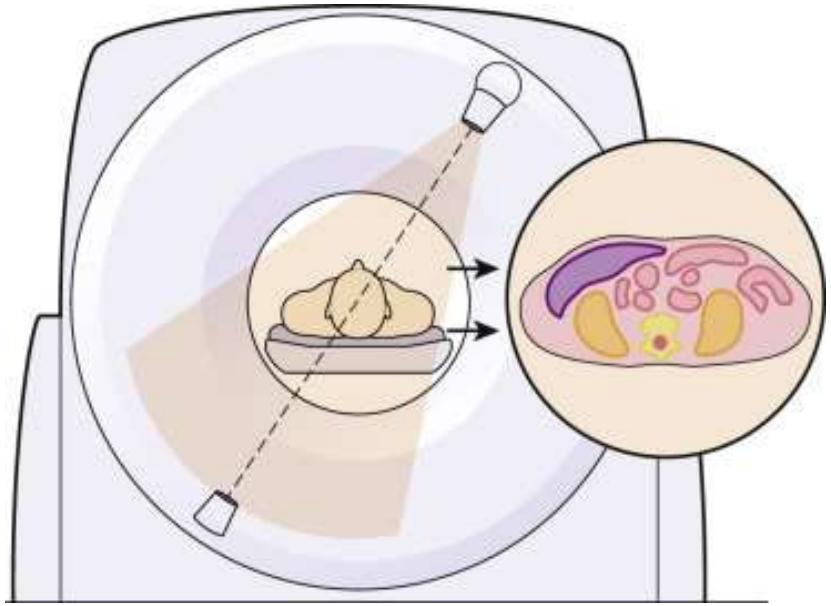
Georges de la Tour, The Education of the Virgin, ca. 1650

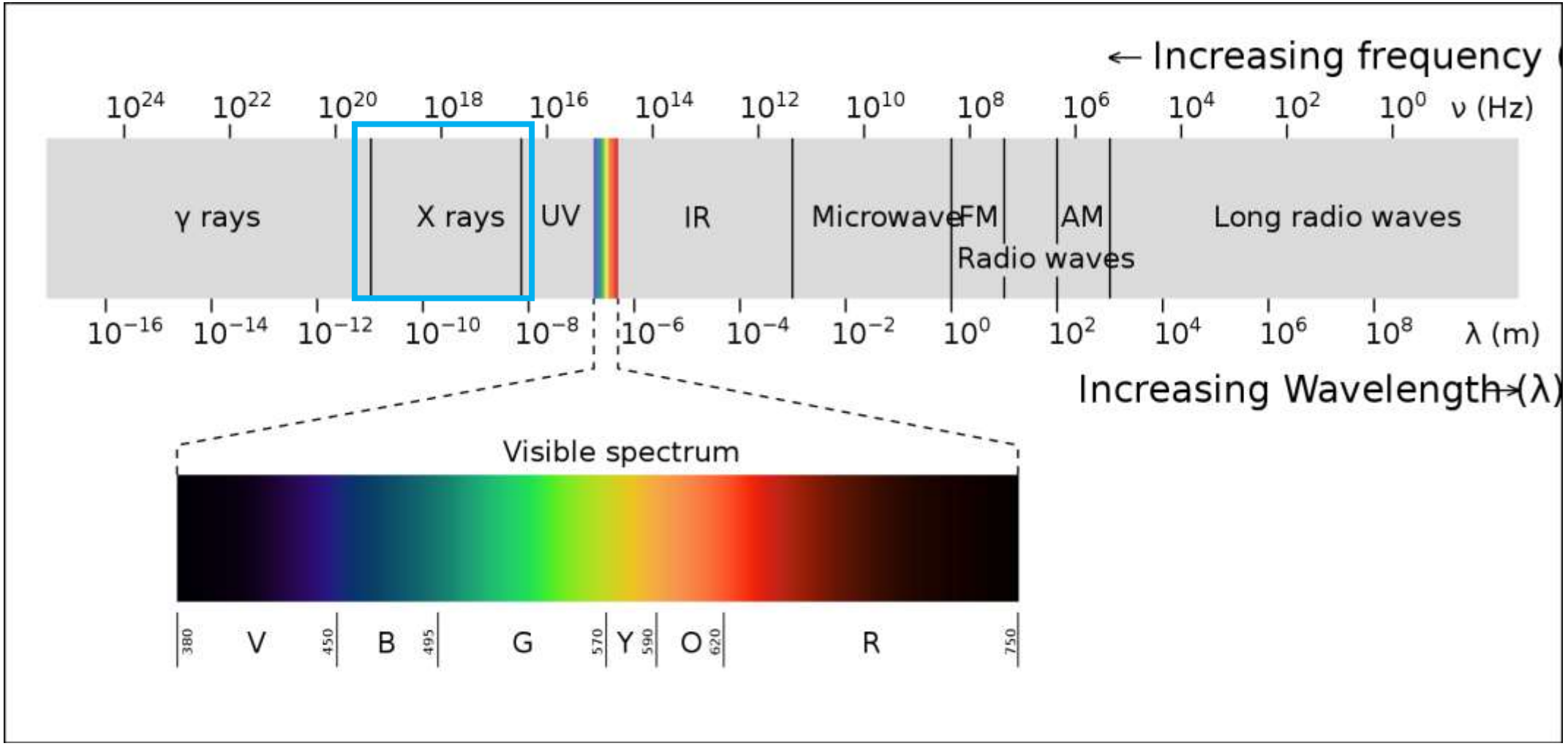
The idea of Computerized Tomography (CT)

Goal: provide spatial information about the inside of a sample by taking measurements from the outside from different perspectives

Here we focus on the application of CT to medical imaging, the technique and process of imaging the interior of a body for clinical analysis and medical intervention



The starting point: imaging via X-rays (10^{-3} to 10nm)



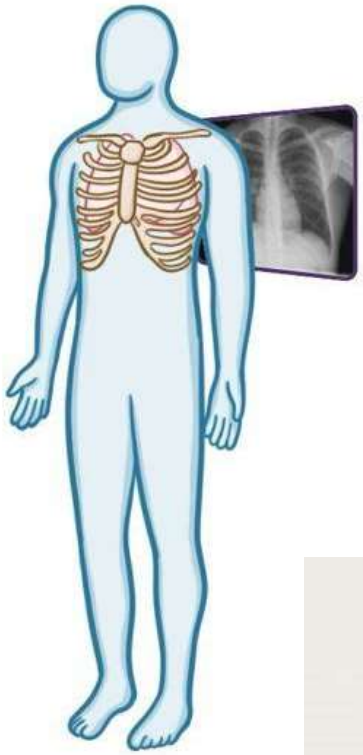
X-rays as investigating signal

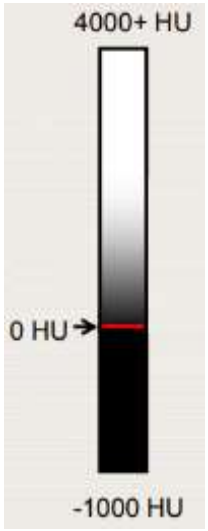
Principle: the energy of the X-ray is attenuated from the tissue that it crosses. The denser the tissue region, the higher the attenuation

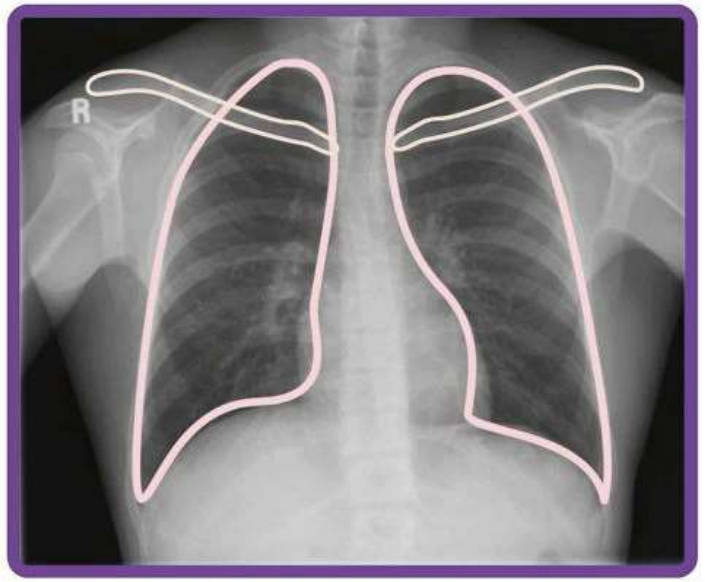
We are interested in a parameter called absorption (=attenuation) coefficient (μ_a). The result is expressed in relative Hounsfield units:

$$HU = \frac{\mu_a - \mu_{a,H_2O}}{\mu_{a,H_2O}} \times 100$$

material	HU
water	0
air	-1000
bone	1086
blood	53
muscle	41



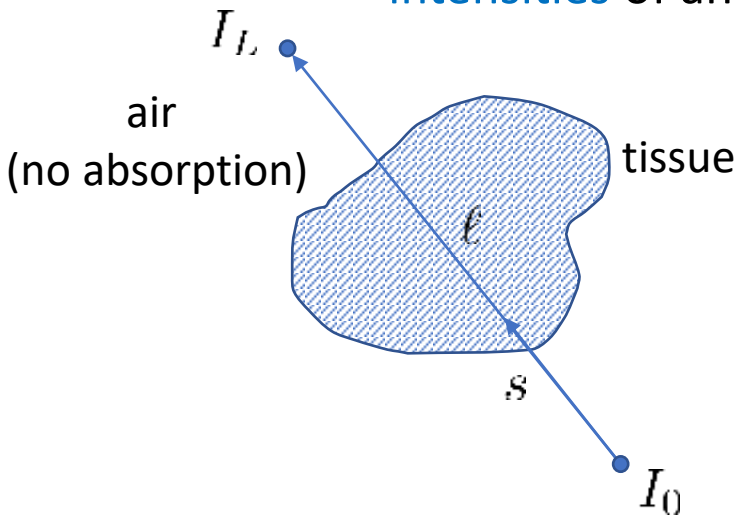




Muscles, tissue and bones attenuate differently the X-ray → different GLs: darker=less attenuation

Beer-Lambert law

The Beer-Lambert law connects the initial (known) and final (measured) intensities of an X-ray:



$$\begin{cases} \dfrac{dI}{ds} = -\mu_a I, & \mu_a = \mu_a(s) \\ I(s=0) = I_0 & \text{source} \\ I(s=L) = I_L & \text{detector} \end{cases}$$

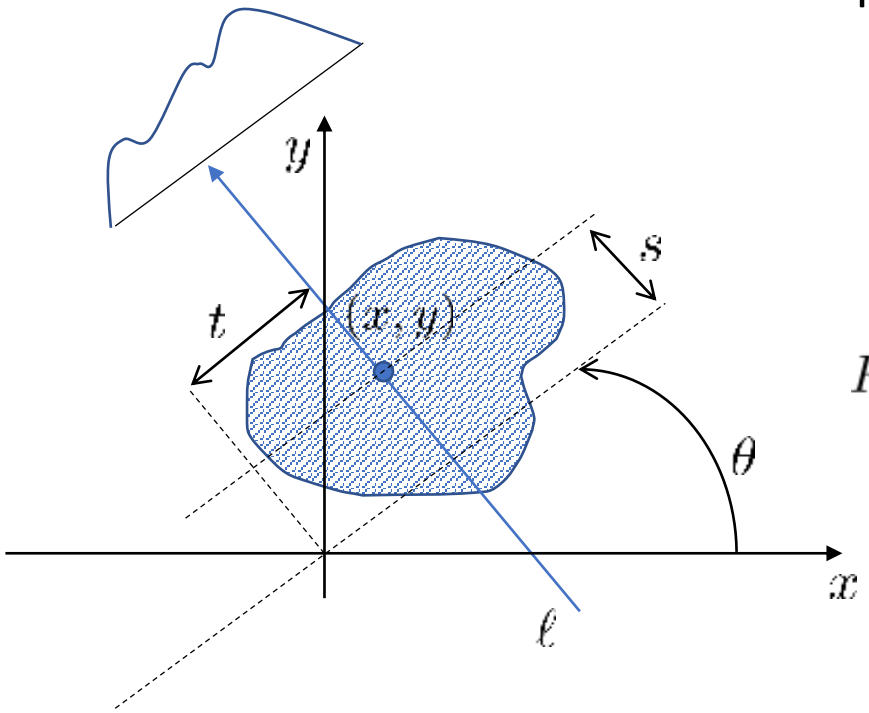
$$I(s) = I_0 \exp\left(-\int_\ell \mu_a(s)\, ds\right) \longrightarrow \int_\ell \mu_a(s)\, ds = -\log\left(\frac{I_L}{I_0}\right)$$

Pb.: we only possess line integrals of the quantity of interest

We need several projections along different directions to reconstruct the 3D spatial structure

Beer-Lambert law and Radon transform

Projection $P_\theta(t)$



The Beer-Lambert law is connected to the Radon transform

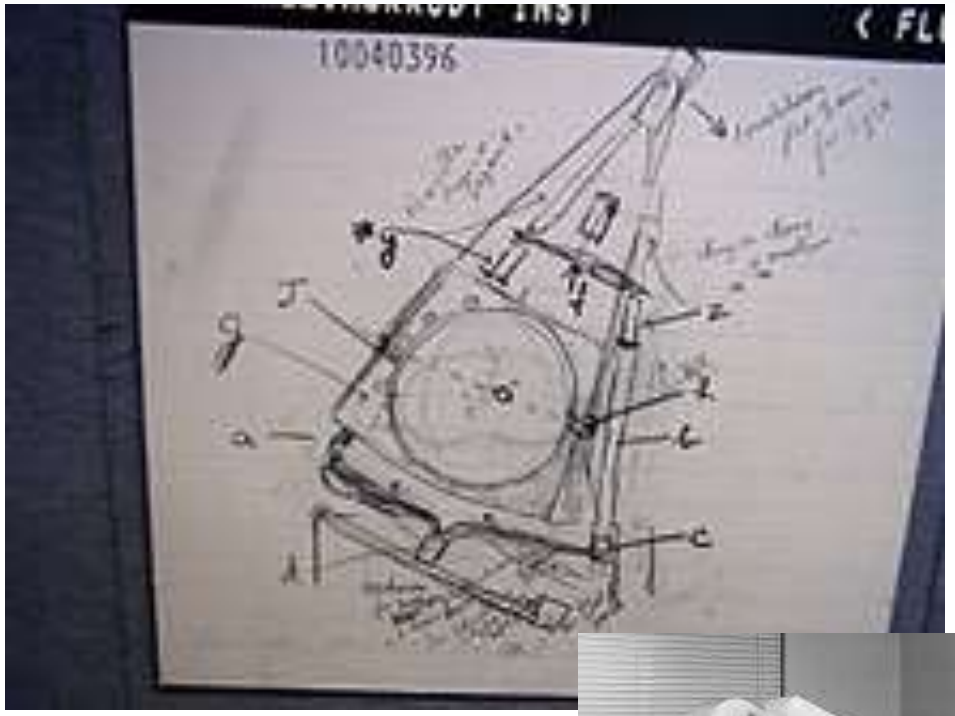
Let $f(x,y) : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ density function

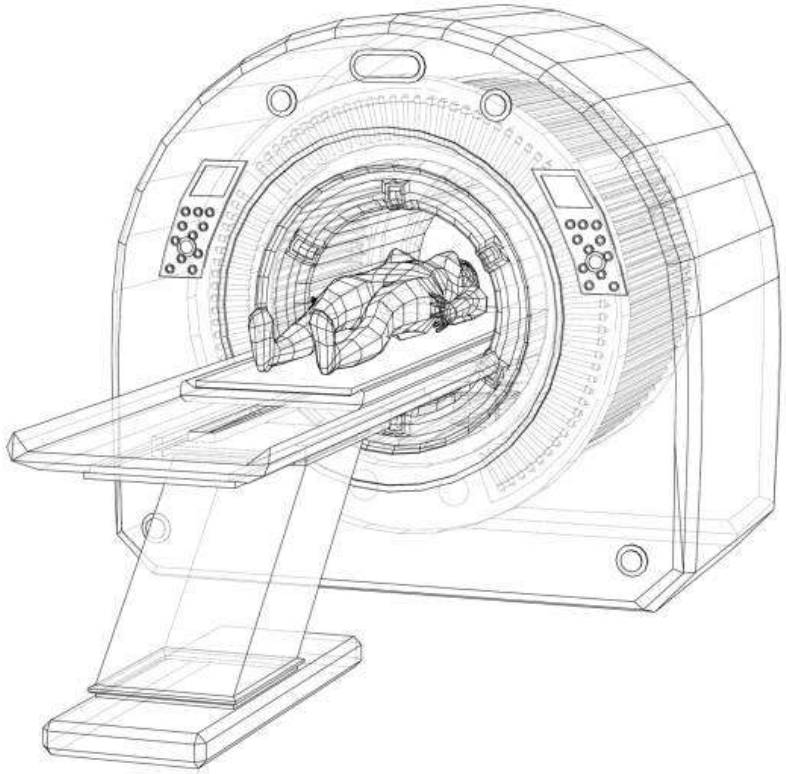
$$P_\theta(t) = \int_\ell f(x,y)\, ds = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(x,y)\delta(x\cos(\theta) + y\sin(\theta) - t)\, dxdy$$

$$\updownarrow$$

$$\int_\ell \mu_a(s)\, ds = -\log\left(\frac{I_L}{I_0}\right)$$

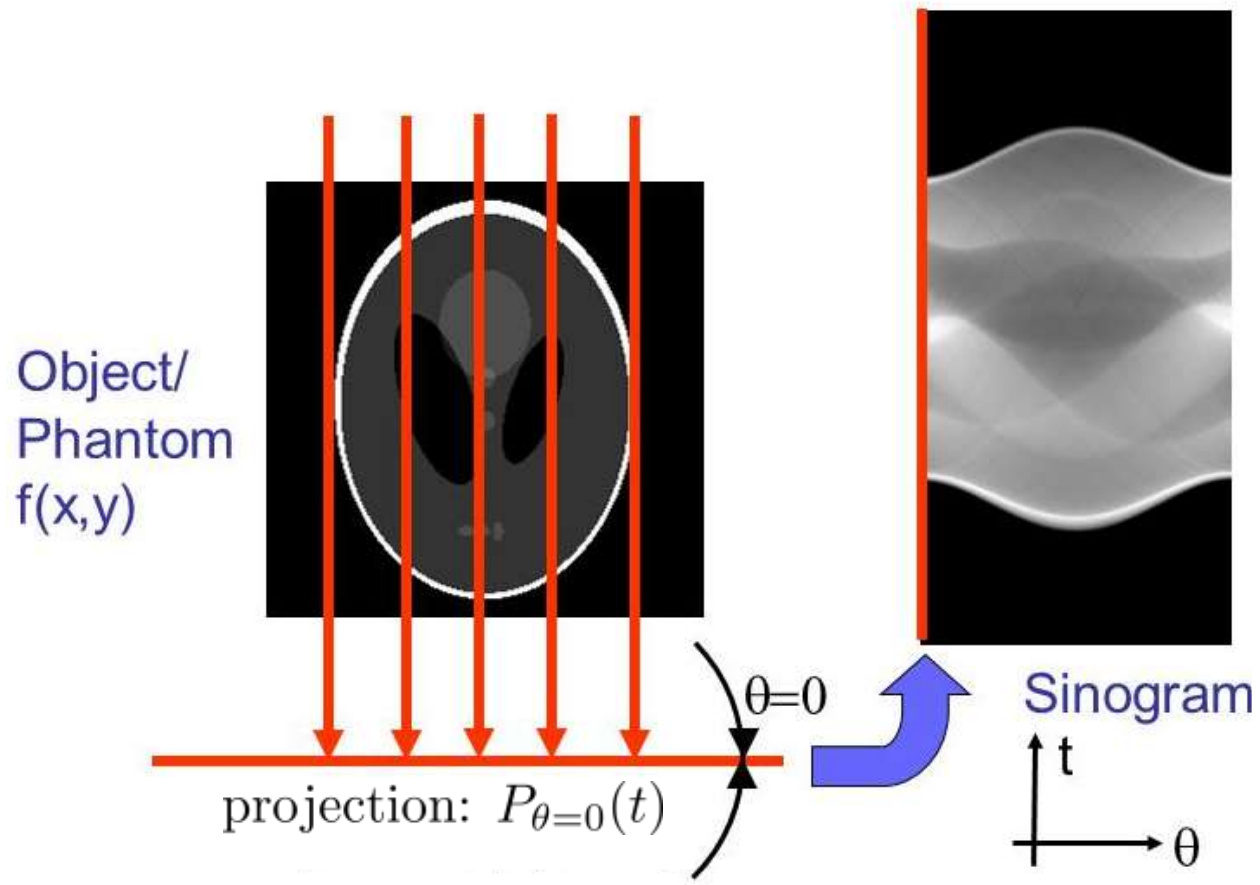
CT scan [1972 Cormack, Hounsfield - Nobel Prize 1979]





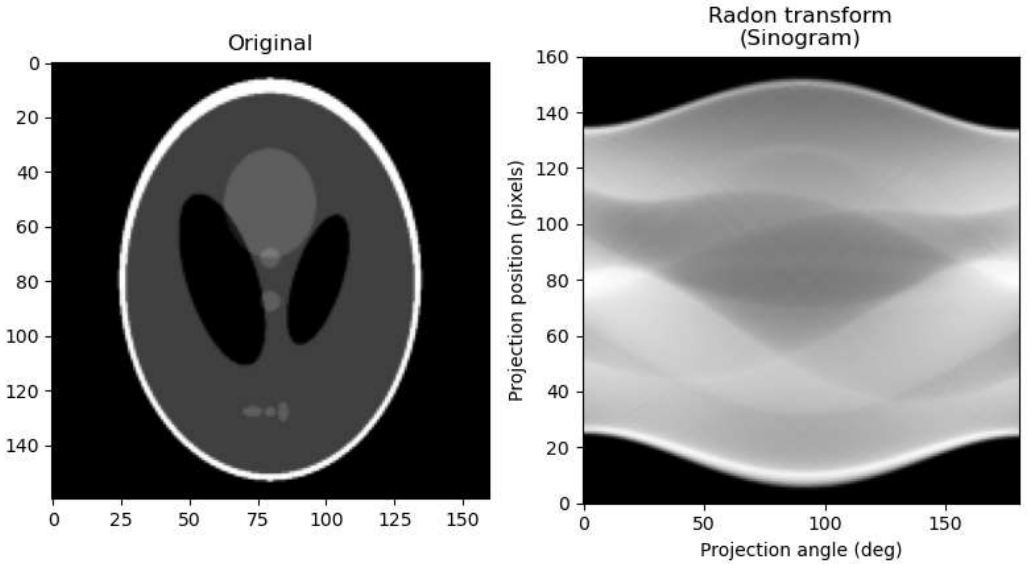
The sinogram

For every angle we obtain a projection, a collection of projections is called sinogram



‘Computational’ version

- Sinogram (measured data): already discrete (finite set of angles, finite set of detectors)
- Sample: discretize in voxels
- Various approaches to discretize (compute or approximate) the line integrals



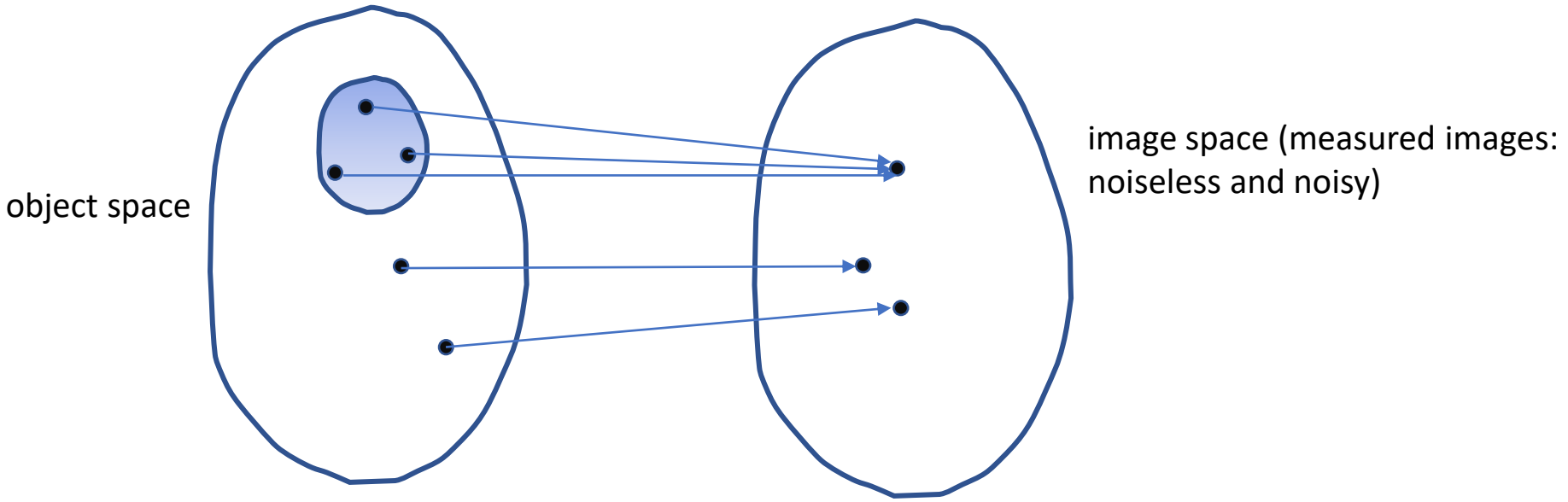
$$\int_{\mathbb{R}^2} \underbrace{\delta(x\cos(\theta)+y\sin(\theta)-t)}_{\mathbf{K}(x,y;\theta,t)} f(x,y)\,dxdy = \mathbf{y}(\theta,t)$$

$$\mathbf{Kf} = \mathbf{y}$$

Mathematical problem in CT reconstruction

Find hidden model parameters (biophysical parameters) given noisy/subsampled indirect observations

- Direct problem: oriented along a cause-effect sequence with a natural ("beneficial") loss of information. The solution is generally smoother than the data (ex: the image provided by a bandlimited system is smoother than the real object imaged)
- Inverse problem: needs to accomplish a transformation which implies a gain of information



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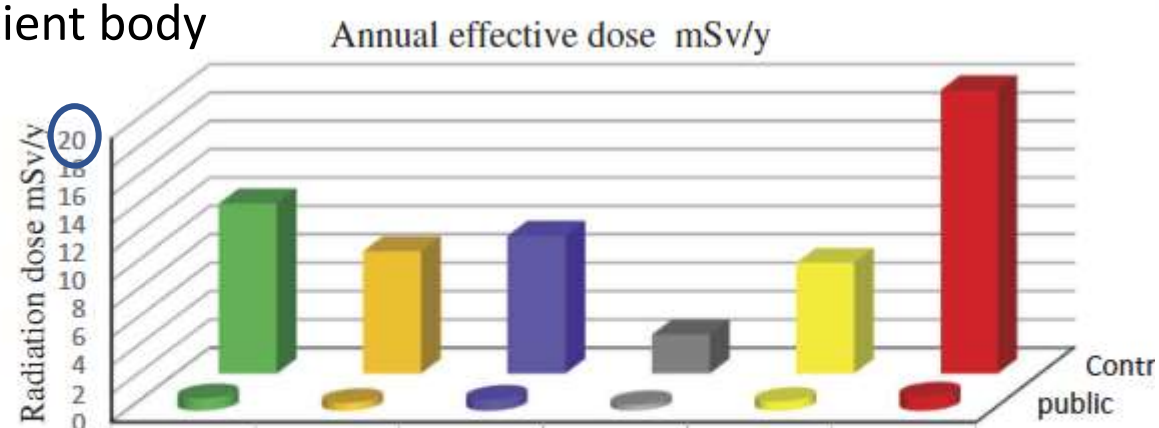
- Inverse problem: needs to accomplish a transformation which implies a gain of information

Ill-posedness:

- small errors in the data may lead to large errors in the model parameters
- several possible model parameter values consistent with the same observations

How much radiation (is too much)?

In a common CT scan in clinical practice, about 1000 angular measurements are taken by rotation of the X-ray tube around the patient body



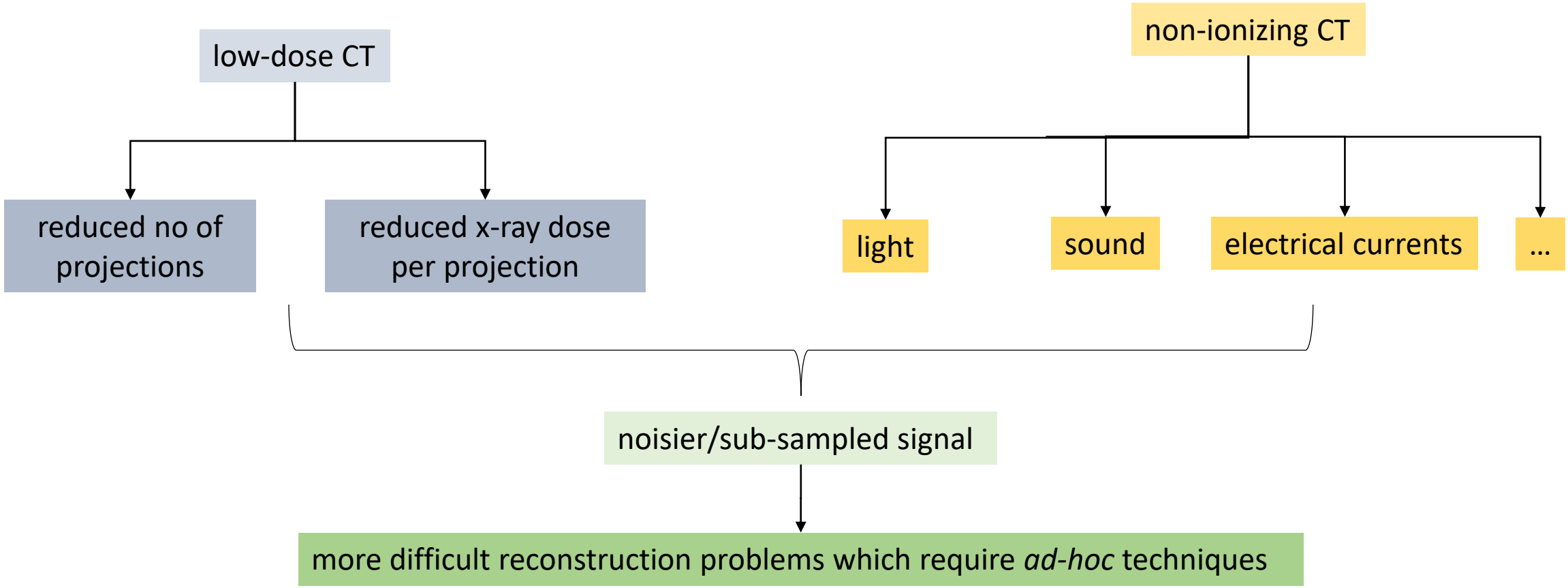
	K	C	D	Q64	Q128	ICRP
public	0.87	0.55	0.77	0.4	0.63	1
Control area	12	8.64	9.72	2.77	7.83	20

Hospitals anonymous names	Parameters		Time (s)	mAs[a]	No. of slices	$CTDI_{vol}$ (mGy)	DLP (mGy cm)	Patient effective dose in mSv
K (AS-64 slices)	Control area Control area	Head	8.14	364	48	59.5	947.2	1.99
		Chest	9.10	193	139	8.5	436.7	6.11
		Abdomen pelvis	5.71	206	168	12.1	550.3	8.25
C (GE-64 slices)		Head	7.25	330	175	55.15	1285	2.70
		Chest	2.87	650	220	8.91	505	7.07
		Abdomen pelvis	3.43	680	236	13.78	1630	24.45
Q (AS-64 slices)		Head	8.01	363	45	57	919.5	1.93
		Chest	9.22	186	139	8.46	432.7	6.06
		Abdomen pelvis	5.31	206	168	12.1	524.6	7.87
Q (ICT-128 slices)		Head	7.73	274	85	62.6	1357	2.85
		Chest	8.00	298	252	8.8	487.1	6.82
		Abdomen pelvis	5.16	486	512	13.79	1415.8	21.24
D (Nesoft-16 slices)		Head	12.3	329	53	74.84	490.6	1.03
		Chest	8.32	274	102	8.98	508.4	7.12
		Abdomen pelvis	8.7	288	40	19.95	481.4	7.22

[Evaluation of arising exposure of ionizing radiation from computed tomography and the associated health concerns JOURNAL OF RADIATION RESEARCH AND APPLIED SCIENCES, 2020]

``Sustainable`` CT

Reduce the exposure to harmful X-rays by pursuing:



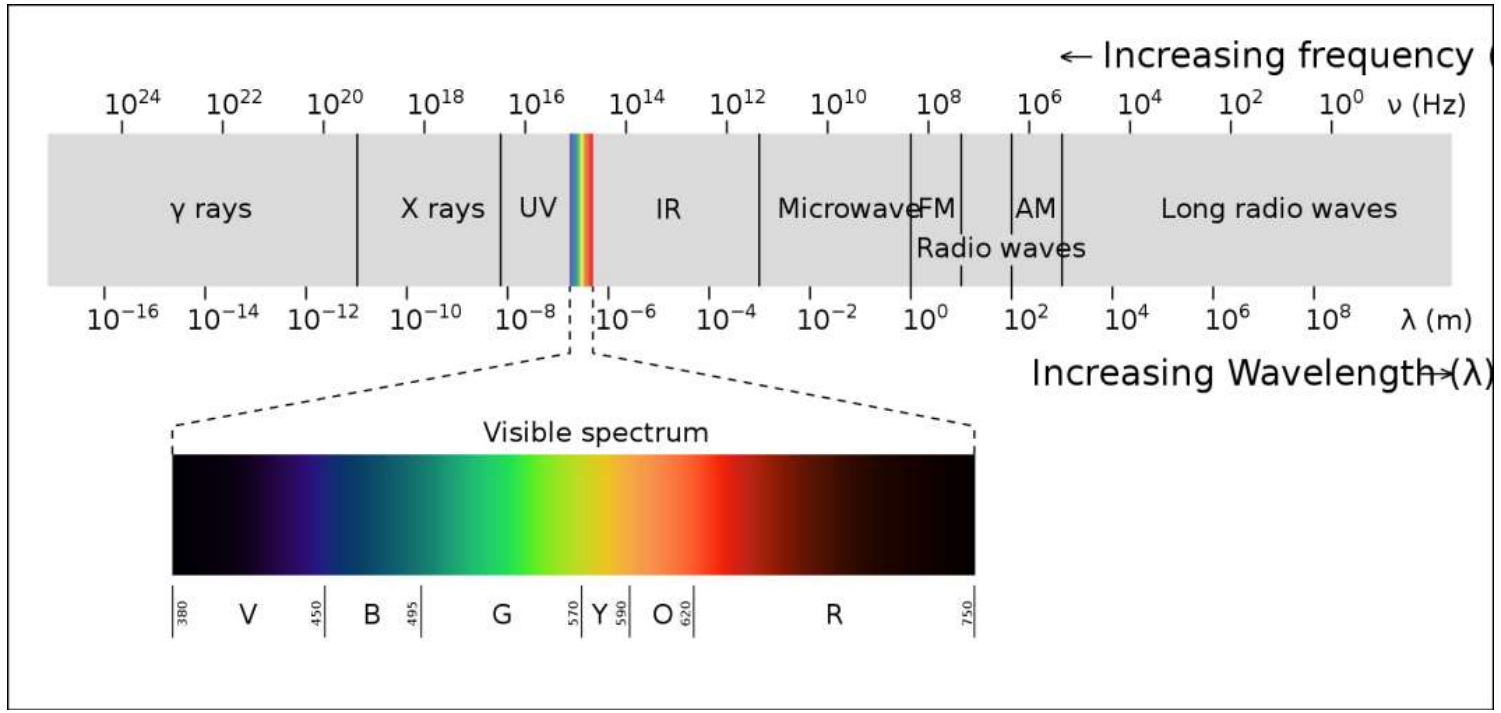
Towards diagnosis by light: Diffuse Optical Tomography (DOT)

Common experience suggests that light can pass travel through biological tissues and be detected on the other side

Traditionally, clinicians have evaluated a patient health condition by his/her complexion (that is, the hue appearing from underneath the skin)

DOT principles

DOT employs near-infrared (NIR, 600 - 900nm) light sources to illuminate the biological tissue *in vivo* from different perspectives, similarly to x-ray tomography, with the aim of inferring the tissue optical parameters



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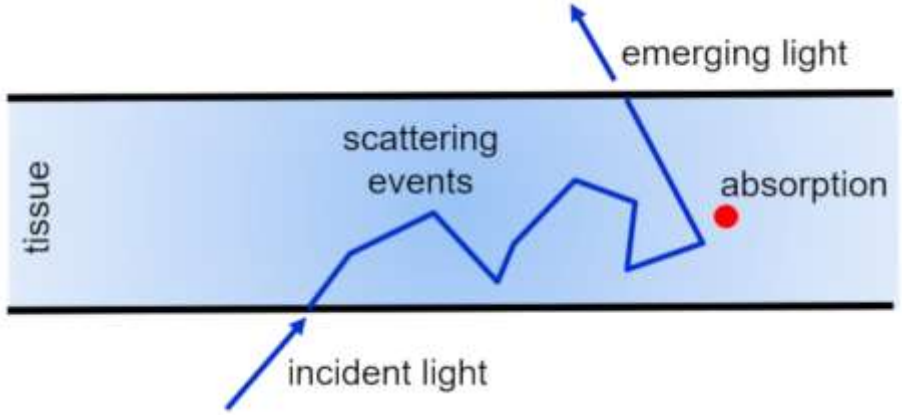
The propagation of light through biological tissues is affected by the optical parameters:

absorption μ_a [1\cm] and scattering μ_s [1\cm]

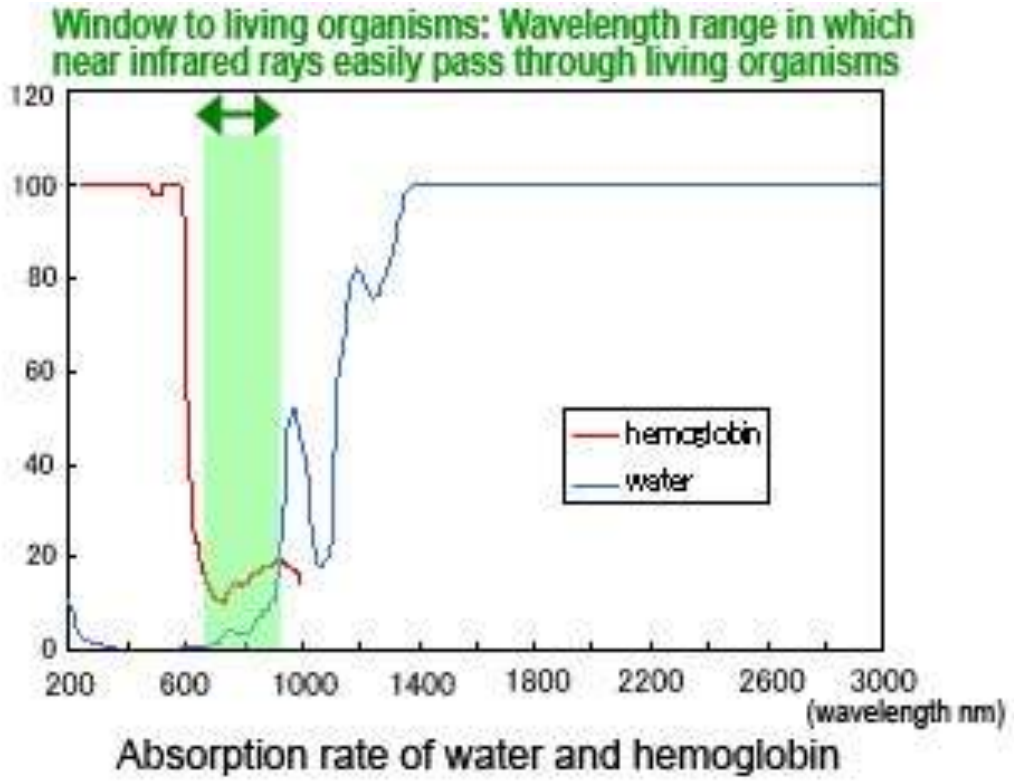
at NIR WLs: scattering>>absorption



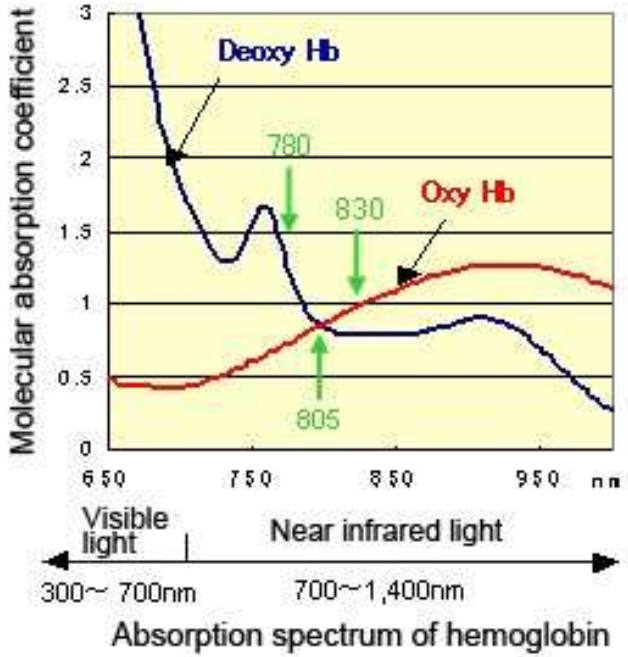
Typical values:

μ_a : 0.01 - 0.30 cm^{-1}
μ_s : 2 - 20 cm^{-1}

The NIR window

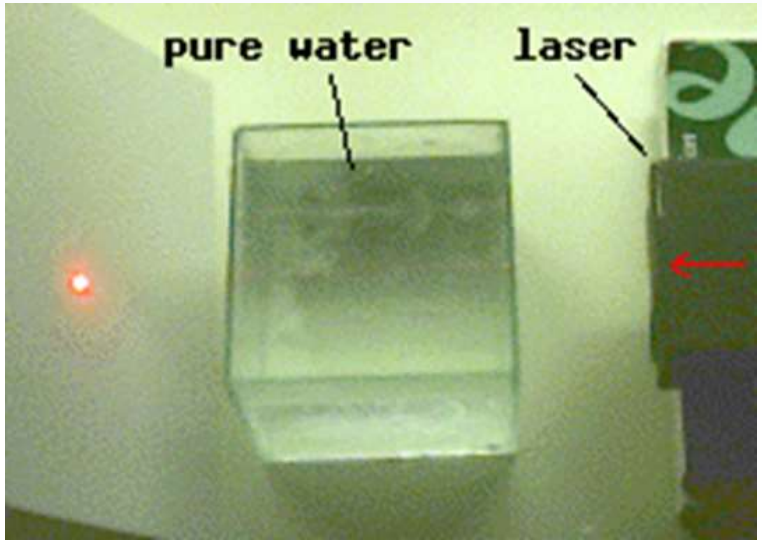


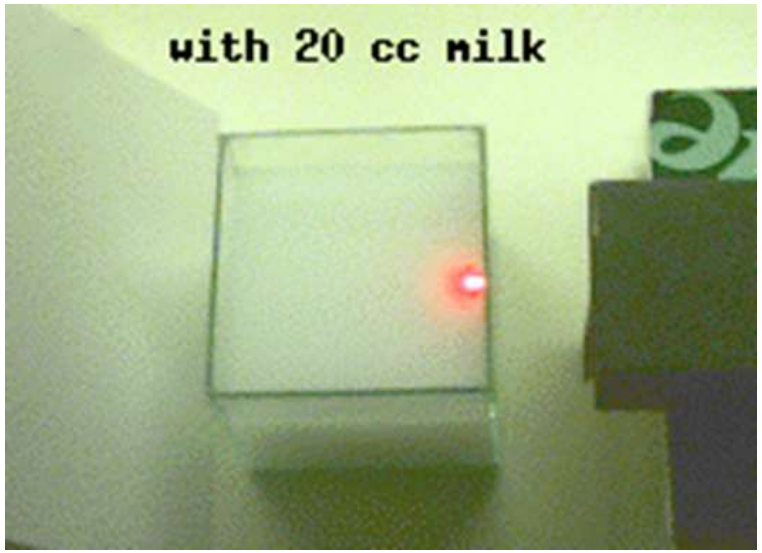
Absorption rate of water and hemoglobin



Absorption spectrum of hemoglobin

…but.. scattering of light in turbid media





https://omlc.org/classroom/scat_demo/

Physics-driven (variational) methods

Reconstruct the model parameters $x \in X$ from measurement $y \in Y$ by solving the minimum problem:

$$x^* = \arg\min_x [d(\mathcal{G}(x), y) + \mathcal{R}(x)]$$

discrepancy (data fidelity)

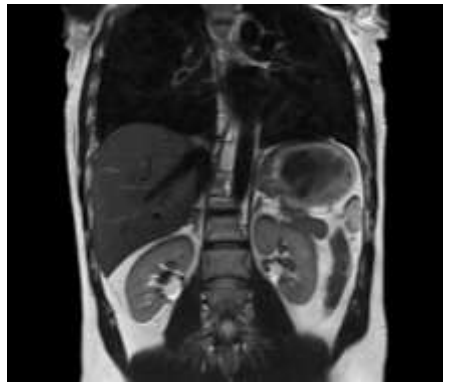
regularization (encodes prior knowledge on the solution)

$\mathcal{G} : X \to Y$ forward (direct) model

discrepancy functional (example)

$$d(\mathcal{G}(x), y) := \frac{1}{2}||\mathcal{G}(x) - y||_2^2$$

hard priors: exams performed with other techniques (ex MRI)



soft priors:

- sparsity of the solution
- range of values
- positivity/negativity

L_2, L_1, TV, …

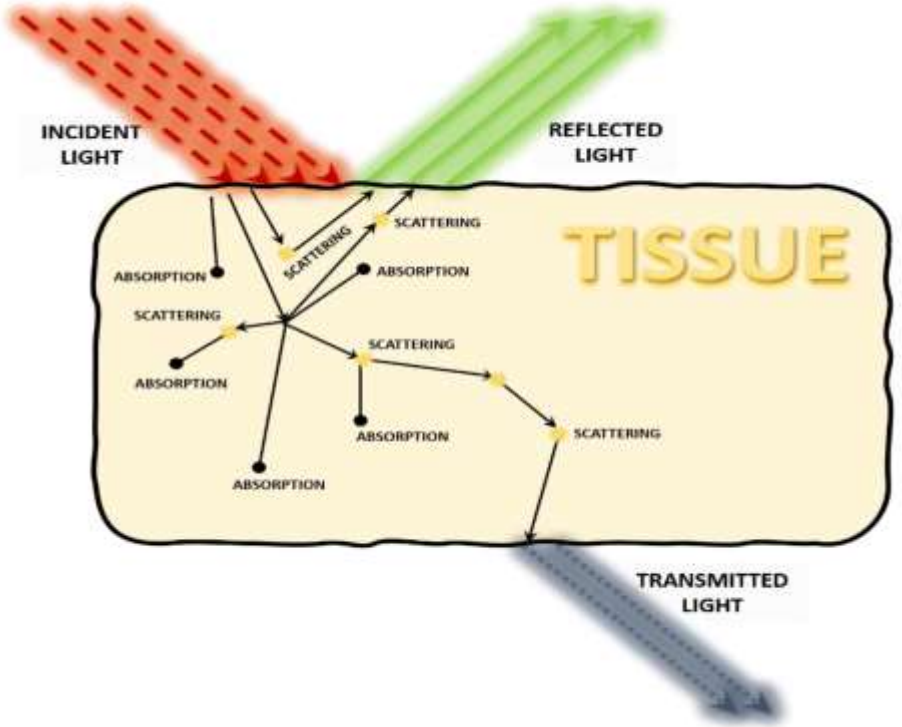
Physical models of light propagation in turbid media

The propagation of light with scattering and absorption is rigorously governed by the radiative transfer equation (RTE)

Key quantity: light radiance $L = L(r, s, t)\ \mathrm{W/cm^2sr}$

light power per unit area traveling in the s direction at position r

Solving the RTE is computationally very intensive
so that many simplifications have been sought in literature



Diffusion approximation: valid when the photon scattering is the governing phenomena in the tissue: true for biological matter such as skin, bone, brain matter or breast tissue (absorption << scattering)

In any case…PDE models (more complex than the Beer law!)

Direct and inverse mathematical models for DOT

Direct problem: given the optical properties $D \simeq \frac{1}{\mu_s}$ and μ_a solve the diffusion equation for photon fluence $\Phi = \Phi(r)\ [\mathrm{W/cm^2}]$

$$\begin{cases} \left[\Delta - \frac{\mu_a}{D(r)}\right]\Phi(r) = -\frac{S(r)}{D} \quad in\ \Omega(= breast), \\ \Phi = 0 \ \ on\ \Gamma_D, \qquad \frac{\partial \Phi}{\partial n} + A\Phi = 0 \ \ on\ \Gamma_R \end{cases}$$

Inverse DOT problem: given photon fluence measurements on the boundary, deduce the optical field μ_a

Rytov approximation

$$\mu_a = \mu_{a,0} + \delta\mu_a$$

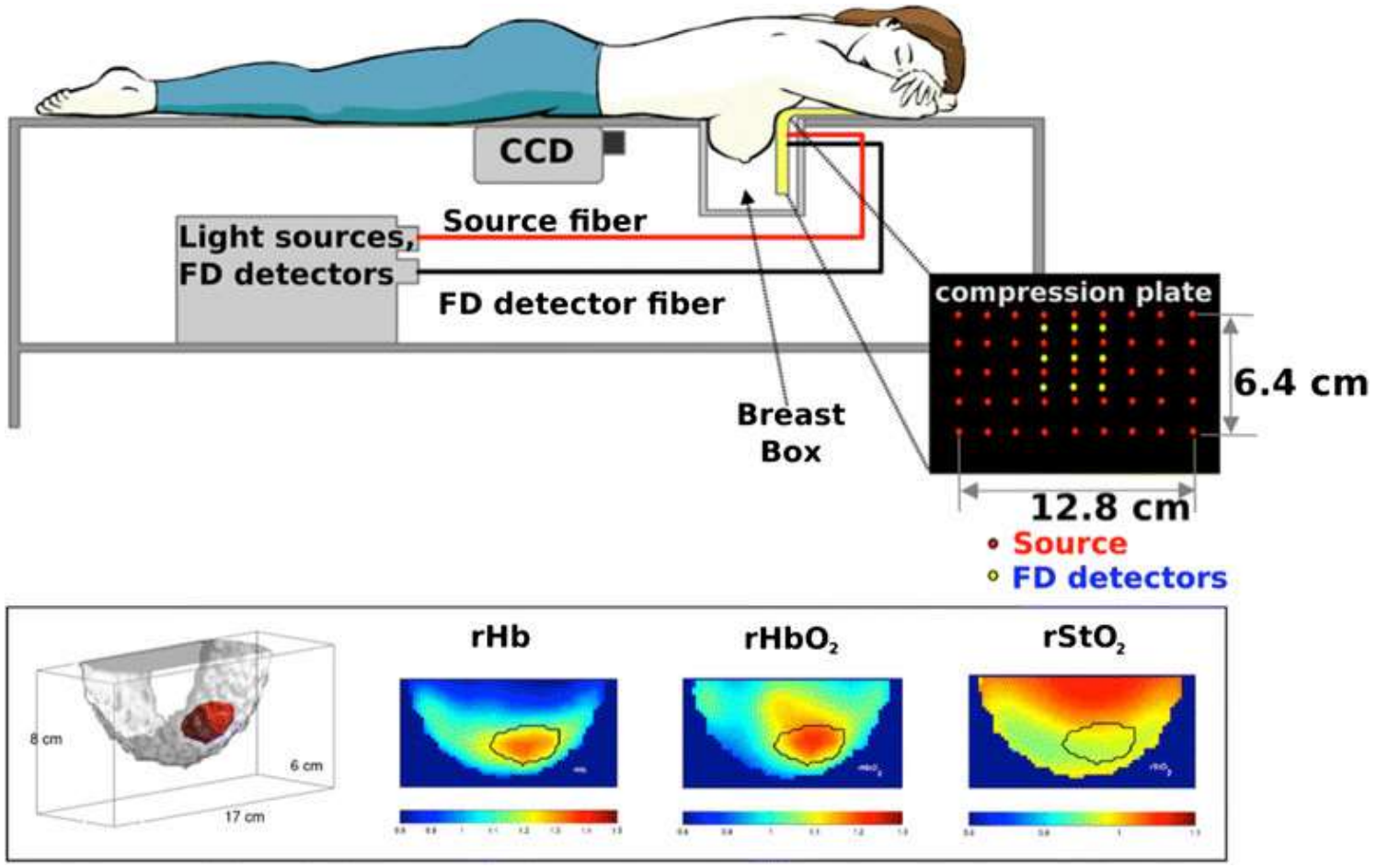
$$\Phi = e^{\Psi_0 + \Psi_1} = \Phi_0 e^{\Psi_1}$$

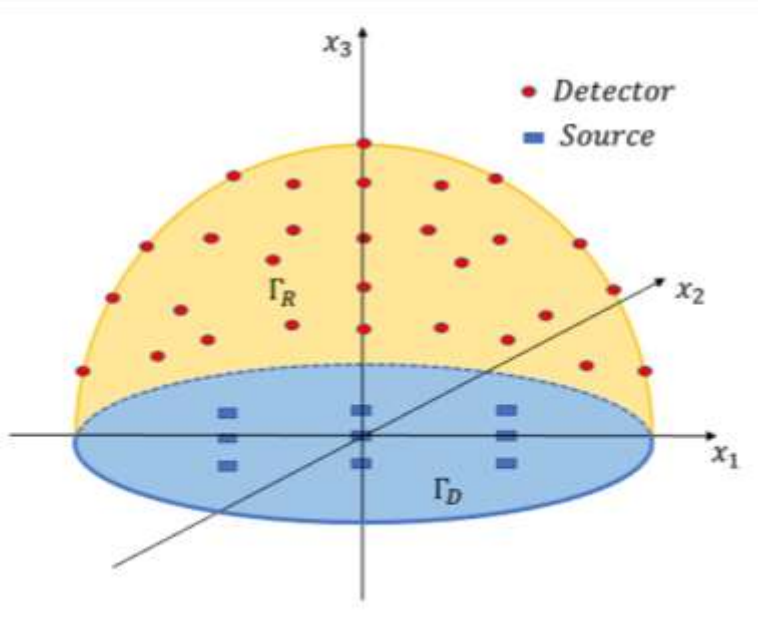
solution via Green's function (meshless)

after some algebra: $$[\Delta - \frac{\mu_{a,0}}{D}](\Phi_0\Psi_1) = \frac{\delta\mu_a}{D}\Phi_0 \qquad \Psi_1 = \frac{1}{\Phi_0 D_0}\int_\Omega G(r-r')\delta\mu_a(r')\Phi_0(r')dr'$$

[S. Arridge, J. Schotland. "Optical tomography: forward and inverse problems", Inverse Problems (2009)]

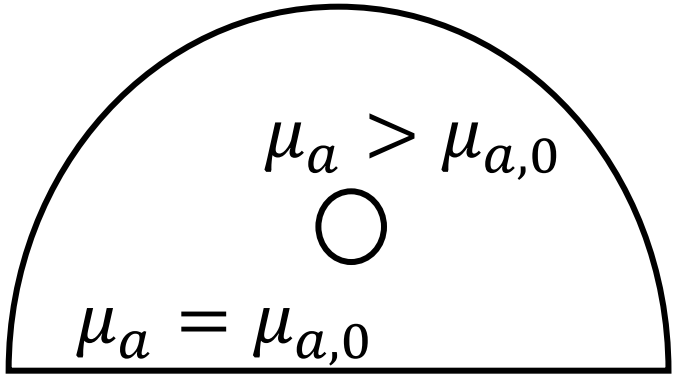
Application of DOT to breast screening



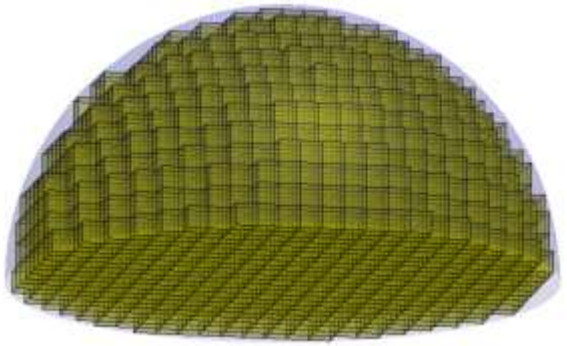


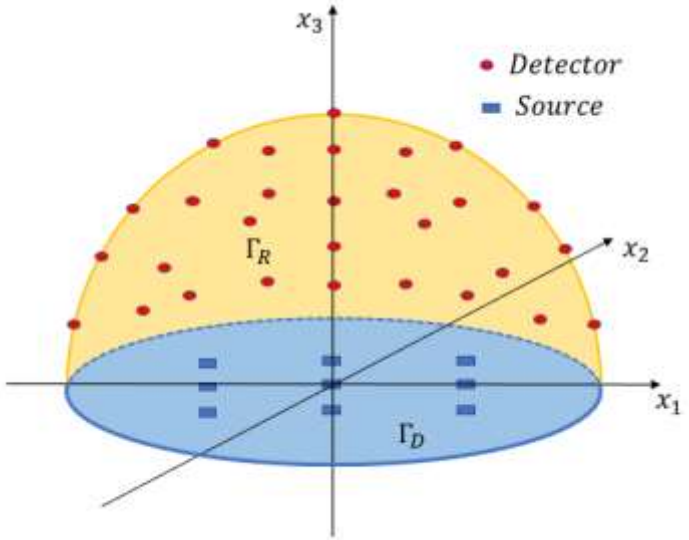
DOT and clinics

Altered optical coefficients might indicate pathogenic processes in the tissue, for example an increased μ_a may be marker of tumoral lesions (related to angiogenesis)



Discretization





Voxelization of the domain (*N* voxels)

$$\psi_1 = \log\left(\frac{\Phi_i}{\Phi_{0,i}}\right)$$

$$\sum_{j=1}^{N} \frac{\Delta V_j}{\mathrm{D}_0 \Phi_0\left(r_i^s, r_i^d\right)} G\left(r_i^d - r_j\right) \Phi_0\left(r_i^s, r_j\right) \delta\mu_a \simeq \log\left(\frac{\Phi_i^M}{\Phi_{0,i}^M}\right)$$

J
sensitivity matrix

y
from measures

$$J\delta\mu_a \simeq y$$

24

Solution and Regularization

Solve $$J\delta\mu_a = y \rightarrow \delta\mu_a = \operatorname{argmin} \left|\left| \mathrm{J}\delta\mathrm{x} - \mathrm{y} \right|\right|_2^2$$

Elastic Net regularization [Zou et al., 05]

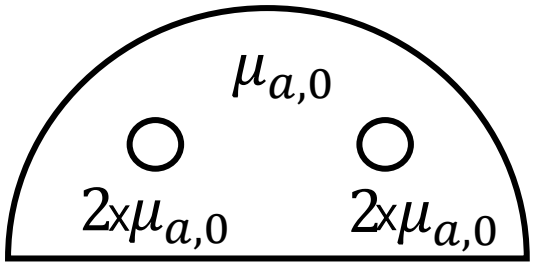
$$\delta\mu_a = \operatorname{argmin} \frac{1}{2} \left|\left| \mathrm{J}\delta\mathrm{x} - \mathrm{y} \right|\right|_2^2 + \lambda \left(\alpha \left|\left| \delta\mu_a \right|\right|_1 + (1-\alpha) ||\delta\mu_a||_2^2 \right), \alpha \in [0,1]$$

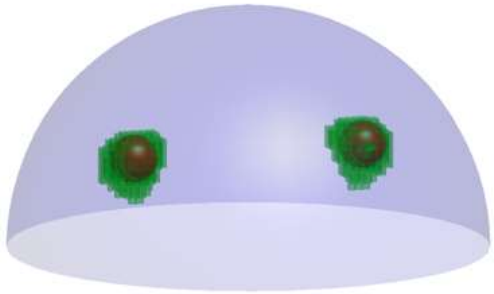
ℓ_1-norm: finds strong components (enhance sparsity)

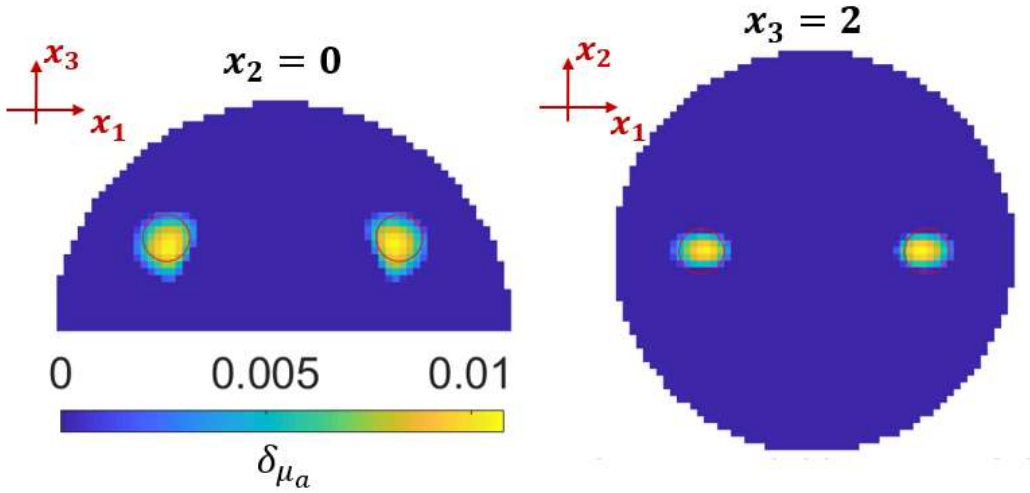
ℓ_2-norm (Tikhonov): robust regularization

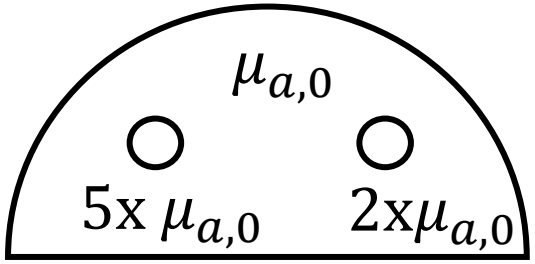
25

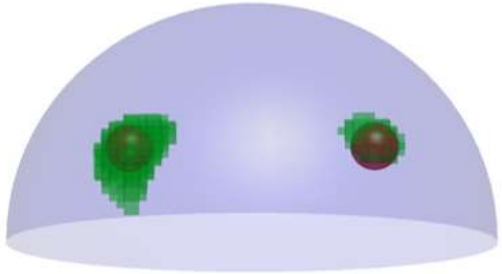
Results on a simulated test case

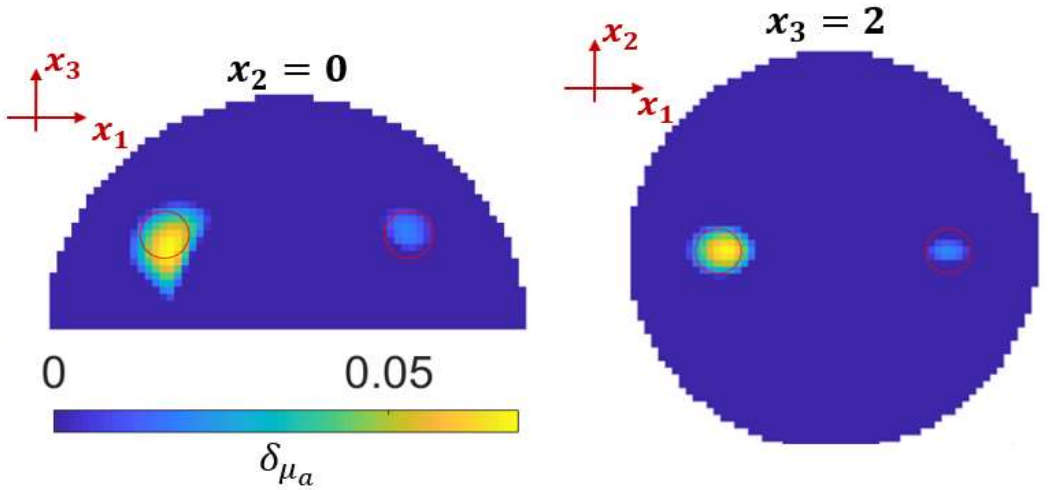










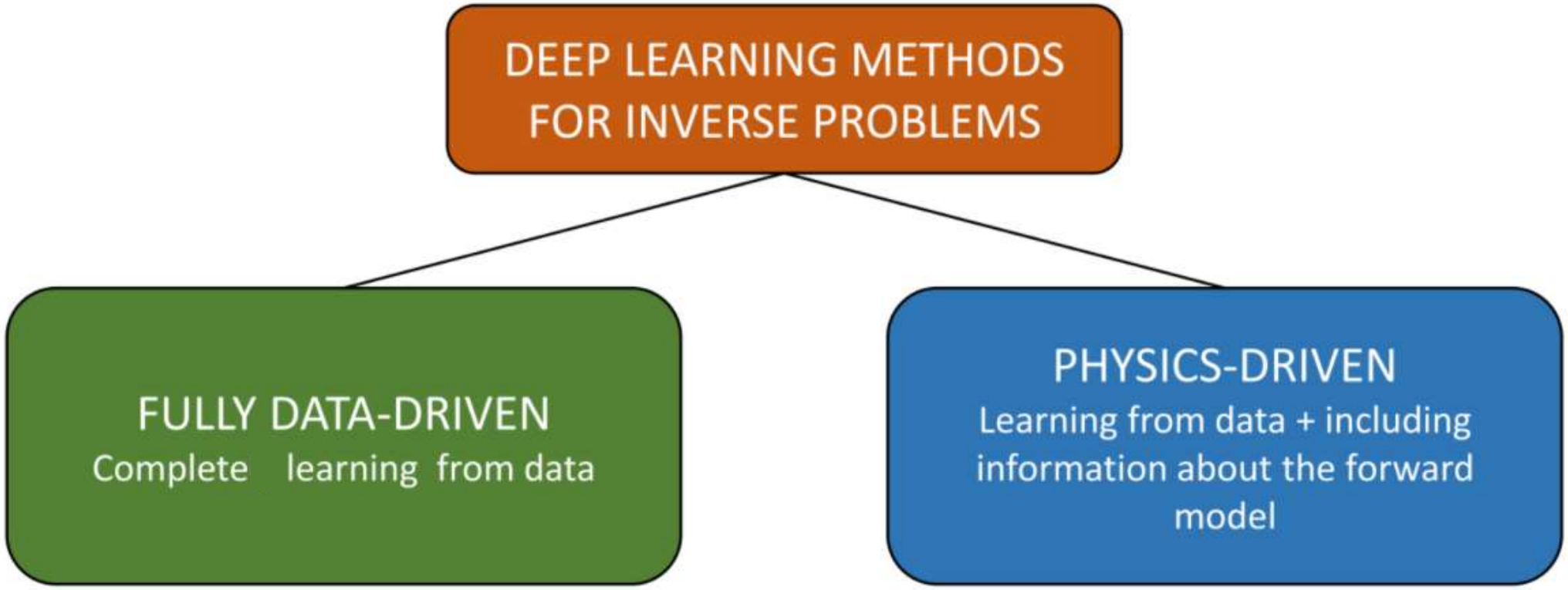


[P.C., M. Lupieri, G. Naldi, RM Weishaeupl. Mathematical and numerical challenges in optical screening of female breast", Int J Numer Method Biomed Eng (2020)]

Variational approach: pro & cons

- well established approach
- difficult choice of the regularization parameters, *ad-hoc* for each single case
- there are conditions that are *never* correctly reconstructed (*e.g.*, more than one inclusion with different absorption coefficients, second row of the previous example)

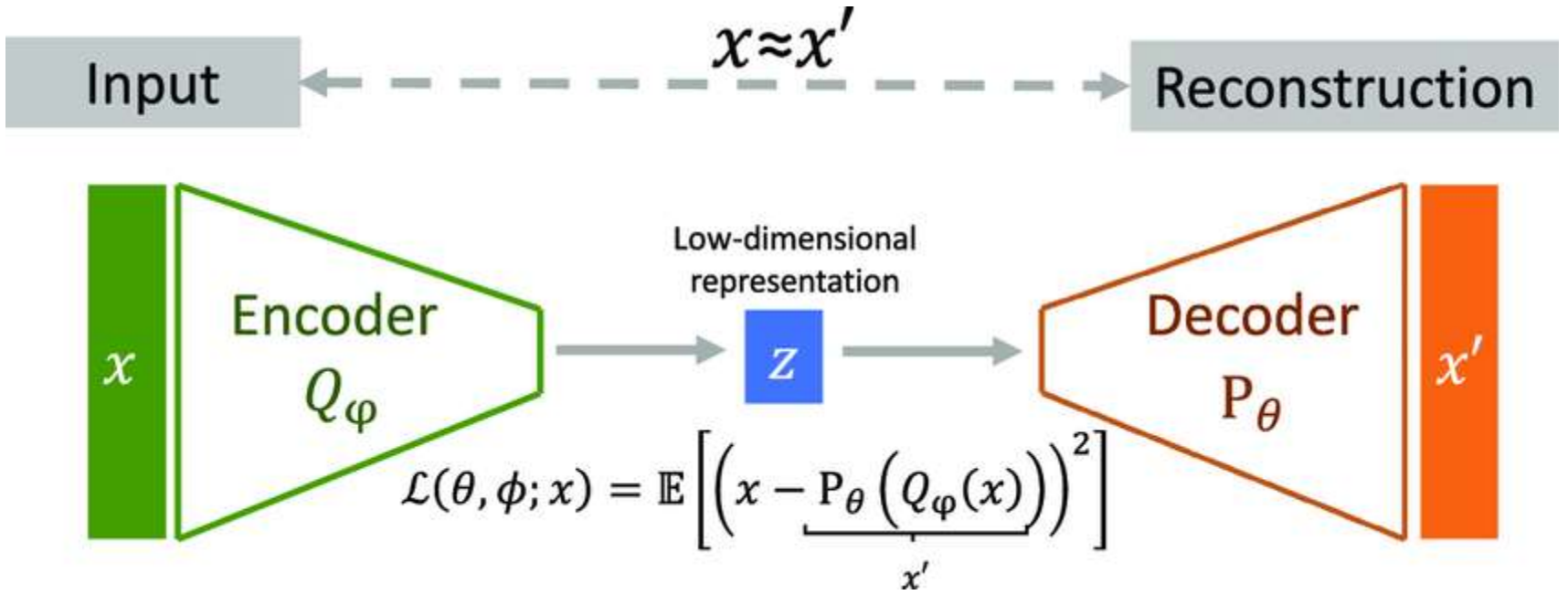
Deep Learning for DOT reconstruction



L-SVD model

Tikh+resCNN

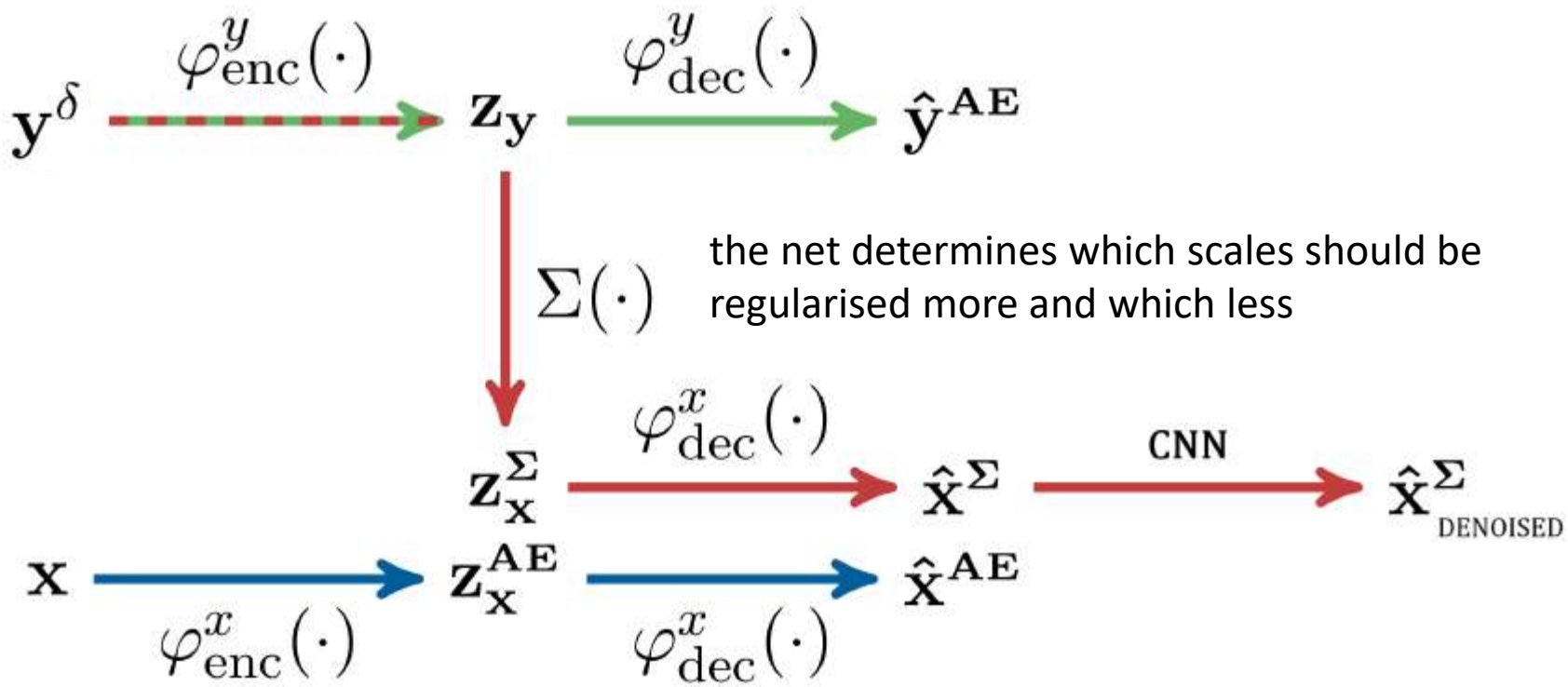
Before starting…: what is an autoencoder?



Fully data drive model: Learned-SVD (L-SVD)

nonlinear learned variant of SVD

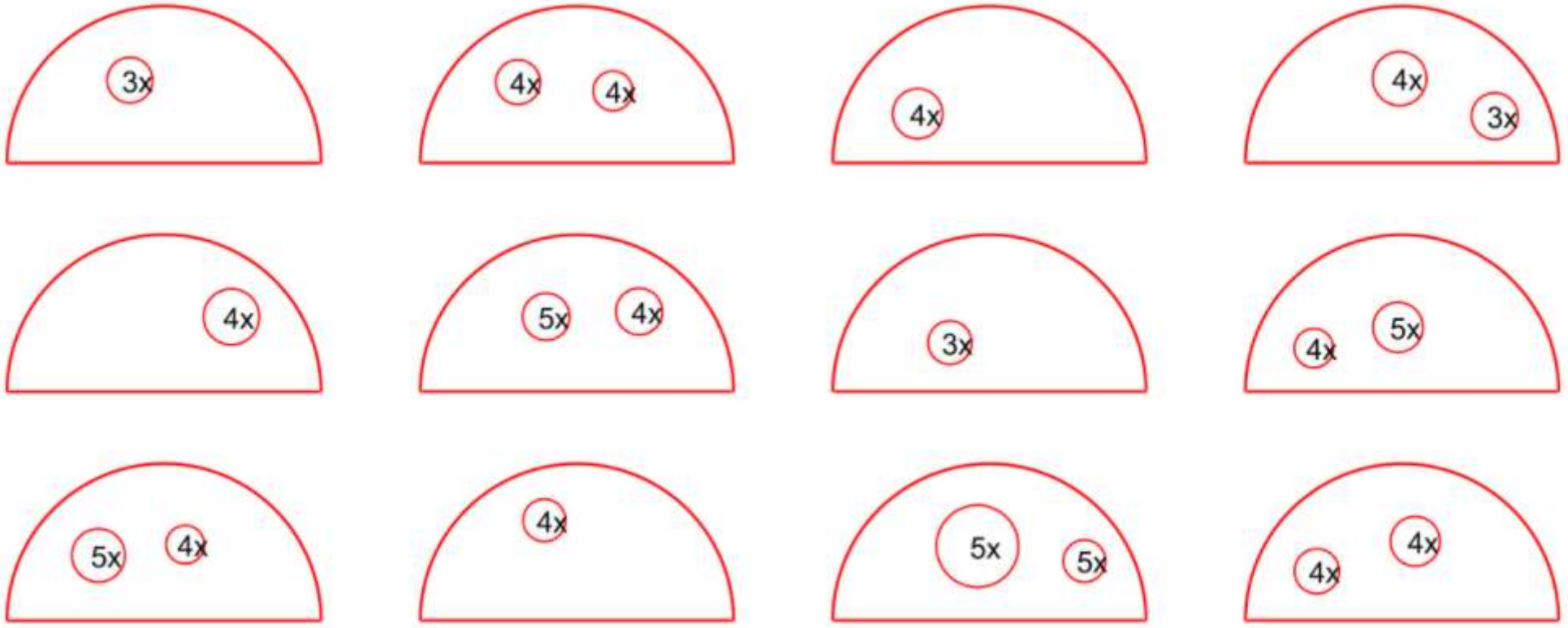
$$\mathcal{N} := \underbrace{\varphi^{y}_{enc} \circ \Sigma \circ \varphi^{x}_{dec}}_{\text{Inversion}} \circ \underbrace{\mathcal{N}_1}_{\text{Denoising CNN}}$$



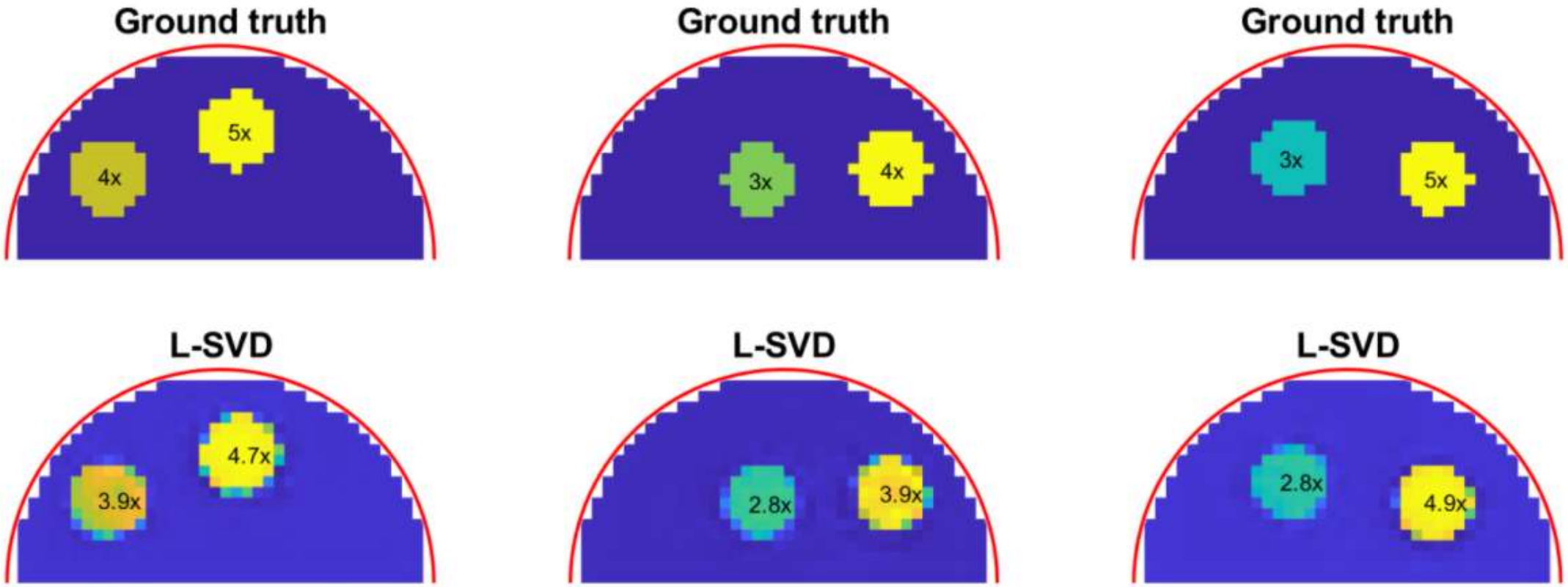
[A. Benfenati, G. Bisazza, P. C. «A Learned SVD approach for Inverse Problem Regularization in Diffuse Optical Tomography» ArXiv preprint no. 2111.13401 (2021)]

Training of the networks

- Synthetic dataset of 1500 samples
- Semi-circular domains with perturbed regions with different: position, location and value of μ_a
- we solved the forward problem (diffusion problem) for each of them, collecting the «measurements» on the boundary



L-SVD model reconstruction (from noiseless data): qualitative evaluation:



Quantitative metrics

We consider the following indices to quantitatively assess the quality of the reconstruction:

Absolute Bias Error
(smaller=better), sum over voxels
rec=reconstructed
GT=ground-truth

$$ABE = \frac{1}{N}\sum_{j=1}^{N} |\mu_{a,rec}^{j} - \mu_{a,GT}^{j}|$$

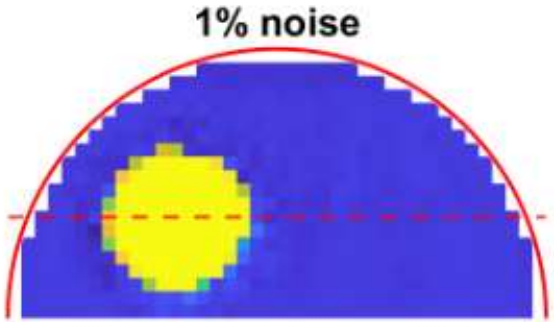
Contrast-to-noise ratio
(higher=better)
PR=perturbed region
BG=background
w_{PR}=perturbed surface/tot surface
overline=average of the region

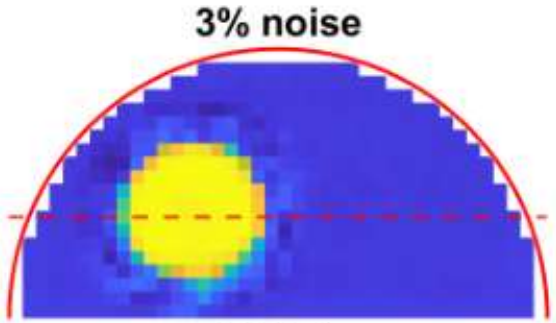
$$CNR = \frac{\overline{\mu}_{a,PR} - \overline{\mu}_{a,BG}}{\sqrt{w_{PR}\sigma_{PR}^{2} + w_{BG}\sigma_{BG}^{2}}}$$

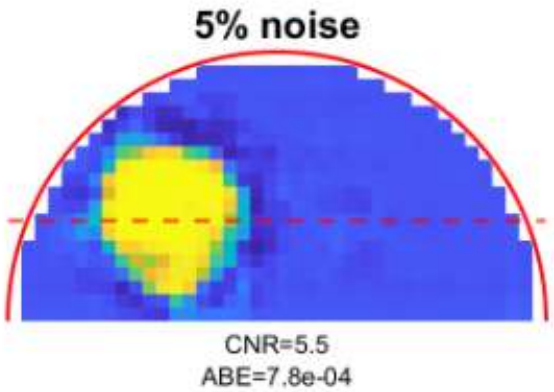
with minor modifications
for more than one PR

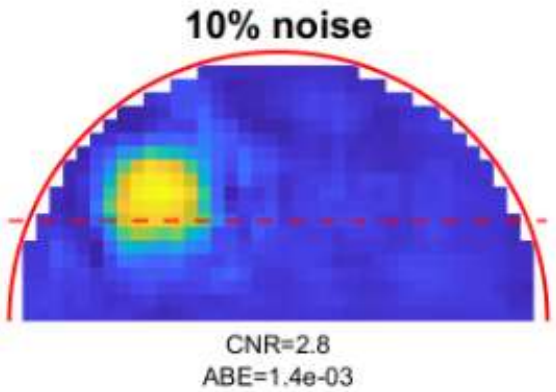
Caveat: one single index may not be adequate to assess the quality, for example CNR>>1 if the absorption coefficient is overestimated in the PR

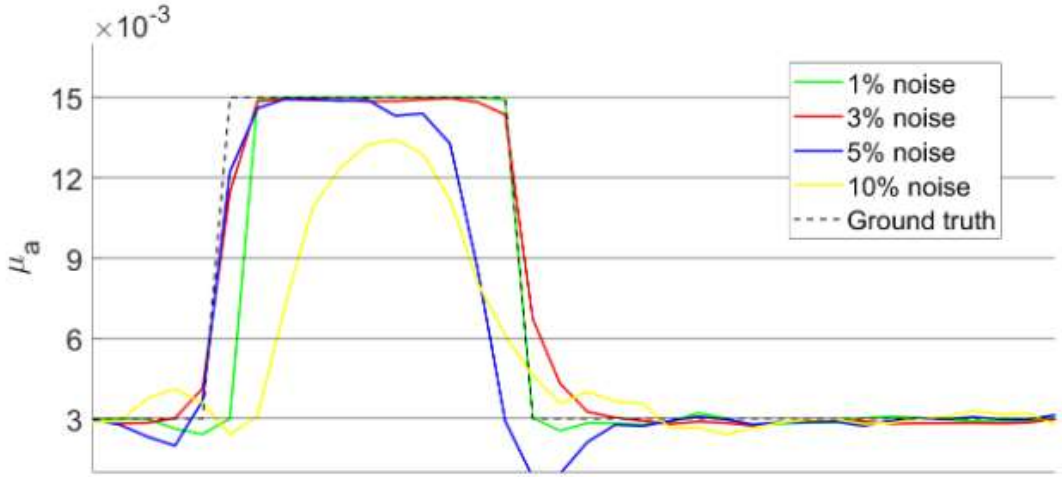
L-SVD model: 1PR- reconstruction from noisy data



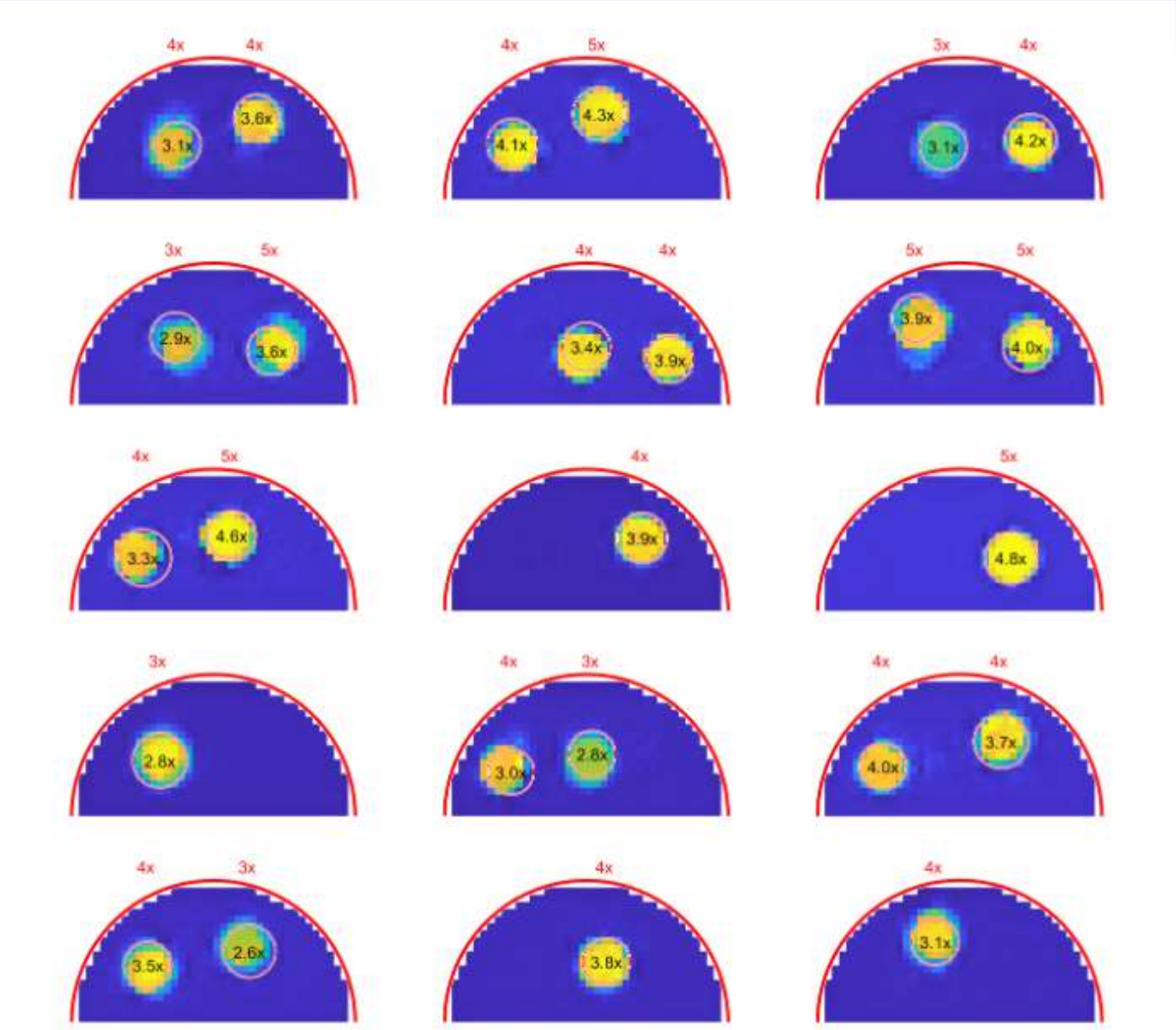






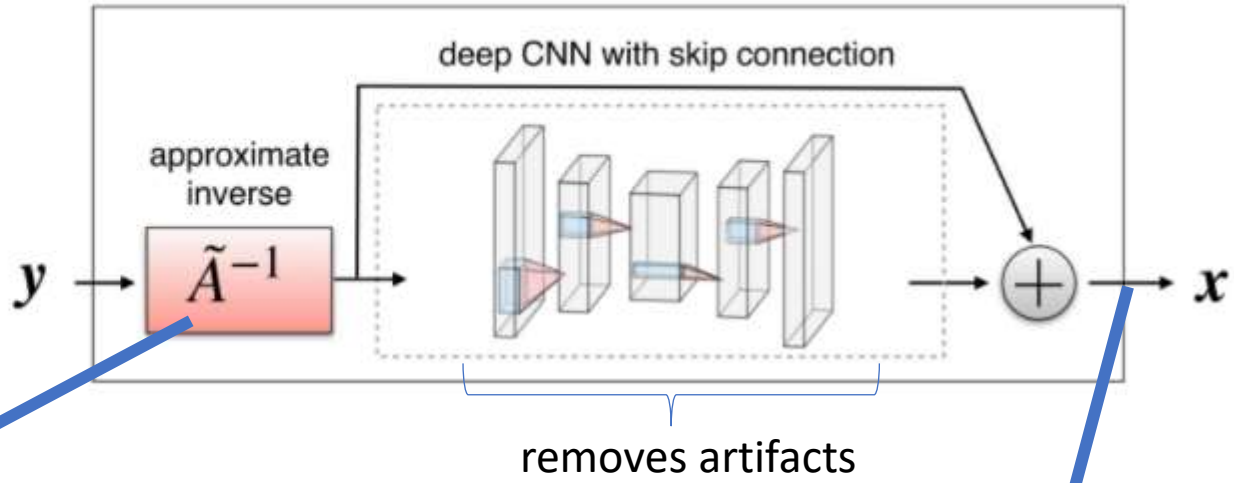


L-SVD model: 1 or 2PRs- reconstruction from 1% noisy data

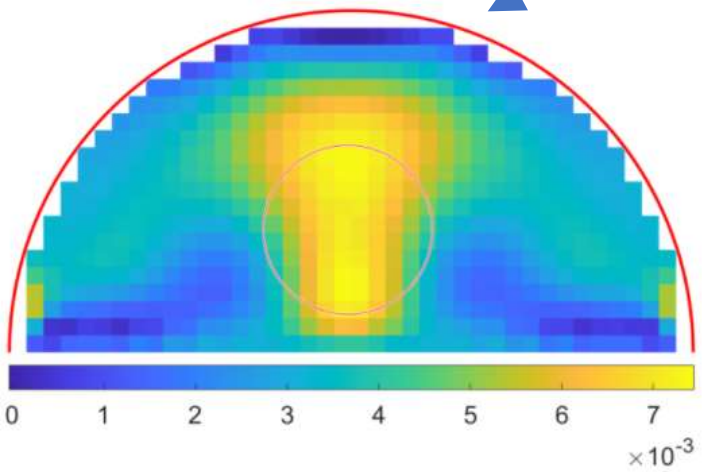


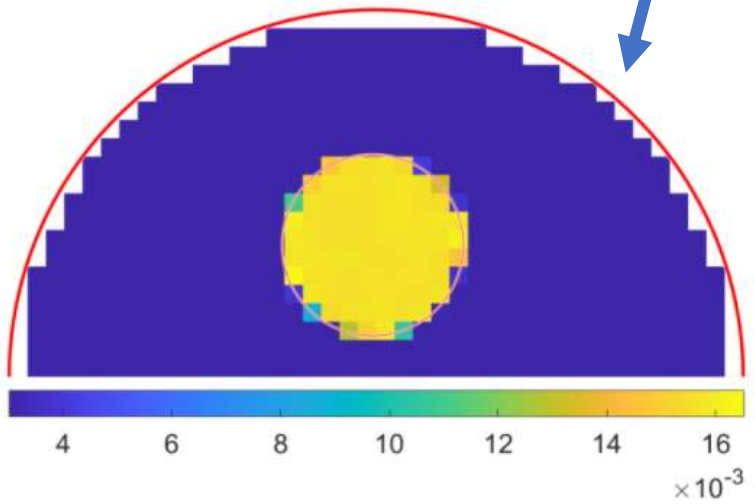
Physics driven model: Tikh+resCNN

$$Y \xrightarrow[\text{not learned}]{\text{Tikhonov}} X \xrightarrow[\text{learned}]{\mathcal{N}} X$$

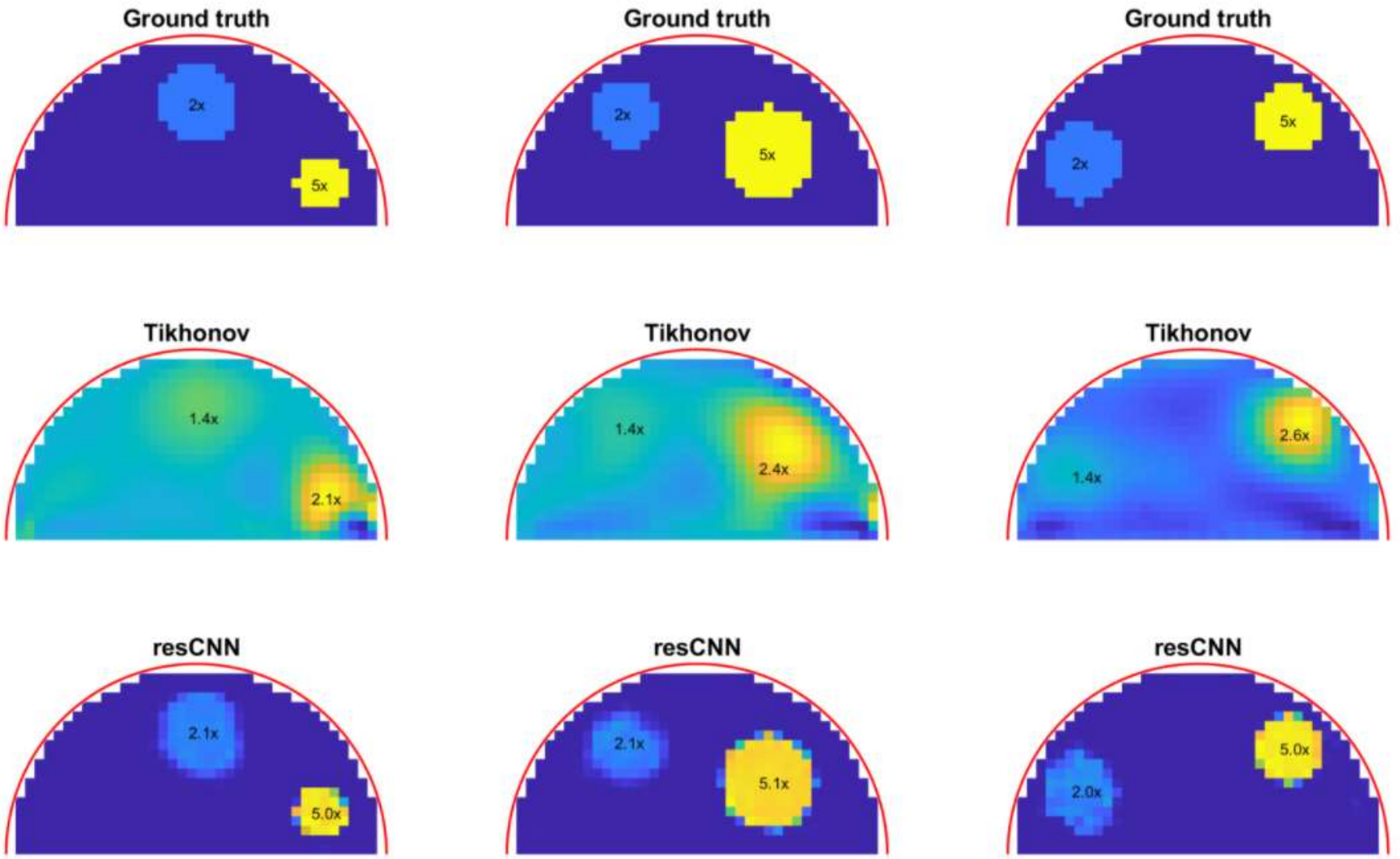


removes artifacts

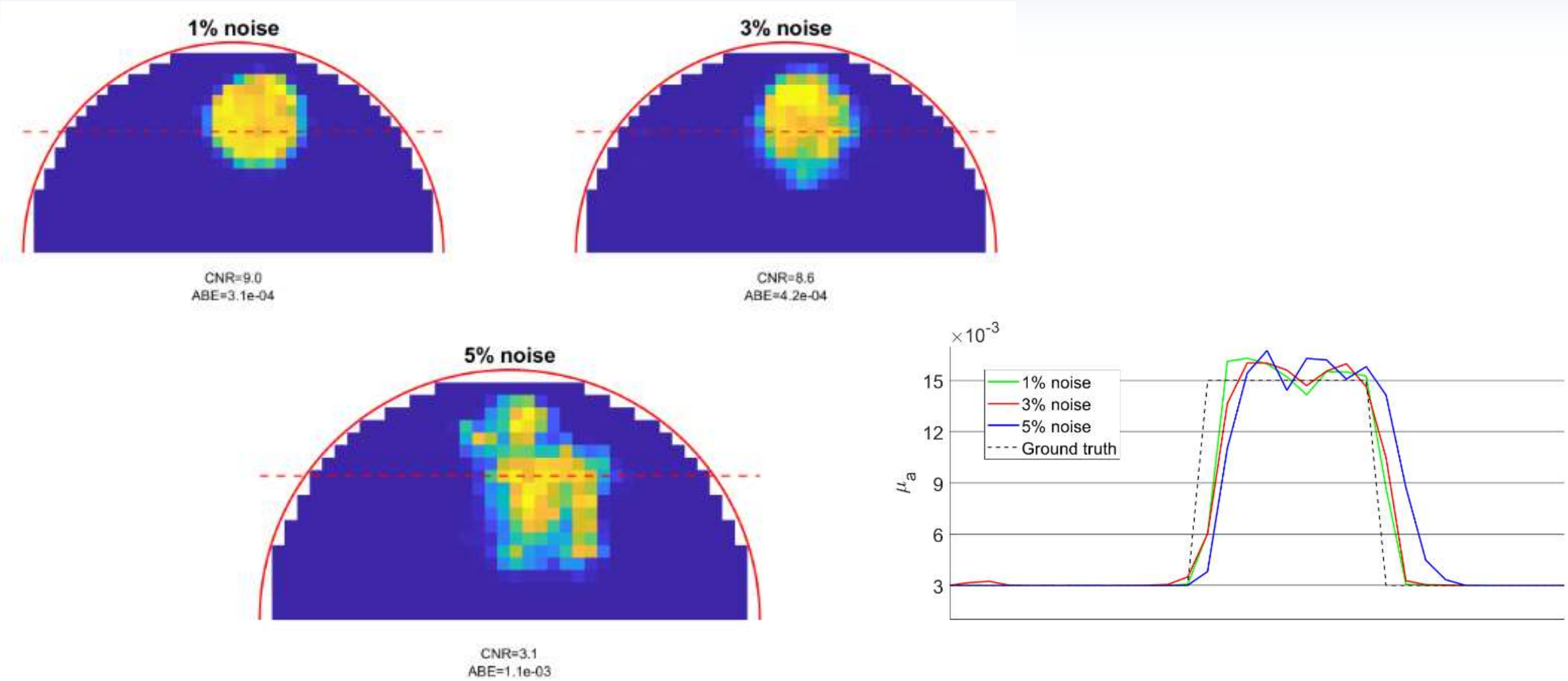




Tikh+resCNN hybrid model reconstruction (from noiseless data)

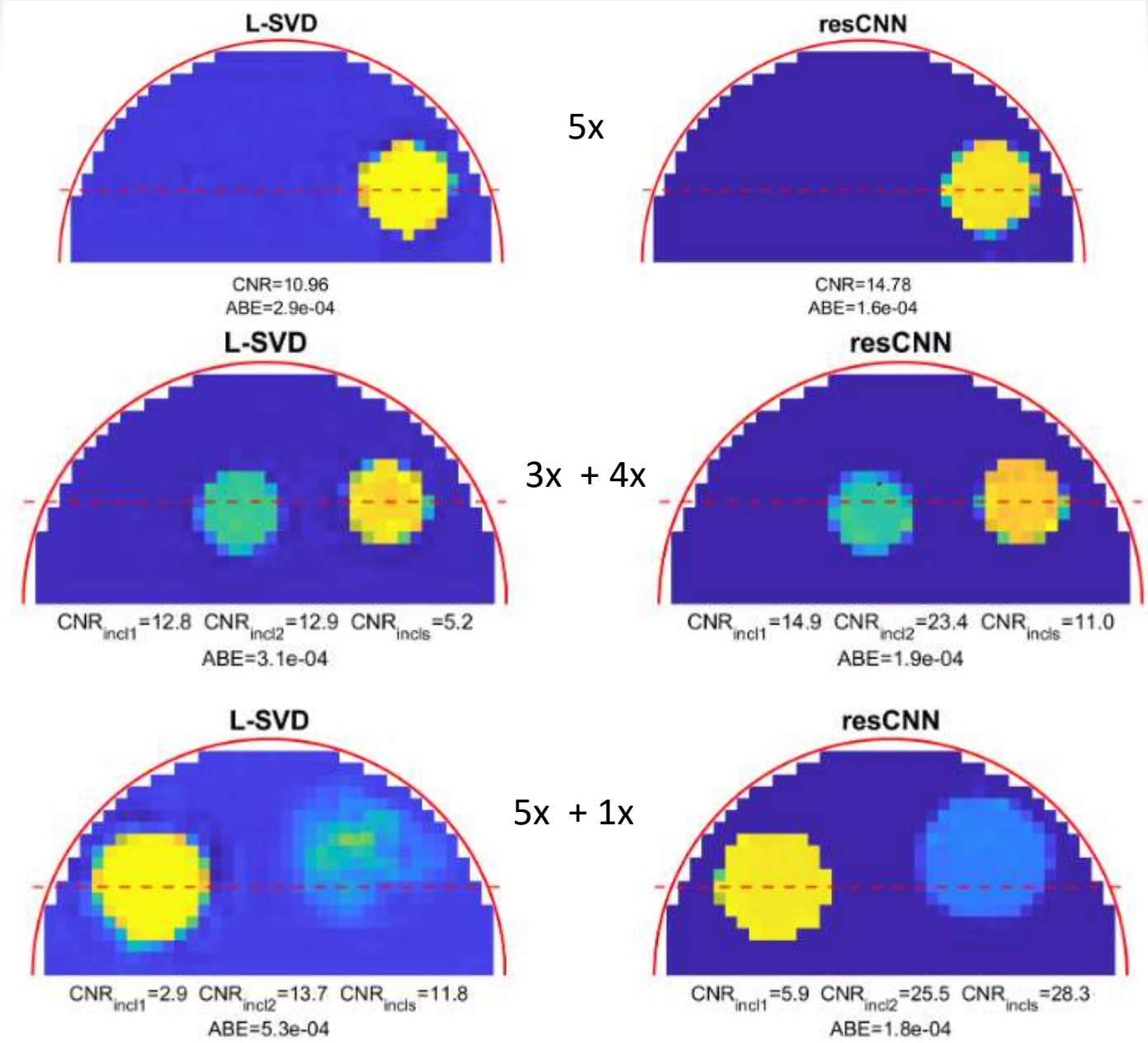


Tikh+resCNN reconstruction from noisy data



Remark: the physics-driven step is very critical, since it propagates noise that must be efficiently cleaned out in the resCNN denoising step

Which model should I choose?



But…

Method	First inversion	Denoising	Total
FC+CNN	43.198.588	5.382	43.203.970
L-SVD+CNN	1.280.388	150.531	1.430.919
resCNN	-	418.419	418.419

noiseless data

Method	Pre-denoising	First inversion	Denoising	Total
FC+CNN	-	43.198.588	5.382	43.203.970
L-SVD+CNN	-	1.280.388	150.531	1.430.919
resCNN	1.904.050	-	418.419	2.322.469

noisy data

Conclusions (I)

- Modern medicine critically relies on imaging for decision making, of which CT scan is a central pillar
- Standard CT scans perform several hundreds of projections, with a significant radiation burden for the patient and the involved medical personnel
- Low-dose and zero-dose CT technologies are main innovation directions to obtain more «sustainable» CT modalities
- A noiser/subsampled signal is typically obtained in less ionizing approaches, that requires ad hoc reconstruction techniques

Conclusions (II)

- Diffuse optical tomography uses near-infrared light as investigating signal: no ionization
- NIR light in the tissue undergoes multiple scattering making the inversion problem severely ill-conditioned
- Classical variational methods have been used for many years but: i) critical choice of the regularization parameters, ii) there are (mildly) complex originating signals that are never correctly reconstructed
- DL techniques (fully data-driven or hybrid physics+data-driven model) can support CT reconstruction but much space is open to improvement