ERROR ESTIMATES FOR THE THREE-FIELD FORMULATION WITH BUBBLE STABILIZATION

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ABSTRACT. In this paper we prove convergence and error estimates for the so-called 3-field formulation using piecewise linear finite elements stabilized with boundary bubbles. Optimal error bounds are proved in L^2 and in the broken H^1 norm for the internal variable u, and in suitable weighted L^2 norms for the other variables λ and ψ.

1. INTRODUCTION

The aim of this paper is to consider the effects of bubble stabilizations of the so-called three-field formulation for domain decomposition methods. Let us briefly recall it. Assume that we have to solve a linear elliptic problem whose variational formulation is given by:

$$\text{find } w \in V \text{ such that} \qquad a(w,v) \;=\; (f,v) \qquad \forall v \in V, \tag{1.1}$$

on a domain Ω. We assume that the problem is associated to a second-order differential operator, so that the space V will be a subspace of $H^1(\Omega)$. By splitting Ω into subdomains Ω^s one introduces suitable subspaces V^s (to be made precise in the sequel) of $H^1(\Omega^s)$, and defines $M^s = H^{-1/2}(\partial\Omega^s)$, $\Sigma = \cup_s \partial\Omega^s$, and $\Phi = \{\text{traces on } \Sigma \text{ of the functions of } V\}$. Setting $V^* := \mathbf{\Pi}_s V^s$ and $M^* := \mathbf{\Pi}_s M^s$, the three-field formulation of (1.1) then reads

$$\left\{\begin{array}{lllll} \text{find } u \in V^*,\ \lambda \in M^*, \text{ and } \psi \in \Phi & \text{such that} & & & \\ \text{i) } a_s(u^s,v) - <\lambda^s, v>_s & = (f,v)_s & \forall v \in V^s & \forall s \\ \text{ii) } <\mu^s, u_s>_s & = <\mu^s, \psi>_s & \forall \mu^s \in M^s & \forall s \\ \text{iii) } \sum_s <\lambda^s, \varphi>_s & = 0 & \forall \varphi \in \Phi & \end{array}\right. \tag{1.2}$$

(the nearly obvious meaning of a_s, $<.,.>_s$ and $(.,.)_s$ will be made precise in the next section.)

This formulation was originally introduced in [8], [9], where it was proved that, under reasonable assumptions, problem (1.2) has a unique solution related to the solution of (1.1) by

$$\left\{\begin{array}{lll} \text{i) } u^s & = w_{|\Omega^s} & \text{for each } s \\ \text{ii) } \lambda^s & = \frac{\partial w}{\partial n_a}_{|\partial\Omega^s} & \text{for each } s \\ \text{iii) } \psi & = w_{|\Sigma}, & \end{array}\right. \tag{1.3}$$

with $\frac{\partial}{\partial n_a}_{|\partial\Omega^s}$ = outward conormal derivative to Ω^s. In order to approximate (1.2) one has to choose, for every s, finite dimensional subspaces V_h^s, M_h^s and Φ_h of V^s, M^s and Φ respectively, to construct $V_h^* := \mathbf{\Pi}_s V_h^s$ and $M_h^* := \mathbf{\Pi}_s M_h^s$, and to consider the following discretized problem.

$$\left\{\begin{array}{lllll} \text{find } u_h \in V_h^*,\ \lambda_h \in M_h^*, \text{ and } \psi_h \in \Phi_h & \text{such that} & & & \\ \text{i) } a_s(u_h^s,v) - <\lambda_h^s, v>_s & = (f,v)_s & \forall v \in V_h^s & \forall s \\ \text{ii) } <\mu^s, u_h^s>_s & = <\mu^s, \psi_h>_s & \forall \mu^s \in M_h^s & \forall s \\ \text{iii) } \sum_s <\lambda_h^s, \varphi>_s & = 0 & \forall \varphi \in \Phi_h. & \end{array}\right. \tag{1.4}$$

In [8], [9] we proved that a finite element discretization of (1.2) on compatible grids (with a suitable interpretation of M_h^s) gives back the global finite element solution of (1.1) with formulae identical to (1.3). To deal with more general approximations, including the use of (possibly) different discretizations for the three variables in the same subdomain, or from one subdomain to another, a wide class of stabilization procedures was introduced and discussed in [2]. However, it is interesting to see if, for particular cases, a simpler and cheaper stabilization could be used, instead of the more general (but heavier and costlier) stabilization of [2]. A typical example of interest is clearly the case where the variables u and ψ are approximated by continuous piecewise linear finite elements, although on different, incompatible grids, and the variable λ by piecewise constants. This was done in [7], where a stabilization procedure was proposed, based on the idea of adding suitable boundary bubbles to the space V_h^s. The underlying concept is reasonably simple: it is clear that in (1.4) the only control on ψ_h can be obtained from (1.4 ii). Therefore, to gain control on ψ_h one can increase the space M_h^s. As each space M_h^s is

made of piecewise constants, this can be done simply by refining the mesh. It has to be pointed out that, once the first refinement has been made, the grid for M_h^s could undergo a further refinement (if convenient for some technical reason) without jeopardizing the control on ψ_h. In its turn, the control on λ_h can only be obtained from equation (1.4 i), since (1.4 iii) is clearly too weak (any monster taking the same value with opposite signs at the interfaces will satisfy (1.4 iii).) Hence, to control λ_h^s we have to enrich the finite element spaces V_h^s, and the cheapest way is to add boundary bubbles. In [7] we showed, on a single domain problem, that an almost arbitrary mismatch between the grid for M_h^s and that for V_h^s can be stabilized (to gain control on λ_h^s) by suitable quadratic boundary bubbles. We also showed that the bubbles, *together with the unknown* λ_h, could be easily eliminated by static condensation, thus remaining with a symmetric (if problem (1.1) is symmetric) positive definite problem in the unknown u_h only. In particular, in [7] we assumed, as a starting point, that two original, unrelated grids were given, one for λ_h and one for u_h. For technical reasons (essentially, in order to be able to perform the static condensation on λ_h^s afterwards) our first step was to refine the grid for M_h^s, by adding all the boundary nodes of the grid for V_h^s. However we had to assume, to avoid technical difficulties, that the points of the two original grids were not too close (otherwise the refined grid for M_h^s might contain very small intervals) and that the two given grids were "comparable" in the following sense

$$h_\lambda^s \;\geq\; \gamma h_u^s \tag{1.5}$$

(with γ a fixed constant) with obvious meaning for the symbols. These assumptions were removed or relaxed, always for the single domain case, in [10], where an optimal estimate for $u-u_h$ was proved, allowing the presence of "very close nodes" and changing (1.5) into the much weaker

$$h_\lambda^s \;\geq\; \gamma (h_u^s)^3 \tag{1.6}$$

(see [10] for further details.)

Here we tackle the whole problem (1.4), proving optimal error estimates for the three variables $u,\ \lambda,\ \psi$. We assume that two independent grids are given, this time for V_h^s and Φ_h. It seems particularly reasonable to assume the grid for Φ_h as given, since in many cases this will be the space were multidomain preconditioners (for the Schur complement) will be applied: to have a convenient grid might then allow the use of more powerful preconditioners (see for instance [13], [15], [3], [5], [20], [14], [18], [19].) Similarly, the given grids for V_h^s might have been chosen, in each subdomain, to match specific difficulties arising in that subdomain, and related to the coefficients of the operator and/or to the right-hand side. On the other hand, the choice of the grid for M_h^s seems to be less crucial, at least in a certain number of applications, and could be made in order to allow an easier stabilization. As in the single domain problem, we work with piecewise linear finite elements for ψ_h and u_h. The bubbles used for stabilization, as well as the unknowns λ_h^s (piecewise constants), can be eliminated by static condensation. For the sake of simplicity we restrict ourselves to the case were the given grids for V_h^s and Φ_h are "comparable", in the above sense. It would surely be interesting to have results under weaker assumptions, at least of the type (1.6). On the other hand, we allow the case of "very close nodes" that can be generated by merging the nodes of the two given independent grids. For a multifield formulation similar to the one discussed here (but using four fields) and related preconditioners we refer to [16].

We now give an outline of the paper. In the next Section we make precise our assumptions on the domain, the operator, the given decompositions and the formulation of the stabilized discrete problem. We shall also recall a few trace theorems that will be used in the subsequent sections. In Section 3 we prove the error estimates: we start with $u-u_h$, proving optimal estimates first in the broken H^1-norm and then in the L^2-norm. Optimal error estimates are then proven for $\psi-\psi_h$ and for $\lambda-\lambda_h$, in suitable weighted L^2-norms. Under our assumptions, these estimates imply optimal error bounds in $L^2(\Sigma)$ for $\psi-\psi_h$. The same is true for $\lambda-\lambda_h$, but only outside the possible "very small intervals". Finally, in Section 4 we underline some numerical aspects, such as the actual implementation of the procedure and its expected numerical performances.

2. Statement of the problem and discretization

2.1. Assumptions on the operator, the domain, and the subdomains. To fix ideas, let us consider the following differential problem

$$\begin{cases} Au &= f & \text{in } \Omega \subset R^2 \\ u &= 0 & \text{on } \partial\Omega \end{cases} \tag{2.1}$$

where Ω is a convex polygon, and

$$Au \equiv -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_j}\left(a_{ij}(x)\frac{\partial u}{\partial x_i}\right) + a_0(x)u. \tag{2.2}$$

We assume that, for almost every x in Ω, $0 \le a_0(x) \le K$ and that the 2×2 matrix $(a_{ij}(x))$ is symmetric and positive definite, with smallest eigenvalue $\ge \alpha > 0$ and largest eigenvalue $\le \beta$, independent of x. To the operator A we assign the bilinear form

$$a(w,v) = \int_\Omega (\sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_j} + a_0(x) w\, v)\, dx, \qquad w,\, v \in H^1(\Omega). \tag{2.3}$$

Let $V = H_0^1(\Omega)$, with norm $\|\cdot\|_V = |\cdot|_{1,\Omega}$. With our assumptions on the coefficients the bilinear form is continuous and $V-$elliptic, i.e.,

$$\exists \alpha > 0: \qquad \alpha |v|_{1,\Omega}^2 \qquad \le a(v,v) \qquad \forall v \in V, \tag{2.4}$$

$$\exists M > 0: \qquad a(w,v) \quad \le M \|w\|_{1,\Omega} \|v\|_{1,\Omega} \qquad \forall w, v \in V. \tag{2.5}$$

Then for, say, $f \in L^2(\Omega)$ problem (1.1) has a unique solution $u \in H_0^1(\Omega) \cap H^2(\Omega)$.

Let now Ω be partitioned into a finite number N of polygonal subdomains Ω^s $(s = 1, N)$, with boundary $\Gamma^s = \partial\Omega^s$ and diameter H_s. We make precise the notation of (1.2) by defining, for $s = 1, N$,

$$\begin{array}{l} V^s = \{v \in H^1(\Omega^s),\ v = 0 \ \text{ on } \Gamma^s \cap \partial\Omega\}, \\ <\cdot,\cdot>_s = \text{ duality pairing between } H^{1/2}(\Gamma^s) \text{ and } H^{-1/2}(\Gamma^s), \\ (\cdot,\cdot)_s = \text{ scalar product in } L^2(\Omega^s), \\ a_s(w,v) = \sum_{i,j=1}^2 (a_{ij}(x) \frac{\partial w}{\partial x_i}, \frac{\partial v}{\partial x_j})_s + (a_0(x) w, v)_s \ \ w, v \in V. \end{array} \tag{2.6}$$

In agreement with the definition of V and $\Phi = \{\text{traces on } \Sigma \text{ of the functions of } V\}$, we will have:

$$\Phi = \{\varphi \in H^{1/2}(\Sigma) : \varphi = 0 \text{ on } \partial\Omega\}. \tag{2.7}$$

With this notation problem (1.1) is equivalent to problem (1.2), and the relationship between the solution w of (1.1) and the solution (u, λ, ψ) of (1.2) is precisely given in (1.3). We explicitly note that (2.5) will also hold in each Ω^s, by replacing Ω, V and a with Ω^s, V^s and a_s respectively.

Finally, we recall some known results related to trace theorems, whose proof can be checked by using trace theorems on the reference element. We have that for every $\theta_0 \le \pi/3$ there exists a constant $C = C(\theta_0)$ such that, for every triangle T with minimum angle bounded from below by θ_0,

$$|v|_{1/2,\partial T} \quad \le C\ |v|_{1,T} \qquad \forall v \in H^1(T), \tag{2.8}$$

$$|\frac{\partial v}{\partial n_a}|_{1/2,\partial T} \quad \le C\ |v|_{2,T} \qquad \forall v \in H^2(T), \tag{2.9}$$

$$|v|_{3/2,\partial T} \quad \le C\ |v|_{2,T} \qquad \forall v \in H^2(T), \tag{2.10}$$

where, as we shall do throughout the whole paper, the usual notation for Sobolev norms and seminorms (see e.g. [12]) is used. On the other hand, we can easily deduce that, if e is an edge of T, and h_T its diameter,

$$\|v\|_{0,e}^2 \le C\, h_T^{-1} \|v\|_{0,T} (\|v\|_{0,T} + h_T |v|_{1,T}) \qquad \forall v \in H^1(T), \tag{2.11}$$

always with a constant C depending only on θ_0. The basic instrument for proving (2.11) is the fundamental theorem of calculus applied to v_n^2, where v_n is a sequence of smooth functions converging to v in $H^1(T)$.

We end this section by recalling a well known property of Sobolev fractional norms and seminorms. We report it for seminorms of fractional order θ in dimension one, but the result is more general. If an interval I is the disjoint union of subintervals I_k, then

$$\sum_k |v|_{\theta,I_k}^2 \le |v|_{\theta,I}^2 \qquad \forall v \in H^\theta(I). \tag{2.12}$$

2.2. **Assumptions on $\mathcal{T}_\psi$, $\mathcal{T}_u^s$ and generation of the $\mathcal{T}_\lambda^s$ decompositions.** In order to discretize problem (1.2), let $\mathcal{T}_\psi$ be a decomposition of Σ into intervals I, and let, for each $s = 1, N$, $\mathcal{T}_u^s$ be a decomposition of Ω^s into triangles T, and $\mathcal{T}_\lambda^s$ a decomposition of Γ^s into intervals I. We shall denote by h_u^s, h_λ^s, h_ψ the mesh size of $\mathcal{T}_u^s$, $\mathcal{T}_\lambda^s$, $\mathcal{T}_\psi$ respectively. Next, we define the finite element spaces

$$V_h^s \quad = \{v \in V^s :\ v_{|T} \in P_1(T)\ \forall T \in \mathcal{T}_u^s\} \qquad \forall s, \tag{2.13}$$

$$M_h^s \quad = \{\mu^s \in L^2(\Gamma^s) :\ \mu^s_{|I} \in P_0(I)\ \forall I \in \mathcal{T}_\lambda^s\} \qquad \forall s, \tag{2.14}$$

$$\Phi_h \quad = \{\varphi \in \Phi :\ \varphi_{|I} \in P_1(I)\ \forall I \in \mathcal{T}_\psi\}, \tag{2.15}$$

and we set, as before, $V_h^* := \mathbf{\Pi}_s V_h^s$ and $M_h^* := \mathbf{\Pi}_s M_h^s$. Then we look for an approximate solution (u_h, λ_h, ψ_h) of (1.2), with $u_h \in V_h^*$, $\lambda_h \in M_h^*$, and $\psi_h \in \Phi_h$. If the three decompositions are totally arbitrary, the discrete problem can be unstable, and even singular. Here we assume that the decompositions $\mathcal{T}_u^s$ and $\mathcal{T}_\psi$ are given and

'untouchable'. To avoid unnecessary complications in the bubble stabilization described below, we just require that, for each s, the decomposition $\mathscr{T}_u^s$ does not include triangles having more than one edge on the boundary of Ω^s. As we have seen in the Introduction, we can however choose the grid $\mathscr{T}_\lambda^s$ in such a way that the problem becomes stabilizable by the simple addition of boundary bubbles to the spaces V_h^s. To do that, however, we shall need, as in [7], some minor assumptions on the two decompositions $\mathscr{T}_u^s$ and $\mathscr{T}_\psi$. Next, we will generate the $\mathscr{T}_\lambda^s$ decomposition and introduce the stabilization through boundary bubbles. We assume that the decomposition $\mathscr{T}_\psi$ is quasi-uniform [12] and that, for all s, the restriction to the boundary Γ^s of $\mathscr{T}_u^s$ is also quasi-uniform. We assume moreover that the decompositions are comparable. In particular we introduce h_{max} as the biggest among the lengths of the intervals of $\mathscr{T}_\psi$ and the lengths of the boundary edges of the decompositions $\mathscr{T}_u^s$, for all s. Similarly, we define h_{min} as the minimum over the same set. We assume that:

$$\exists \gamma_1 > 0 \text{ such that} \qquad h_{max} \leq \gamma_1 h_{min}. \tag{2.16}$$

In the sequel, to avoid heaviness in the notation we shall write h in place of h_{min} and $\overline{h}$ for the maximum diameter over $\mathscr{T}_\psi$ and over all $\mathscr{T}_u^s$'s (including internal triangles). We apologize for the rather unusual notation but, as we shall see, h will be used much more often than $\overline{h}$.

Remark 1. *The assumptions on the $\mathscr{T}_u$ grid are rather heavy, and will forbid the application of the present theory to several interesting cases. We remark nevertheless that, in a certain number of cases, grid refinements and self adaptive procedures might occur only in the interior of the subdomains, thus escaping our present limitations. On the other hand, as we shall see in the last Section, we believe that the above assumptions are just technical ones, and that the results (and the applicability of the whole method) are valid in more general circumstances.*

We generate now the grid for the λ_h's as

$$\mathscr{T}_\lambda^s = \text{ merge } \{(\mathscr{T}_u^s)_{|\Gamma^s}, (\mathscr{T}_\psi)_{|\Gamma^s}\} \qquad \forall s. \tag{2.17}$$

More precisely, for all s, we take as nodes for $\mathscr{T}_\lambda^s$ the union of the nodes of $\mathscr{T}_\psi$ and the nodes of $\mathscr{T}_u^s$ belonging to Γ^s. In doing this it may occur that $\mathscr{T}_\lambda^s$ has very small intervals, whenever two nodes of $(\mathscr{T}_u^s)_{|\Gamma^s}$ and $(\mathscr{T}_\psi)_{|\Gamma^s}$ get too close. Such "small intervals" will be allowed in our discussion (neglecting possible programming problems), but will need a special treatment from the theoretical point of view. For this we introduce a constant β such that $0 < \beta < 1$ and we define, following essentially the notation of [10], "irregular" the intervals whose length is smaller than βh. More precisely, for $I_\lambda^s \in \mathscr{T}_\lambda^s$, we say that

$$I_\lambda^s \text{ is "irregular" if } |I_\lambda^s| \leq \beta h. \tag{2.18}$$

Otherwise I_λ^s will be called "regular". Therefore, we have

$$\exists \gamma_2 > 0 \text{ such that, } \forall s \; \forall I_\lambda^s \in \mathscr{T}_\lambda^s \qquad \text{if } I_\lambda^s \text{ is "regular", then } |I_\lambda^s| \geq \gamma_2 h. \tag{2.19}$$

Note that one endpoint of an irregular interval must be a node of $\mathscr{T}_u^s$, and the other one a node of $\mathscr{T}_\psi$.

2.3. **Introduction of the bubbles.** For each s, we add to the discretization of u^s as many bubbles as the intervals of $\mathscr{T}_\lambda^s$. More precisely, let T be a triangle having an edge T' on Γ^s, so that $T' = \cup_k I_k$, $I_k \in \mathscr{T}_\lambda^s$. Accordingly, $T = \cup_k T_k$. (See Fig. 1). A boundary bubble b_k is a function $b_k \in H^1(\Omega^s)$ such that $supp(b_k) \subset T_k$ and $\int_{I_k} b_k \neq 0$. (See [7] for more details.) The optimal shape of these bubbles is still under investigation. As a simple example, one can think of a quadratic function vanishing on the two internal edges of T_k (see Fig. 1). As the choice has to be made up to a scaling factor, we can then assume that b_k has value 1/4 at the midpoint of I_k. With this choice one can easily compute that

$$\int_{T_k} b_k \, dx = |T_k|/12, \qquad \int_{I_k} b_k \, ds = |I_k|/6, \qquad \int_{T_k} |\nabla b_k|^2 \, dx = \frac{\sum_{i=1}^3 |e_{k,i}|^2}{48|T_k|}, \tag{2.20}$$

where $e_{k,i}$ are the edges of T_k. Consequently, we can deduce

$$|b_k|^2_{1,T_k} \equiv \int_{T_k} |\nabla b_k|^2 \, dx \leq \gamma h_T / |I_k|, \tag{2.21}$$

with h_T = diameter of T, $|I_k|$ = length of I_k, and γ a positive constant independent of h_T and $|I_k|$.

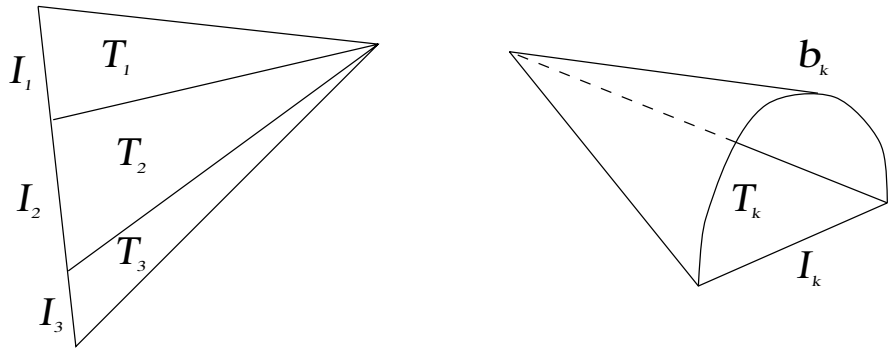


FIGURE 1

2.4. **The stabilized problem.** From now on we shall always assume that the decompositions $\mathscr{T}_\psi$ and $\mathscr{T}_u^s$ are given and satisfy (2.16), that the decompositions $\mathscr{T}_\lambda^s$ are generated according to (2.17), and that the spaces V_h^s, M_h^s and Φ_h are defined as in (2.13), (2.14) and (2.15), respectively. Let moreover, for every s, B_h^s be the space of bubbles introduced above. We shall replace the space V_h^s by the augmented space

$$\widetilde{V}_h^s \;=\; V_h^s \oplus B_h^s \qquad \forall s \tag{2.22}$$

and then set $\widetilde{V}_h^* = \mathbf{\Pi}_s \widetilde{V}_h^s$, $M_h^* = \mathbf{\Pi}_s M_h^s$. The stabilized discrete problem is now

$$\left\{\begin{array}{lllll} \text{find } u_h \in \widetilde{V}_h^*,\ \lambda_h \in M_h^* \text{ and } \psi_h \in \Phi_h & \text{such that} & & & \\ \text{i) } a_s(u_h^s, v) - <\lambda_h^s, v>_s & = (f,v)_s & \forall v \in \widetilde{V}_h^s & \forall s \\ \text{ii) } <\mu^s, u_h^s>_s & = <\mu^s, \psi_h>_s & \forall \mu^s \in M_h^s & \forall s \\ \text{iii) } \textstyle\sum_s <\lambda_h^s, \varphi>_s & = 0 & \forall \varphi \in \Phi_h. & \end{array}\right. \tag{2.23}$$

Remark 2. *As pointed out in* [7] *and further developed in* [10] *for the single-domain case, the bubbles and the multipliers λ_h^s can be eliminated by static condensation, leaving, as unknowns, only ψ_h and the piecewise linear part of u_h. This will be further discussed in the last Section.*

3. ERROR ESTIMATES

Comparing (1.2) and (2.23) we easily obtain the error equations

$$\left\{\begin{array}{lllll} \text{i) } a_s(u^s - u_h^s, v) - <\lambda^s - \lambda_h^s, v>_s & = 0 & \forall v \in \widetilde{V}_h^s & \forall s \\ \text{ii) } <\mu^s, u^s - u_h^s>_s & = <\mu^s, \psi - \psi_h>_s & \forall \mu^s \in M_h^s & \forall s \\ \text{iii) } \textstyle\sum_s <\lambda^s - \lambda_h^s, \varphi>_s & = 0 & \forall \varphi \in \Phi_h. & \end{array}\right. \tag{3.1}$$

Let $u_I^s \in \widetilde{V}_h^s$, $\lambda_I^s \in M_h^s$, $\psi_I \in \Phi_h$ be some interpolants of $u^s \in V^s$, $\lambda^s \in M^s$, $\psi \in \Phi$ respectively, to be specified later. Then, by adding and subtracting these interpolants in (3.1) we obtain

$$\begin{array}{rll} \alpha \textstyle\sum_s |u^s - u_h^s|_{1,\Omega^s}^2 & \le \underbrace{\displaystyle\sum_s a_s(u^s - u_h^s, u^s - u_I^s)}_{I} & + \textstyle\sum_s a_s(u^s - u_h^s, u_I^s - u_h^s) \qquad \text{(use (3.1 } i\text{))} \\ & = & + \textstyle\sum_s <\lambda^s - \lambda_h^s, u_I^s - u_h^s>_s . \end{array} \tag{3.2}$$

For the second term in (3.2), by adding and subtracting u^s we have

$$\begin{array}{rll} \textstyle\sum_s <\lambda^s - \lambda_h^s, u_I^s - u_h^s>_s & = \underbrace{\displaystyle\sum_s <\lambda^s - \lambda_h^s, u_I^s - u^s>_s}_{II} & + \textstyle\sum_s <\lambda^s - \lambda_h^s, u^s - u_h^s>_s \\ & & + \textstyle\sum_s <\lambda^s - \lambda_h^s, u^s - u_h^s>_s . \end{array} \tag{3.3}$$

By inserting now λ_I^s in the last term of (3.3) and using (3.1 ii) we have

$$\begin{array}{rll} \textstyle\sum_s <\lambda^s - \lambda_h^s, u^s - u_h^s>_s & = \underbrace{\displaystyle\sum_s <\lambda^s - \lambda_I^s, u^s - u_h^s>_s}_{III} & + \textstyle\sum_s <\lambda_I^s - \lambda_h^s, u^s - u_h^s>_s \\ & & + \textstyle\sum_s <\lambda_I^s - \lambda_h^s, \psi - \psi_h>_s . \end{array} \tag{3.4}$$

Finally, inserting λ^s in the last term of (3.4) and using (3.1 iii) we have

$$\begin{aligned} \textstyle\sum_s < \lambda_I^s - \lambda_h^s, \psi - \psi_h >_s &= \underbrace{\sum_s < \lambda_I^s - \lambda^s, \psi - \psi_h >_s}_{IV} + \underbrace{\sum_s < \lambda^s - \lambda_h^s, \psi - \psi_I >_s}_{V.} \\ &= IV + V. \end{aligned} \tag{3.5}$$

We will now define proper interpolants to estimate the different pieces of (3.2), (3.3), (3.4), and (3.5). From now on we will omit the superscript s unless it is really necessary.

Let $\psi_I \in \Phi_h$ be the usual continuous piecewise linear interpolant of ψ, and, for each s, let $u_I \in V_h^s$ be the usual continuous piecewise linear interpolant of u. The standard interpolation estimates are [12]

$$||u - u_I||_{r,T} \leq C h_T^{2-r} |u|_{2,T} \qquad \forall T \in \mathscr{T}_u^s \qquad (r = 0, 1), \tag{3.6}$$

$$||u - u_I||_{r,e} \leq C|e|^{3/2-r} |u|_{3/2,e} \qquad \forall e \text{ edge of } T,\ \forall T \in \mathscr{T}_u^s \qquad (r = 0, 1), \tag{3.7}$$

$$||\psi - \psi_I||_{r,I} \leq C|I|^{3/2-r} |\psi|_{3/2,I} \qquad \forall I \in \mathscr{T}_\psi \qquad (r = 0, 1). \tag{3.8}$$

(Here and in the sequel C will denote a constant independent of the mesh size, not necessarily the same in the various occurrences). For the $\lambda-$variable two different 'interpolants' will be needed. For each s, let us denote by λ_I the piecewise constant function *on the* $\mathscr{T}_\psi-$*grid* defined by

$$\int_{I_\psi} (\lambda - \lambda_I)\, ds = 0 \qquad \forall I_\psi \in \mathscr{T}_\psi, \tag{3.9}$$

and by $\widetilde{\lambda} \in M_h^s$ the piecewise constant function *on the* $\mathscr{T}_\lambda-$*grid* defined by

$$\int_{I_{\widetilde{\lambda}}^s} (\lambda - \widetilde{\lambda}) b\, ds = 0 \qquad \forall I_\lambda^s \in M_h^s,\ \forall b \in B_h^s. \tag{3.10}$$

We recall that the following interpolation estimates hold (see [12])

$$||\lambda - \lambda_I||_{-r,I_\psi} \leq C|I_\psi|^{1/2+r} |\lambda|_{1/2,I_\psi}, \qquad 0 \leq r \leq 1/2, \tag{3.11}$$

$$||\lambda - \widetilde{\lambda}||_{-r,I_\lambda^s} \leq C|I_\lambda^s|^{1/2+r} |\lambda|_{1/2,I_\lambda^s} \qquad 0 \leq r \leq 1/2. \tag{3.12}$$

We also recall that (1.3) implies that λ takes opposite values on the two sides of each subdomain interface. In its turn λ_I, being defined on the $\mathscr{T}_\psi-$grid, inherits the same property. Hence,

$$\sum_s < \lambda, \varphi >_s = \sum_s < \lambda_I, \varphi >_s = 0 \qquad \forall \varphi \in \Phi, \tag{3.13}$$

as $\varphi \in \Phi$ is single-valued on Σ. On the other hand, $\widetilde{\lambda}$ does not enjoy this property.

Some comments are in order before proceeding to estimate the various pieces in (3.2)-(3.5). In doing this, two types of difficulties will arise. One is connected with the fact that, on the left-hand side of (3.2), we have the broken $H^1 - seminorm$ of the error while, in estimating some of the pieces, it would be easier to use the broken $H^1 - norm$. Hence, we will need to bound the norm of the error with its seminorm. This is not a major problem, and is dealt with in Lemma 4. The second source of difficulties lies in the estimate of pieces II and V, whose treatment is actually very similar, where an estimate of $\lambda - \lambda_h$ in terms of $u - u_h$ is needed. This is done in Lemma 1.

Lemma 1. *Let T be a boundary triangle of $\mathscr{T}_u^s$, and let I_k be a boundary edge as in Fig. 1. Then we have*

$$||\lambda_h - \widetilde{\lambda}||_{0,I_k} \leq C {h_T}^{1/2} |I_k|^{-1} ||u - u_h||_{1,T_k}, \tag{3.14}$$

where (u, λ, ψ) is the solution of (1.2) and (u_h, λ_h, ψ_h) that of (2.23).

Proof.

$$\begin{aligned} |\lambda_h - \widetilde{\lambda}|_{|I_k} &= |\textstyle\int_{I_k} (\lambda_h - \widetilde{\lambda}) b_k\, ds| / \int_{I_k} b_k\, ds && \text{(use (3.10))} \\ &= |\textstyle\int_{I_k} (\lambda_h - \lambda) b_k\, ds| / \int_{I_k} b_k\, ds && \text{(use (3.1 } i\text{))} \\ &= |a(u - u_h, b_k)| / \textstyle\int_{I_k} b_k\, ds && \text{(use (2.5))} \\ &\leq M||u - u_h||_{1,T_k} ||b_k||_{1,T_k} / \textstyle\int_{I_k} b_k\, ds && \text{(use (2.20) – (2.21))} \\ &\leq C {h_T}^{1/2} |I_k|^{-3/2} ||u - u_h||_{1,T_k}. \end{aligned} \tag{3.15}$$

The result follows then by observing that

$$||\lambda_h - \widetilde{\lambda}||_{0,I_k} = |\lambda_h - \widetilde{\lambda}|_{|I_k}|I_k|^{1/2} \tag{3.16}$$

□

The unsatisfactory part of (3.14) is clearly the presence of $|I_k|^{-1}$. To deal with it in the treatment of II and V we shall need an estimate for $u - u_I$ and $\psi - \psi_I$, respectively, where $|I_k|$ appears as a factor. This is done in the following Lemma.

Lemma 2. *Let $e \subset \Gamma^s$ be either an edge of a triangle of $\mathcal{T}_u^s$ or an interval of $\mathcal{T}_\psi$, and let I_k be an interval of $\mathcal{T}_\lambda^s$ included in e . Let g be a function in $H^{3/2}(e)$, and let g^I be its linear interpolant on e (that is, $g^I \in P_1(e)$, and $g^I = g$ at the endpoints of e). Then the following estimate holds*

$$||g - g_I||_{0,I_k} \leq C|e|^{1/2}|I_k||g|_{3/2,e}. \tag{3.17}$$

Proof. If I_k is a regular interval, the result follows immediately from usual interpolation estimates [12] and (2.16), (2.19). Indeed we have

$$||g - g_I||_{0,I_k} \leq C|e|^{3/2}|g|_{3/2,e} \leq C|e|^{1/2}|I_k||g|_{3/2,e}. \tag{3.18}$$

If instead I_k is irregular, then one of its endpoints must coincide with an endpoint of e. Setting then $d = g - g_I$, and choosing a coordinate system on e such that $I_k = (0, |I_k|)$, and $d(0) = 0$, the fundamental theorem of calculus and Cauchy-Schwarz inequality give

$$d(s) = \int_0^s d'(t)\,dt \leq s^{1/2}(\int_0^s (d'(t))^2\,dt)^{1/2}. \tag{3.19}$$

From (3.19) we then have

$$\begin{aligned} ||d||_{0,I_k}^2 &\equiv \textstyle\int_0^{|I_k|} d^2(s)\,ds \leq \int_0^{|I_k|} s(\int_0^s (d'(t))^2\,dt)\,ds \\ &\leq |I_k| \textstyle\int_0^{|I_k|}(\int_0^{|I_k|}(d'(t))^2\,dt)\,ds = |I_k|^2|d|_{1,I_k}^2, \end{aligned} \tag{3.20}$$

so that

$$||g - g_I||_{0,I_k} \leq |I_k|\,|g - g_I|_{1,I_k}. \tag{3.21}$$

Finally, the result follows from (3.21) and interpolation estimates [12]

$$|g - g_I|_{1,I_k} \leq |g - g_I|_{1,e} \leq C|e|^{1/2}|g|_{3/2,e}. \tag{3.22}$$

□

Lemma 2 is the crucial step where we could not avoid the assumption (2.16), at least in the present line of proof. Indeed, for intervals $|I_k|$ which are away from the endpoints of e, we cannot estimate $||g - g_I||_{0,I_k}$ better than $|e|^2|I_k|^{1/2}$, even using a $W^{2,\infty}$ regularity for g (which, in the applications, will be either u or ψ.) This is not enough to compensate for the $|I_k|^{-1}$ term in (3.14), unless $|e|$ is bounded in terms of $|I_k|$. The use of a higher regularity for g and/or a nonlinear bound of $|e|$ in terms of $|I_k|$ would strongly increase the ugliness of the paper without seriously improving the quality of the results. An ideal result would be to derive optimal error bounds without assuming any relationship between $\mathcal{T}_{u|\Sigma}$ and $\mathcal{T}_\psi$, but we did not succeed so far. In view of this, from now on we shall use (2.16) whenever this can simplify the proofs, even when it is not strictly necessary.
The combined use of Lemma 1 and Lemma 2 allows us to prove the following Lemma 3 and Corollary 1, which provide the basic instruments for estimating the terms II and V, respectively.

Lemma 3. *Let T be a boundary triangle of $\mathcal{T}_u^s$ with a boundary edge T', and let I_k be an interval of $\mathcal{T}_\lambda^s$ included in T'. Then*

$$|\int_{I_k} (\lambda - \lambda_h)(u - u_I)\,ds| \leq C(h_T^2|\lambda|_{1/2,I_k} + h_T||u - u_h||_{1,T})|u|_{2,T}, \tag{3.23}$$

where (u, λ, ψ) is the solution of (1.2), (u_h, λ_h, ψ_h) that of (2.23), and u_I is the linear interpolant of u.

Proof. Using Cauchy-Schwarz inequality, the triangle inequality, the interpolation estimate (3.12) and then (3.14) and (3.17) with $g = u$ we easily deduce

$$
\begin{aligned}
&|\int_{I_k}(\lambda-\lambda_h)(u-u_I)\,ds| \le \|\lambda-\lambda_h\|_{0,I_k}\|u-u_I\|_{0,I_k}\\
&\le (\|\lambda-\widetilde{\lambda}\|_{0,I_k}+\|\widetilde{\lambda}-\lambda_h\|_{0,I_k})\|u-u_I\|_{0,I_k}\\
&\le (C|I_k|^{1/2}|\lambda|_{1/2,I_k}+\|\widetilde{\lambda}-\lambda_h\|_{0,I_k})\|u-u_I\|_{0,I_k}\\
&\le C(|I_k|^{1/2}|\lambda|_{1/2,I_k}+h_T^{1/2}|I_k|^{-1}\|u-u_h\|_{1,T})h_T^{1/2}|I_k|\ |u|_{3/2,T'},
\end{aligned}
\tag{3.24}
$$

and the result follows immediately from (2.10) and $|I_k| \le Ch_T$. □

With similar arguments we can prove the following result.

Corollary 1. *In the same hypotheses of Lemma 3 we have*

$$
|\int_{I_k}(\lambda-\lambda_h)(\psi-\psi_I)\,ds| \le C(|I_k|^{3/2}|\lambda|_{1/2,I_k}+h_T^{1/2}\|u-u_h\|_{1,T})|I_\psi|^{1/2}|\psi|_{3/2,I_\psi},
\tag{3.25}
$$

where (u,λ,ψ) is the solution of (1.2), (u_h,λ_h,ψ_h) that of (2.23), I_ψ is the interval $\in \mathcal{T}_\psi$ such that $I_k \subseteq I_\psi$, and ψ_I is the linear interpolant of ψ .

Proof. The proof mimics exactly that of Lemma 3 with $u = \psi,\ u_I = \psi_I$. □

We have now all the necessary instruments to estimate the pieces (3.2)-(3.5). To simplify the following notation, we set, for s=1,N

$$
E_{h,s}^2 = \|u-u_h\|_{1,\Omega^s}^2; \qquad E_{I,s}^2 = \textstyle\sum_{T\subset\Omega^s} h_T^2|u|_{2,T}^2 \tag{3.26}
$$

$$
E_h^2 = \sum_s E_{h,s}^2; \qquad E_I^2 = \textstyle\sum_s E_{I,s}^2. \tag{3.27}
$$

We can now estimate the different pieces in (3.2)-(3.3)-(3.4)-(3.5). We begin with (3.2):

$$
\begin{aligned}
I &= \textstyle\sum_s a_s(u-u_h,u-u_I) && \text{(use (2.5))}\\
&\le M\textstyle\sum_s \|u-u_h\|_{1,\Omega^s}\|u-u_I\|_{1,\Omega^s} && \text{(C-S)}\\
&\le M(\textstyle\sum_s \|u-u_h\|_{1,\Omega^s}^2)^{1/2}(\sum_s \|u-u_I\|_{1,\Omega^s}^2)^{1/2} && \text{(use (3.26) and (3.27))}\\
&= ME_hE_I,
\end{aligned}
\tag{3.28}
$$

where, as well as in the following, C-S indicates the use of Cauchy-Schwarz inequality. We now estimate (3.3). We recall that (2.16) implies that for every triangle T in $\mathcal{T}_u^s$ having an edge T' in Γ^s there are only a finite number of subintervals I_k of $\mathcal{T}_\lambda^s$ contained in T'. Then:

$$
\begin{aligned}
II &= \textstyle\sum_s <\lambda-\lambda_h, u_I-u>_s\\
&= \textstyle\sum_s\sum_T\sum_k \int_{I_k}(\lambda-\lambda_h)(u_I-u)\,d\Gamma && \text{(use (3.23) and (2.12))}\\
&\le C\textstyle\sum_s\sum_T(h_T^2|\lambda|_{1/2,T'}+h_T\|u-u_h\|_{1,T})|u|_{2,T} && \text{(use (1.3) and (2.9))}\\
&\le C\textstyle\sum_s\sum_T(h_T^2|u|_{2,T}+h_T\|u-u_h\|_{1,T})|u|_{2,T} && \text{(C-S)}\\
&\le C(E_I^2+E_hE_I)
\end{aligned}
\tag{3.29}
$$

The estimate of (3.4) is also simple, provided that we introduce the following notation. For every s, and for every I_ψ in Γ^s we denote by T_ψ a generic triangle (not necessarily in $\mathcal{T}_u^s$ but respecting the minimum angle condition)

belonging to Ω^s and having I_ψ as an edge. Then,

$$
\begin{aligned}
III &= \textstyle\sum_s < \lambda - \lambda_I, u - u_h >_s \\
&= \textstyle\sum_s \sum_{I_\psi} \int (\lambda - \lambda_I)(u - u_h)\, d\Gamma && \text{((3.9) and duality)} \\
&\le \textstyle\sum_s \sum_{I_\psi} ||\lambda - \lambda_I||_{-1/2, I_\psi} |u - u_h|_{1/2, I_\psi} && \text{(use (3.11) and (2.8))} \\
&\le C \textstyle\sum_s \sum_{I_\psi} |I_\psi| |\lambda|_{1/2, I_\psi} |u - u_h|_{1, T_\psi} && \text{(use (2.9) and C-S)} \\
&\le C \textstyle\sum_s (\sum_T h_T^2 |u|_{2,T})^{1/2} |u - u_h|_{1, \Omega^s} && \text{(C-S)} \\
&\le C E_I E_h.
\end{aligned}
\tag{3.30}
$$

The term IV would be a potential source of major difficulties, as it requires in principle an estimate of $\psi - \psi_h$ in terms of $u - u_h$. Our choice of λ_I, however, makes it vanish thanks to (3.13):

$$
IV = \sum_s < \lambda_I^s - \lambda^s, \psi - \psi_h >_s = 0. \tag{3.31}
$$

Before estimating V we need further notation. For every triangle T in $\mathscr{T}_u^s$, we define $I_\psi(T)$ as the union of the intervals I_ψ in $\mathscr{T}_\psi$ which intersect ∂T on a set of nonzero length. Note that there are only a finite number of them, depending only on the constant γ_1 in (2.16). We also denote by T_ψ^s any triangle (with the minimum angle property) in Ω^s having $I_\psi(T)$ as an edge. Then the estimate of the last piece proceeds as follows.

$$
\begin{aligned}
V &= \textstyle\sum_s < \lambda - \lambda_h, \psi - \psi_I >_s \\
&= \textstyle\sum_s \sum_T \sum_k \int_{I_k} (\lambda - \lambda_h)(\psi - \psi_I)\, d\Gamma && \text{(use (3.25))} \\
&\le C \textstyle\sum_s \sum_T \sum_k (|I_k|^{3/2} |\lambda|_{1/2, I_k} + h_T^{1/2} ||u - u_h||_{1,T}) |I_\psi|^{1/2} |\psi|_{3/2, I_\psi} && \text{(use (2.12) and (2.16))} \\
&\le C \textstyle\sum_s \sum_T (h_T^2 |\lambda|_{1/2, T'} + h_T ||u - u_h||_{1,T}) |\psi|_{3/2, I_\psi(T)} && \text{(use (2.9) and (2.10))} \\
&\le C \textstyle\sum_s \sum_T (h_T^2 |u|_{2,T} + h_T ||u - u_h||_{1,T}) |u|_{2, T_\psi} && \text{(C-S)} \\
&\le C(E_I^2 + E_h E_I).
\end{aligned}
\tag{3.32}
$$

We can now collect equations (3.2)-(3.5) and the estimates (3.28)- (3.32), thus obtaining

$$
\alpha \sum_s |u_s - u_h^s|_{1, \Omega^s}^2 \le C(E_I^2 + E_I E_h). \tag{3.33}
$$

The final estimate would then be achieved if we could bound the term $E_h = \sum_s ||u - u_h||_{1,\Omega^s}^2$ appearing in the right-hand side of (3.33) in terms of the left-hand side. For this we need the following lemma.

Lemma 4. *If (u, λ, ψ) and (u_h, λ_h, ψ_h) are the solutions of (1.2) and (2.23) respectively, then*

$$
\sum_s ||u - u_h||_{1,\Omega^s}^2 \le C \sum_s |u - u_h|_{1,\Omega^s}^2. \tag{3.34}
$$

Proof. For each s, let q_s be defined as

$$
q_s = (\int_{\Omega^s} (u - u_h)\, dx)/|\Omega^s|, \tag{3.35}
$$

and let q be the piecewise constant function defined on Ω by $q_{|\Omega^s} = q_s$. Then set

$$
w = u - u_h - q. \tag{3.36}
$$

Since w has zero mean value in each Ω^s we easily have

$$
\sum_s ||w||_{1,\Omega^s}^2 \le C \sum_s |w|_{1,\Omega^s}^2 = C \sum_s |u - u_h|_{1,\Omega^s}^2. \tag{3.37}
$$

Let now $\chi \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of $-\Delta\chi = q$ in Ω; set $\tau = \nabla\chi$ and set, for every $s = 1, N$, $\mu_s^* = \frac{\partial \chi}{\partial n_s}_{|\Gamma^s} = \tau \cdot n_{s|\Gamma^s}$. We consider now a new, artificial triangulation in each Ω^s that agrees with $\mathscr{T}_\psi$ on Γ^s and has maximum diameter smaller than or equal to $\overline{h}$ (plus, clearly, the usual minimum angle condition).

Note that, in this way, we obtain a compatible decomposition of the whole Ω. Let now τ^I be the lowest order Raviart-Thomas interpolant of τ (see e.g. [6]). We have

$$||\tau^I||_{H(div;\Omega)} \leq C||\tau||_{1,\Omega} \leq C||\chi||_{2,\Omega} \leq C||q||_{0,\Omega}. \tag{3.38}$$

We notice that, for each s, if μ_I^s is the interpolant of μ_s^* defined as in (3.9), then

$$\mu_I^s = \tau^I \cdot n_{s|\Gamma^s}. \tag{3.39}$$

With classical arguments we have then

$$\begin{aligned} ||q||_{0,\Omega}^2 &= -\int_\Omega \Delta\chi q\, dx = -\sum_s \int_{\Omega^s} \Delta\chi q_s\, dx = -\sum_s \int_{\Gamma^s} \mu_s^* q_s\, d\Gamma = -\sum_s \int_{\Gamma^s} \mu_I^s q_s\, d\Gamma \\ &= -\sum_s \int_{\Gamma^s} \mu_I^s (q_s + w)\, d\Gamma + \sum_s \int_{\Gamma^s} \mu_I^s w\, d\Gamma = -\sum_s \int_{\Gamma^s} \mu_I^s (u - u_h)\, d\Gamma + \sum_s \int_{\Gamma^s} \mu_I^s w\, d\Gamma \\ &= -\sum_s \int_{\Gamma^s} \mu_I^s (\psi - \psi_h)\, d\Gamma + \sum_s \int_{\Gamma^s} \mu_I^s w\, d\Gamma. \end{aligned} \tag{3.40}$$

The first term in the last line of (3.40) vanishes thanks to (3.13), and the second can be bounded as follows

$$\begin{aligned} \sum_s \int_{\Gamma^s} \mu_I^s w\, d\Gamma &= \sum_s \int_{\Gamma^s} \tau^I \cdot n_s w\, d\Gamma = \sum_s (\int_{\Omega^s} \mathrm{div}\, \tau^{\mathrm{I}}\, \mathrm{w}\, \mathrm{dx} + \int_{\Omega^s} \tau^{\mathrm{I}} \cdot \nabla \mathrm{w}\, \mathrm{dx}) \\ &\leq ||\tau^I||_{H(div;\Omega)}\, (\sum_s ||w||_{1,\Omega^s}^2)^{1/2} \leq C||q||_{0,\Omega}\, (\sum_s ||w||_{1,\Omega^s}^2)^{1/2}, \end{aligned} \tag{3.41}$$

where, in the first step, we used (3.39), and in the last step we used (3.38). Hence, from (3.40),(3.41) and (3.37),

$$||q||_{0,\Omega} \leq C(\sum_s ||w||_{1,\Omega^s}^2)^{1/2} \leq C(\sum_s |u - u_h|_{1,\Omega^s}^2)^{1/2}. \tag{3.42}$$

Since

$$\sum_s ||u - u_h||_{1,\Omega^s}^2 = \sum_s ||w + q||_{1,\Omega^s}^2 = \sum_s (||w||_{1,\Omega^s}^2 + ||q||_{0,\Omega^s}^2), \tag{3.43}$$

using (3.42) in (3.43) we have the result . □

From (3.33) and (3.34) we deduce the first convergence theorem.

Theorem 1. *Let (u, λ, ψ) and (u_h, λ_h, ψ_h) be the solutions of (1.2) and (2.23) respectively. Then the following estimate holds*

$$E_h = (\sum_s ||u - u_h||_{1,\Omega^s}^2)^{1/2} \leq CE_I \leq C\overline{h}|u|_{2,\Omega}. \tag{3.44}$$

We are now able to prove a convergence result in $L^2(\Omega)$ for the u variable.

Theorem 2. *Let (u, λ, ψ) and (u_h, λ_h, ψ_h) be the solutions of (1.2) and (2.23) respectively. Then the following estimate holds*

$$||u - u_h||_{0,\Omega} \leq C\overline{h}E_I \leq C\overline{h}^2|u|_{2,\Omega}. \tag{3.45}$$

Proof. Let $w \in H_0^1(\Omega) \cap H^2(\Omega)$ be the solution of the adjoint problem of (2.1) $A^*w = u - u_h$ in Ω. Let, for every $s = 1, N$, $w_I^s \in V_h^s$ be the piecewise linear interpolant of w, so that interpolation estimate (3.6) holds, and let $\widetilde{w}_I \in \Phi_h$ be the piecewise linear interpolant of w on Σ, for which interpolation estimate (3.8) holds. Then, integrating by parts, inserting w_I^s, using (3.1 i), and inserting $\widetilde{w}_I$ we get

$$\begin{aligned} ||u - u_h||_{0,\Omega}^2 &= < u - u_h, A^*w > = -\sum_s < u - u_h, \tfrac{\partial w}{\partial n_a} >_s + \sum_s a_s(u - u_h, w) \\ &= I + \sum_s a_s(u - u_h, w - w_I^s) + \sum_s a_s(u - u_h, w_I^s) = I + II + \sum_s < \lambda - \lambda_h, w_I^s >_s \\ &= I + II + \sum_s < \lambda - \lambda_h, w_I^s - w >_s + \sum_s < \lambda - \lambda_h, w - \widetilde{w}_I >_s + \sum_s < \lambda - \lambda_h, \widetilde{w}_I >_s \\ &= I + II + III + IV + V. \end{aligned} \tag{3.46}$$

The term I will be estimated later on. To estimate II we proceed as for (3.28) and use continuous dependence of w on $u - u_h$ and (3.44). Thus,

$$II \leq C\overline{h}|w|_{2,\Omega}(\sum_s ||u - u_h||_{1,\Omega^s}^2)^{1/2} \leq C\overline{h}E_h||u - u_h||_{0,\Omega}. \tag{3.47}$$

The term V vanishes thanks to (3.1 iii), while III and IV can be treated exactly as (3.29) and (3.32) respectively:

$$\begin{aligned} III,\ IV &\leq C\sum_s \sum_T (h_T^2|u|_{2,T} + h_T||u-u_h||_{1,T})|w|_{2,T} \\ &\leq C\sum_s \sum_T (h_T|u|_{2,T} + ||u-u_h||_{1,T})\overline{h}|w|_{2,T} \\ &\leq C(E_I^2+E_h^2)^{1/2}\ \overline{h}|w|_{2,\Omega} \leq C\overline{h}E_I||u-u_h||_{0,\Omega}. \end{aligned} \tag{3.48}$$

It remains to estimate the term I. For that, set, for every $s=1,N$, $\mu_s^* = \frac{\partial w}{\partial n_a}_{|\Gamma^s}$. Let μ_I^s be the interpolant of μ_s^* defined as in (3.9). Inserting μ_I^s, using (3.1 ii), (3.13), (3.11), and (2.8) we have

$$\begin{aligned} I &= -\sum_s < u-u_h, \mu_s^* - \mu_I^s >_s - \sum_s < u-u_h, \mu_I^s >_s \\ &= -\sum_s < u-u_h, \mu_s^* - \mu_I^s >_s - \sum_s < \psi-\psi_h, \mu_I^s >_s \\ &= -\sum_s < u-u_h, \mu_s^* - \mu_I^s >_s \leq \sum_s |u-u_h|_{1/2,\Gamma^s}||\mu_s^* - \mu_I^s||_{-1/2,\Gamma^s} \\ &\leq C\sum_s |u-u_h|_{1,\Omega^s}\overline{h}|\mu_s^*|_{1/2,\Gamma^s} \leq C\overline{h}E_h|w|_{2,\Omega} \leq C\overline{h}E_I||u-u_h||_{0,\Omega}. \end{aligned} \tag{3.49}$$

Finally, using (3.47)-(3.49) in (3.46) we conclude. □

So far, we have obtained optimal error estimates for the variable u. In a sense, this might be considered as sufficient, in order to assess the accuracy of the three-field formulation. However, it might be interesting to see if similar error estimates could be proved for the other two variables λ and ψ. In the sequel we are going to present some of those error estimates, restraining ourselves, for the sake of simplicity, to the ones whose proof is more elementary. In particular we shall obtain optimal error estimates for both ψ and λ in the weighted L^2-norms

$$|||\varphi|||_{\Sigma_\psi} = (\sum_{I_\psi} |I_\psi|\,||\varphi||_{0,I_\psi}^2)^{1/2} \tag{3.50}$$

and

$$|||\mu|||_{\Sigma_\lambda} = (\sum_s \sum_k |I_k|\,||\mu||_{0,I_k}^2)^{1/2} \tag{3.51}$$

In order to make the following exposition more fluent, we anticipate the following technical fact, whose proof is totally elementary and will therefore be left to the reader. Assume that I is an interval, ρ a real number with $0<\rho<1$, and I_ρ a subinterval having length $|I_\rho| \geq \rho|I|$. Let m_ρ be the midpoint of I_ρ. Then, for every p, polynomial of degree ≤ 1 that does not change sign in I, we have

$$\int_{I_\rho} p(m_\rho)p(t)\,dt = |I_\rho|p^2(m_\rho) \geq \frac{3\rho^3}{4}\int_I p^2(t)\,dt. \tag{3.52}$$

We can now prove the following Inf-Sup condition for the spaces M_h^s and $\Phi_{h|\Gamma^s}$.

Lemma 5. *Let s be fixed. Assume that the interval $I_\psi \in \mathcal{T}_{\psi|\Gamma^s}$ contains at least two regular intervals $I_\lambda \in \mathcal{T}_\lambda^s$. Then, for every $\varphi_h \in \Phi_h$ not identically zero on I_ψ, there exists a $\mu_h^s \in M_h^s$, not identically zero on I_ψ, such that*

$$||\mu_h^s||_{0,I_\psi}^2 = \int_{I_\psi} \mu_h^s\varphi_h\,d\Gamma \geq (\frac{3\rho_{min}^3}{4})^{1/2}||\varphi_h||_{0,I_\psi}||\mu_h^s||_{0,I_\psi}, \tag{3.53}$$

where

$$\rho_{min} = \min\{\frac{\beta h}{|I_\psi|},\quad I_\psi \subset \mathcal{T}_{\psi|\Gamma^s}\}, \tag{3.54}$$

and β is defined in (2.18).

Proof. As φ_h is not identically zero, there exists at least one regular interval I_λ where φ_h does not change sign. Let m be the midpoint of this I_λ, and set $\mu_h^s = \varphi_h(m)$ in I_λ, $\mu_h^s = 0$ in the rest of I_ψ. We have then

$$\int_{I_\psi} \mu_h^s\varphi_h\,d\Gamma = ||\mu_h^s||_{0,I_\psi}^2. \tag{3.55}$$

On the other hand, using (3.52) we also have

$$\int_{I_\psi} \mu_h^s\varphi_h\,d\Gamma \geq \frac{3\rho_{min}^3}{4}||\varphi_h||_{0,I_\psi}^2, \tag{3.56}$$

as the regularity of I_λ implies $|I_\lambda| \geq \beta h \geq \rho_{min}|I_\psi|$. Now, (3.55)-(3.56) give (3.53). □

We can now prove our convergence theorem for the variable ψ.

Theorem 3. *Let (u,λ,ψ) and (u_h,λ_h,ψ_h) be the solutions of (1.2) and (2.23) respectively, and assume that for all $s=1,N$ each interval $I_\psi \in \mathcal{T}_{\psi|\Gamma^s}$ contains at least two regular intervals $I_\lambda \in \mathcal{T}_\lambda^s$. Then the following estimate holds*

$$|||\psi-\psi_h|||_{\Sigma_\psi} = (\sum_s \sum_{I_\psi} |I_\psi|\,||\psi-\psi_h||^2_{0,I_\psi})^{1/2} \le C\overline{h}E_I \le C\overline{h}^2|u|_{2,\Omega}. \tag{3.57}$$

Proof. . Let ψ_I be the interpolant of ψ as in (3.7), and use, in every I_ψ, Lemma 5 with $\varphi_h=\psi_h-\psi_I$. We obtain

$$|I_\psi|^{1/2}\int_{I_\psi}\mu_h^s(\psi_h-\psi_I)\,d\Gamma \ge (\frac{3\rho^3_{min}}{4})^{1/2}|I_\psi|^{1/2}||\psi_h-\psi_I||_{0,I_\psi}\,||\mu_h^s||_{0,I_\psi}. \tag{3.58}$$

On the other hand,

$$|I_\psi|^{1/2}\int_{I_\psi}\mu_h^s(\psi_h-\psi_I)\,d\Gamma = |I_\psi|^{1/2}\int_{I_\psi}\mu_h^s(\psi_h-\psi)\,d\Gamma + |I_\psi|^{1/2}\int_{I_\psi}\mu_h^s(\psi-\psi_I)\,d\Gamma \;= I \;+\; II \tag{3.59}$$

In order to bound the first term, we set as in (3.30) T_ψ a triangle in Ω^s having I_ψ as an edge, and we apply (2.11) in T_ψ. We obtain

$$\begin{aligned} I \;&= |I_\psi|^{1/2}\textstyle\int_{I_\psi}\mu_h^s(\psi_h-\psi)\,d\Gamma = |I_\psi|^{1/2}\int_{I_\psi}\mu_h^s(u_h-u)\,d\Gamma \le |I_\psi|^{1/2}||\mu_h^s||_{0,I_\psi}\;||u_h-u||_{0,I_\psi} \\ &\le C|I_\psi|^{1/2}||\mu_h^s||_{0,I_\psi}\,(|I_\psi|^{-1}||u-u_h||^2_{0,T_\psi}+||u-u_h||_{0,T_\psi}||u-u_h||_{1,T_\psi})^{1/2} \end{aligned} \tag{3.60}$$

The second term in (3.59) is easily bounded using (3.8) and (2.10) as

$$II \le C||\mu_h^s||_{0,I_\psi}|I_\psi|^2|\psi|_{3/2,I_\psi} \le C||\mu_h^s||_{0,I_\psi}|I_\psi|^2|u|_{2,T_\psi}. \tag{3.61}$$

Inserting (3.60)-(3.61) in (3.59) and then in (3.58) and taking the square gives, on the intervals I_ψ where $\psi_h-\psi_I$ does not vanish identically:

$$|I_\psi|||\psi_h-\psi_I||^2_{0,I_\psi} \le C(||u-u_h||^2_{0,T_\psi}+|I_\psi|||u-u_h||_{0,T_\psi}||u-u_h||_{1,T_\psi}+|I_\psi|^4|u|^2_{2,T_\psi}). \tag{3.62}$$

Summation of (3.62) gives, using (3.44) and (3.45)

$$\sum_s\sum_{I_\psi}|I_\psi|\,||\psi_h-\psi_I||^2_{0,I_\psi} \le C\overline{h}^2E_I^2 \tag{3.63}$$

and the result follows by (3.8) and the triangle inequality. □

Remark 3. *We notice that the additional assumption required on $\mathcal{T}_\lambda^s$ for proving the error bound (3.57) is not difficult to realize in practice. Indeed, one can start by defining $\mathcal{T}_\lambda^s$ as in (2.17). If there is an I_ψ that does not contain two regular intervals of $\mathcal{T}_\lambda^s$, we just add a node to $\mathcal{T}_\lambda^s$, by splitting in two equal parts the longest I_λ contained in I_ψ.*

Remark 4. *The norm (3.50) used in estimate (3.57) is clearly weaker than the norm in $L^2(\Sigma)$. However it is optimal, compared with the regularity assumptions on the solution. In particular, using (2.16) and (3.57) one can easily deduce*

$$||\psi-\psi_h||_{0,\Sigma} \le C\overline{h}^{3/2}|u|_{2,\Omega} \tag{3.64}$$

for quasi-uniform meshes.

As far as the λ variable is concerned, we already had in (3.15) that

$$|\lambda_h-\widetilde{\lambda}|_{|I_k} \le Ch_T{}^{1/2}|I_k|^{-3/2}||u-u_h||_{1,T} \tag{3.65}$$

for every $I_k \subset \mathcal{T}_\lambda^s$ (s=1,N), where T is the triangle in $\mathcal{T}_u^s$ having I_k as part of its boundary. If I_k is regular, this implies

$$|\lambda_h-\widetilde{\lambda}|_{|I_k} \le C|I_k|^{-1}||u-u_h||_{1,T}. \tag{3.66}$$

We would like to prove that (3.66) is essentially true for irregular intervals as well. Assume therefore that I_k is irregular. We need some notation. Let Q be the endpoint of I_k that is a vertex of $\mathcal{T}_u^s$ (there must be one). Let T be the triangle of $\mathcal{T}_u^s$ having $I_k \subset \partial T$. Let P and R be the boundary vertices in $\mathcal{T}_u^s$ sharing a triangle with Q.

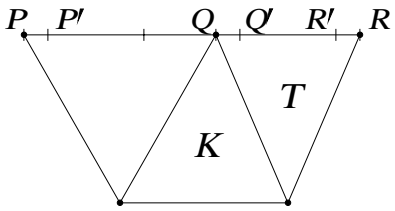


FIGURE 2

Let Q' be the other endpoint of I_k and assume, to fix ideas, that $Q' \in \overline{QR}$. As it will be the worst case, we also assume that there exist two other points, $P' \in \overline{PQ}$ and $R' \in \overline{QR}$ such that $\overline{PP'}$ and $\overline{RR'}$ are irregular intervals of $\mathcal{T}_\lambda^s$. See Fig. 2. Let now v_h be a function in V_h^s (hence, piecewise linear on $\mathcal{T}_u^s$) such that $v_h(Q) = 1$ and $v_h = 0$ at the other vertices, and K=supp(v_h). In the case of Fig. 2, K will be the union of the three triangles.

We are ready. Splitting the integral in five pieces we have

$$\begin{aligned} |\int_Q^{Q'} (\lambda_h - \widetilde{\lambda}) v_h \, d\Gamma| &= |\int_P^R \cdot d\Gamma - \int_P^{P'} \cdot d\Gamma - \int_{P'}^{Q} \cdot d\Gamma - \int_{Q'}^{R'} \cdot d\Gamma - \int_{R'}^{R} \cdot d\Gamma| \\ &\leq |I| \ + \ |II| \ + \ |III| \ + \ |IV| + \ + |V| \end{aligned} \tag{3.67}$$

and we bound separately the five pieces. To start with, we remark that III and IV are made of regular integrals. Therefore from (3.66) we have immediately

$$|III| \ + \ |IV| \leq C \|u - u_h\|_{1,K}, \tag{3.68}$$

as $|v_h| \leq 1$. On the other hand, using elementary calculus and then (3.65) we have

$$|II| = |\lambda_h - \widetilde{\lambda}|_{PP'} \cdot |PP'| \cdot v_h(\frac{P+P'}{2}) = |\lambda_h - \widetilde{\lambda}|_{PP'} \cdot \frac{(|PP'|)^2}{2|PQ|} \leq C \, \|u - u_h\|_{1,K}. \tag{3.69}$$

The term V is then estimated in an identical way

$$|V| \leq C \, \|u - u_h\|_{1,K}. \tag{3.70}$$

so that we have only to deal with I. Adding and subtracting λ, using (3.12), (3.1,i), (2.9) and estimating the norms of v_h (that is $\|v_h\|_{0,PR} \leq Ch^{1/2}$ and $\|v_h\|_{1,K} \leq C$) we obtain:

$$\begin{aligned} \int_P^R (\lambda_h - \widetilde{\lambda}) v_h \, d\Gamma &= \int_P^R (\lambda_h - \lambda) v_h \, d\Gamma + \int_P^R (\lambda - \widetilde{\lambda}) v_h \, d\Gamma \leq Ch^{1/2} |\lambda|_{1/2,PR} \|v_h\|_{0,PR} \ + \ a(u - u_h, v_h) \\ &\leq Ch|u|_{2,K} \ + \ M\|u - u_h\|_{1,K} \|v_h\|_{1,K} \leq C(h|u|_{2,K} \ + \ \|u - u_h\|_{1,K}). \end{aligned} \tag{3.71}$$

Summing all the five contributions from (3.68), (3.70), (3.69) and (3.71) we have

$$|\int_Q^{Q'} (\lambda_h - \widetilde{\lambda}) v_h \, d\Gamma| \leq C(h|u|_{2,K} + \|u - u_h\|_{1,K}). \tag{3.72}$$

On the other hand, the same integral can be estimated from below:

$$\begin{aligned} |\int_Q^{Q'} (\lambda_h - \widetilde{\lambda}) v_h \, d\Gamma| &= |\lambda_h - \widetilde{\lambda}|_{I_k} \cdot |QQ'| \cdot v_h(\tfrac{Q+Q'}{2}) = |\lambda_h - \widetilde{\lambda}|_{I_k} \cdot |QQ'| \cdot \tfrac{2|QR| - |QQ'|}{2|QR|} \\ &\geq C|I_k| \, |\lambda_h - \widetilde{\lambda}|_{I_k} \end{aligned} \tag{3.73}$$

Comparing (3.72) and (3.73) we conclude with the following theorem.

Theorem 4. *Let (u, λ, ψ) and (u_h, λ_h, ψ_h) be the solutions of (1.2) and (2.23) respectively, and let $\widetilde{\lambda}$ be defined by (3.10). For all s=1,N let I_k be an interval in $\mathcal{T}_\lambda^s$, and let K be the union of those triangles of $\mathcal{T}_u^s$ that have at least one point in common with I_k. We have*

$$|\lambda_h - \widetilde{\lambda}|_{I_k} \leq C|I_k|^{-1}(h|u|_{2,K} + \|u - u_h\|_{1,K}). \tag{3.74}$$

From (3.74), the previous error estimate (3.44), the interpolation estimate (3.12) and the triangle inequality we have then the final error estimate for λ, in the norm (3.51).

Theorem 5. *Let (u, λ, ψ) and (u_h, λ_h, ψ_h) be the solutions of (1.2) and (2.23) respectively. We have*

$$|||\lambda-\lambda_h|||_{\Sigma_\lambda}=(\sum_s\sum_k|I_k|\,||\lambda-\lambda_h||^2_{0,I_k})^{1/2}\le CE_I\le C\overline{h}|u|_{2,\Omega}. \tag{3.75}$$

4. Numerical considerations

In this Section we shall address a certain number of numerical considerations, in order to cast some light on different issues, like the actual implementation of the procedure and its expected numerical performances. To present a full set of numerical experiments (that might show the accuracy and stability of the method, the quality of the preconditioners that can be used, and the overall performance in terms of accuracy versus computer time in different typical situations of industrial interest) is far beyond the scopes of this paper.

On the other hand, since a full numerical evidence in support of the present approach is presently lacking, it is worth spending a few comments in order to explain the reasons why we believe in its potential.

We then go back to the stabilized problem (2.23), that we report here for the convenience of the reader:

$$\left\{\begin{array}{llll}\text{find } u_h\in\widetilde{V}_h^*,\ \lambda_h\in M_h^*\text{ and }\psi_h\in\Phi_h & \text{such that} & & \\ \text{i) } a_s(u_h^s,v)-<\lambda_h^s,v>_s & =\ (f,v)_s & \forall v\in\widetilde{V}_h^s & \forall s\\ \text{ii) } <\mu^s,u_h^s>_s & =<\mu^s,\psi_h>_s & \forall\mu^s\in M_h^s & \forall s\\ \text{iii) } \sum_s<\lambda_h^s,\varphi>_s & =0 & \forall\varphi\in\Phi_h. & \end{array}\right. \tag{4.1}$$

First of all, we note that the formulation (4.1) does not seem to be very well suited for a global solution, as one would have by applying an iterative algorithm (conjugate gradient, GMRES, etc) to the whole system. Even the procedure that eliminates the bubbles and the Lagrange multipliers λ_h^s (that we are going to describe in detail in the sequel) will not produce a system which is well suited for that. On the other hand, in most cases, this is not the preferred way for solving a domain decomposition problem. Looking at (4.1) it is clear that, for fixed ψ_h, the first two equations can be solved independently and *in parallel*. We can therefore consider the mapping $\mathscr{S}_h=(\mathscr{S}_u,\mathscr{S}_\lambda)$ that associates to the pair (f,ψ_h) the solution (u_h,λ_h) of the first two equations of (4.1). With this notation, problem (4.1) can be written as:

$$\text{find }\psi_h\in\Phi_h\text{ such that}\qquad\sum_s<\mathscr{S}_\lambda(f,\psi_h),\varphi>_s=0,\qquad\forall\varphi\in\Phi_h. \tag{4.2}$$

It is clear that in (4.2) a crucial role is played by the linear operator S_h, from Φ_h to its dual space, defined by:

$$<S_h(\psi_h),\varphi>:=\sum_s<\mathscr{S}_\lambda(0,\psi_h),\varphi>_s, \tag{4.3}$$

which is commonly called Schur complement, and whose spectral properties have a paramount relevance in solving (4.2) by iterative methods. It is easy to see that S_h is the discretization of a pseudo-differential operator S, of order 1, on Σ_h. In order to precondition (4.2) one has to find a cheaply computable operator that could be regarded as an approximation of S^{-1}. Several choices for that are found in the literature. For instance one could solve local Neumann problems (not necessarily related to the actual form of the operator A or to the $\mathscr{T}_u^s$ grids) in order to get an approximation of the local Steklov-Poincaré operators. Or one can define a sort of $H^{1/2}$ inner product on Φ_h, and invert the associated linear operator. We refer to the survey [11], and to the impressive set of proceedings of the various meetings on Domain Decomposition Methods, whose last volume is [4]. It is clear, however, that in doing that the task is made much easier if the grid on Σ_h is uniform. For instance one could use fast solvers for the local Neumann problems, or find easy expressions for the $H^{1/2}$ inner product, and so on. This is the main reason why we believe that an approach allowing the use of a uniform grid on Σ_h (independently of the grids that are used in the subdomains) is worth investigating.

Let us now discuss the actual computation of the operator $\mathscr{S}_h$, and in particular the solution of the local problems in each Ω^s. For this, we concentrate on a single domain. We assume ψ_h to be given, and we see how to compute the corresponding u_h^s and λ_h^s. For the sake of simplicity, we might drop at this point the superscript s, as the same identical procedure will be applied, in parallel, in each subdomain. With an abuse of notation, we are also going to call Ω (instead of Ω^s) the current subdomain, and so on. No confusion should arise. With this simplified notation the local problem becomes:

$$\left\{\begin{array}{llll}\text{given }\psi_h\text{ on }\partial\Omega,\text{ find }u_h\in\widetilde{V}_h & \text{and }\lambda_h\in M_h & \text{such that} & \\ \text{i) } a(u_h,v)-<\lambda_h,v> & =\ (f,v) & \forall v\in\widetilde{V}_h & \\ \text{ii) } <\mu,u_h> & =<\mu,\psi_h> & \forall\mu\in M_h, & \end{array}\right. \tag{4.4}$$

where M_h is made of piecewise constants on $\mathscr{T}_\lambda$, and $\widetilde{V}_h$ is obtained as the sum of piecewise linear functions on $\mathscr{T}_u$ (that we will denote by V_L,) and the space of boundary bubbles B_h. Accordingly, u_h will be written as

$$u_h(x) = u_L(x) + \sum_k u_B^k \, b_k(x). \tag{4.5}$$

From the second equation of (4.4), by taking μ as the characteristic function of an interval I_k of $\mathscr{T}_\lambda$, we obtain:

$$u_B^k = \left(\int_{I_k} \psi_h \mathrm{d}s - \int_{I_k} u_L \mathrm{d}s \right) / \int_{I_k} b_k \mathrm{d}s. \tag{4.6}$$

We remark that both u_L and ψ_h are linear in each I_k, so that, indicating by m_k the midpoint of I_k, and setting

$$\gamma_k^I = \int_{I_k} b_k \mathrm{d}s, \tag{4.7}$$

we immediately obtain from (4.6) that

$$u_B^k = (\psi_h(m_k) - u_L(m_k))|I_k|/\gamma_k^I. \tag{4.8}$$

We can now use the first equation of (4.4), with $v = b_k$, in order to express λ_k as a function of the other variables:

$$\lambda_k = \left(a(u_L, b_k) + u_B^k \, a(b_k, b_k) - (f, b_k) \right) / \gamma_k^I. \tag{4.9}$$

Assume now, for simplicity, that in (2.2) we have $a_0 = 0$, and that the other coefficients $a_{i,j}$ are piecewise constants. Assume also that the right-hand side f is piecewise constant. The last two assumptions are not really restrictive, as in most cases both $a_{i,j}$ and f are approximated anyhow by piecewise constants in the actual implementation. The first assumption, on a_0, is more restrictive, but, as we shall see, is there only to have a nicer final formula, and a piecewise constant a_0 could also be taken very easily into account. We remark now that, integrating by parts, for every $v_L \in V_L$ and for every k we have:

$$a(v_L, b_k) = a(b_k, v_L) = \int_{I_k} \frac{\partial v_L}{\partial n_a} b_k \mathrm{d}s = \frac{\partial v_L}{\partial n_a} \gamma_k^I, \tag{4.10}$$

where clearly $\partial v_L / \partial n_a$ is the conormal derivative of v_L with respect to the bilinear form a, as given in (2.3). We also set:

$$\gamma_k^T = \int_{T_k} b_k \mathrm{d}x, \qquad \gamma_k^a = a(b_k, b_k), \tag{4.11}$$

so that (4.9) becomes

$$\lambda_k = \frac{\partial u_L}{\partial n_a} + \frac{\left(u_B^k \gamma_k^a - f_k \gamma_k^T \right)}{\gamma_k^I}. \tag{4.12}$$

With some manipulations, using (4.8) and (4.10), we obtain

$$a(\sum_k u_B^k \, b_k, v_L) = \int_{\partial\Omega} (\psi_h - u_L) \frac{\partial v_L}{\partial n_a} \mathrm{d}s. \tag{4.13}$$

Taking now the first equation of (4.4) for $v = v_L$, using (4.8) (4.12) and (4.10), we obtain with easy computations that

$$\begin{aligned} &a(u_L, v_L) - < u_L, \frac{\partial v_L}{\partial n_a} > - < v_L, \frac{\partial u_L}{\partial n_a} > + \textstyle\sum_k \left(\frac{|I_k|}{\gamma_k^I} \right)^2 \gamma_k^a \, u_L(m_k) v_L(m_k) = \\ &(f, v_L) - \textstyle\sum_k f_k \left(\frac{\gamma_k^T |I_k|}{\gamma_k^I} \right) v_L(m_k) - < \psi_h, \frac{\partial v_L}{\partial n_a} > + \textstyle\sum_k \left(\frac{|I_k|}{\gamma_k^I} \right)^2 \gamma_k^a \, \psi_h(m_k) v_L(m_k). \end{aligned} \tag{4.14}$$

Equation (4.14) can be further simplified. In particular, if v_L is a basis function having value 1 at a boundary node and 0 at the other nodes of $\mathscr{T}_u$, and if f is piecewise constant on $\mathscr{T}_u$, using (2.20) we rather easily get:

$$\sum_k f_k \left(\frac{\gamma_k^T |I_k|}{\gamma_k^I} \right) v_L(m_k) = \sum_k f_k \frac{|T_k|}{2} v_L(m_k) = \frac{3}{4} \int f \, v_L \mathrm{d}x \tag{4.15}$$

where the last integral is made over the triangles having an edge on $\partial\Omega$. By linearity, the formula will then hold for every v_L vanishing at all internal nodes. Hence, we can also consider the new scalar product, defined for piecewise linear functions u_L and v_L as

$$< u_L, v_L >_h := \sum_k \left(\frac{|I_k|}{\gamma_k^I} \right)^2 \gamma_k^a \, u_L(m_k) v_L(m_k), \tag{4.16}$$

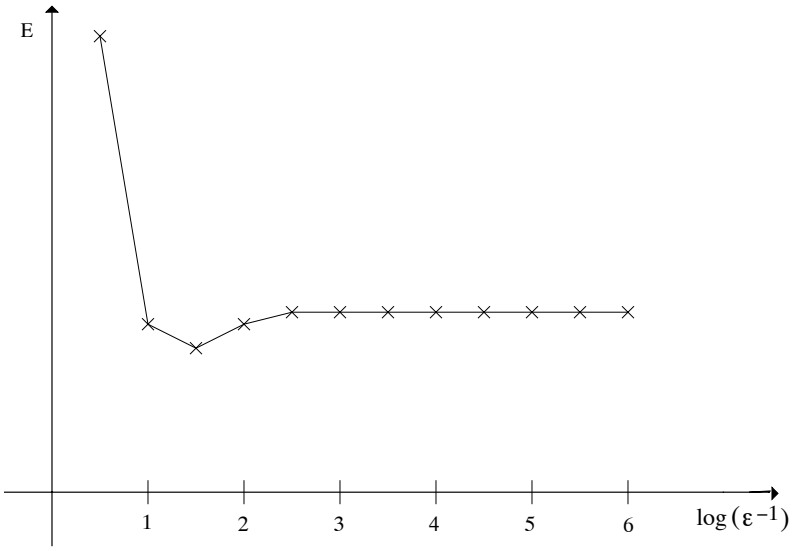


FIGURE 3

and, in agreement with (4.15),

$$(f, v_L)_* := \frac{3}{4} \int f\, v_L^* \mathrm{d}x, \tag{4.17}$$

where the integral is extended over the elements having an edge on $\partial\Omega$, and v_L^* is obtained from v_L by setting all the values at internal nodes equal to zero. Equation (4.14) becomes now

$$\begin{aligned} &a(u_L, v_L) - < u_L, \frac{\partial v_L}{\partial n_a} > - < v_L, \frac{\partial u_L}{\partial n_a} > + < u_L, v_L >_h = \\ &(f, v_L) - (f, v_L)_* - < \psi_h, \frac{\partial v_L}{\partial n_a} > + < \psi_h, v_L >_h \qquad \forall v_L \in V_L. \end{aligned} \tag{4.18}$$

The nature of the resulting local problem should now be clear. We remark first that the method we obtained by adding and eliminating bubbles is very close to the Nitsche's method [17], which roughly corresponds to

$$\begin{aligned} &a(u_L, v_L) - < u_L, \frac{\partial v_L}{\partial n_a} > - < v_L, \frac{\partial u_L}{\partial n_a} > + \frac{\alpha}{h} < u_L, v_L > = \\ &(f, v_L) - < \psi_h, \frac{\partial v_L}{\partial n_a} > + \frac{\alpha}{h} < \psi_h, v_L > \qquad \forall v_L \in V_L, \end{aligned} \tag{4.19}$$

and whose implementation is immediate. In (4.19) α is a suitable constant which has to be $\geq \alpha_0$ depending on the minimum angle of the triangles. We also point out that, for reasonably smooth f, the term $(f, v_L)_*$, which appears in (4.18) and not in (4.19), will actually be negligible (but in any case is easy to compute). On the other hand the terms $\frac{\alpha}{h} < u_L, v_L >$ and $< u_L, v_L >_h$ compare very well in terms of powers of h. The effect of having smaller intervals I_k, or even irregular intervals, should correspond, roughly speaking, to choose a local value for α which is bigger than necessary. In these circumstances, both (4.18) and (4.19) exhibit, in practical computations, the typical behavior of penalty methods, as depicted in Fig. 3: for smaller and smaller values of the penalty parameter ε, the relative error stabilizes to a value which is slightly bigger that the optimal one (corresponding, in general, to $\varepsilon \sim h$) that is well within the bounds of optimality (in terms of powers of h.) For this reason, we believe that our assumptions on the grids, and in particular the quasi-uniformity of $\mathscr{T}_u$ on the boundary and the comparability of the two grids $\mathscr{T}_u$ and $\mathscr{T}_\psi$, are rather technical assumptions, and do not correspond to an actual weakness of the method.

Finally, we want to point out that, in the spirit of [1], one could think of changing the shape of the bubbles b_k so that the actual computation of the coefficients appearing in the final reduced system become easier. Essentially, within certain limitations, one might *prescribe* the value of the coefficients and use them in the computation, just knowing that *there exist* bubbles (still providing optimal error bounds) that will produce, after elimination, the prescribed coefficients. The actual shape of these *virtual bubbles* does not need to be known explicitly, as only the final coefficients enter the computation. These matters are currently under investigation.

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