An abstract theory for mixed approximations of second order elliptic problems

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ABSTRACT - We present an abstract theory for discretizations of second order elliptic problems by mixed finite elements, based on minimal hypotheses. We prove that our theory includes the existing mixed approximations and we show that it gives rise to new mixed discretizations, either conforming or non-conforming, that can be useful in applications.

1. Introduction

The object of this paper is to present an abstract theory for discretizations of second order elliptic problems by means of mixed finite element methods. More precisely, consider the following boundary value problem:

$$\begin{cases} -\, div(a\underline{\nabla} u) \;+\; \nu\, u = f & in \quad \Omega, \\ \qquad\qquad\qquad\quad u = 0 & on \quad \Gamma \;=\; \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a convex polygonal domain in $\mathbf{R}^2$, $f \in L^2(\Omega)$, $a = a(x) \geq a_0 > 0$ is a smooth function on the closure of Ω, and ν is a nonnegative constant. It is known that the basic idea of mixed formulations of (1.1) is to introduce, together with the solution u of (1.1), the vector field $\underline{\sigma} = -a(x)\underline{\nabla} u$ as an independent variable and to derive a suitable variational formulation. The spaces W and V in which u and $\underline{\sigma}$ are sought must be properly related (see [2]) to guarantee existence and uniqueness of the solution. For a problem of the type (1.1), W is $L^2(\Omega)$, while V is $H(div, \Omega)$; i.e., it contains those vectors belonging to $[L^2(\Omega)]^2$ whose divergence is in $L^2(\Omega)$. When dealing with discretizations of a mixed formulation, again proper relations between the discrete analogous W_h and V_h have to be satisfied in order to guarantee uniqueness of the

discrete solution and optimal error bounds [2]. Moreover, a conforming approximation implies that $W_h \subset W$ and $V_h \subset V$. While the requirement $W_h \subset W$ causes no difficulty, the inclusion $V_h \subset V$ implies some regularity on the elements of V_h; more precisely, the normal components of a vector in V_h must be continuous across the interelement boundaries. In the literature there are four families of mixed finite elements [3], [12], [8] satisfying all the requirements and giving optimal error bounds. The key properties of these families are the internal approximation property (i.e., $W_h \subset W, V_h \subset V$) and the commutativity of the divergence operator with suitably defined "projections" from the continuous spaces onto the discrete ones [8], [9], [3], [6]. These families of discretizations lead to a linear algebraic system whose matrix is not positive definite. A way to circumvent this inconvenience is the use of Lagrange multipliers. The idea, introduced by Fraeijis de Veubeke [10], is to relax the continuity requirement on the normal components and then to enforce it via the use of Lagrange multipliers. This leads to the introduction of a third discrete space, say Λ_h, properly related to W_h and V_h (see e.g. [1],[3],[6]). Using this procedure, both scalar and vector variables can be eliminated element by element by static condensation with virtually no cost, and the problem reduces to the solution of a linear system in the Lagrange multipliers only. This Lagrange multiplier formulation, often called a hybridization of the mixed formulation, is the starting point of the present paper. In section 2 we briefly recall the mixed formulation of problem (1.1). In section 3 we state three minimal hypotheses ($(H1) - (H3)$) and construct an abstract theory based on them. We point out that these hypotheses imply neither commuting properties nor internal approximation properties. In particular, non-conforming approximations for the vector variable are allowed ($V_h \not\subset V$). This allows more generality in the choice of the discretizations, giving rise to new mixed approximations of non-conforming type. In section 4 we give abstract error bounds making use of our minimal hypotheses. The error estimates depend on the interpolation estimates and also on the possible non-conformity of the approximation. In section 5 we recall the four known families of mixed finite elements and show that our abstract theory applies. Moreover, over a triangular decomposition, we introduce two new mixed finite elements which satisfy the abstract theory, and we give optimal error bounds. We underline an interesting property of our new elements: they produce a final matrix

which is an M-matrix if the triangulation is of weakly acute type (every angle of every triangle is $\leq \frac{\pi}{2}$). Note that no element in the known families provides an M-matrix for the problem (1.1) if $\nu \neq 0$. The M-matrix property is desirable in view of a discrete maximum principle. In particular, in applications to semiconductor device simulations, the M-matrix property is crucial (see [4],[5]). We refer to [11] for a discussion of the subject.

Throughout the paper we shall use the following notation:

$(.\ ,.)$, scalar product in $L^2(\Omega)$;

$\|.\|_s$ and $|.|_s$,norm and seminorm in the Sobolev space $H^s(\Omega)$;

$\|.\|_{s,\mathrm{K}}$ and $|.|_{s,\mathrm{K}}$, norm and seminorm in the Sobolev space $H^s(\mathrm{K})$, $\mathrm{K} \subset \Omega$.

2. The continuous problem

Let us recall the mixed formulation of problem (1.1). Define the spaces

$$V = H(div;\Omega) = \{\underline{\tau} \in [L^2(\Omega)]^2 \mid div\ \underline{\tau} \in L^2(\Omega)\}, \tag{2.1}$$

$$W = L^2(\Omega), \tag{2.2}$$

and let the norm in V be given by

$$\|\underline{\tau}\|_V^2 = \|\underline{\tau}\|_0^2 + \|div\ \underline{\tau}\|_0^2 \ . \tag{2.3}$$

Let us introduce the bilinear forms

$$a(\underline{\sigma},\underline{\tau}) = \int_\Omega a^{-1}\underline{\sigma}\cdot\underline{\tau}\ dx\ , \qquad \underline{\sigma},\ \underline{\tau} \in V, \tag{2.4}$$

$$b(\underline{\tau},\phi) = \int_\Omega div\ \underline{\tau}\ \phi\ dx\ , \qquad \underline{\tau} \in V,\ \ \phi \in W. \tag{2.5}$$

The usual mixed formulation of (1.1) is:

$$\begin{cases} find\ (\underline{\sigma},\ u) \in V \times W\ such\ that \\ a(\underline{\sigma},\underline{\tau}) - b(\underline{\tau},u) = 0, & \forall \underline{\tau} \in V \\ b(\underline{\sigma},\phi) + \nu(u,\phi) = (f,\phi), & \forall \phi \in W. \end{cases} \tag{2.6}$$

Remark 2.1. It is easy to verify that

$$\exists \alpha > 0 \ : \ \ a(\underline{\tau}, \underline{\tau}) \ \geq \ \alpha ||\underline{\tau}||_0^2 \ , \quad \underline{\tau} \in V. \tag{2.7}$$

Moreover, the *Inf Sup* condition [2] holds:

$$\exists \beta > 0 \ : \ \ \forall \phi \in W - \{0\}, \ \ \exists \underline{\tau} \in V - \{0\} : \ \ \frac{b(\underline{\tau}, \phi)}{||\underline{\tau}||_V \ ||\phi||_0} \ \geq \ \beta. \tag{2.8}$$

Hence, problem (2.6) has a unique solution $(\underline{\sigma}, u)$ with

$$\underline{\sigma} \ = \ -a \underline{\nabla} u. \tag{2.9}$$

Note that, due to the regularity of the solution u of (1.1), one has $\underline{\sigma} \in \widetilde{V}$, with

$$\widetilde{V} \ = \ [H^1(\Omega)]^2 \ . \tag{2.10}$$

Moreover, the *Inf Sup* condition (2.8) holds with $\widetilde{V}$ instead of V:

$$\exists \beta > 0 \ : \ \ \forall \phi \in W - \{0\}, \ \exists \underline{\tau} \in \widetilde{V} - \{0\} \ : \ \ \frac{b(\underline{\tau}, \phi)}{||\underline{\tau}||_{\widetilde{V}} \ ||\phi||_0} \ \geq \ \beta. \tag{2.11}$$

3. The discrete formulation

Let T_h be a regular decomposition of Ω into triangles or rectangles K, (see [7]), and let E_h be the set of edges of T_h. We construct our finite element spaces as follows. First, for all $\mathrm{K} \in T_h$, introduce the finite dimensional spaces $\Sigma(\mathrm{K})$ (space of vectors) and $P(\mathrm{K})$ (scalar space); for all $e \in E_h$ introduce the finite dimensional space $R(e)$ (scalar space). The spaces $\Sigma(\mathrm{K})$ and $P(\mathrm{K})$ will consist of regular functions on K, while $R(e)$ will denote functions regular on each edge e. Often, the notation $R(\partial \mathrm{K})$ will be used, with the meaning $R(\partial \mathrm{K}) = \prod_e R(e)$. For the sake of simplicity we might assume that there exists an integer N such that, on each K, $\Sigma(\mathrm{K})$, $\mathrm{P}(\mathrm{K})$, and $R(\partial \mathrm{K})$ consist of polynomials of degree $\leq N$. We will assume the following hypotheses to be verified:

$(H1)$ $\quad dim \ (\ div \ (\Sigma(\mathrm{K}))) \ = \ dim \ P(\mathrm{K});$

$(H2)$ $\quad \gamma_0(P(\mathrm{K})) \subseteq R(\partial \mathrm{K});$

$(H3)$ $\quad \forall \phi \in P(\mathrm{K}), \quad \forall \mu \in R(\partial \mathrm{K}) \quad s.t. \ ||\underline{\nabla} \phi||_{0,\mathrm{K}} + ||\mu||_{0,\partial \mathrm{K}} \neq \ 0, \quad \exists \underline{\tau} \in \Sigma(\mathrm{K}) :$

$$\int_{\mathrm{K}} \underline{\tau} \cdot \underline{\nabla} \phi \ dx \ + \int_{\partial \mathrm{K}} \mu \underline{\tau} \cdot \underline{n} \ ds \ \neq \ 0,$$

where γ_0 denotes the trace on ∂K.

Remark 3.1. Clearly, $(H3)$ implies each of the following conditions separately:

$$\forall \phi \in P(\mathrm{K})\ ,\ \phi \neq 0, \quad \exists \underline{\tau} \in \Sigma(\mathrm{K})\ :\quad \int_{\mathrm{K}} div\ \underline{\tau}\ \phi\ dx \neq\ 0, \tag{3.1}$$

$$\forall \mu \in R(\partial \mathrm{K}),\ \mu \neq 0, \quad \exists \underline{\tau} \in \Sigma(\mathrm{K})\ :\quad \int_{\partial \mathrm{K}} \mu \underline{\tau} \cdot \underline{n}\ ds \neq\ 0. \tag{3.2}$$

Actually, due to $(H2)$, take $\mu = -\gamma_0 \phi$ in $(H3)$ and integrate by parts to get (3.1); (3.2) follows from $(H3)$ with $\phi = 1$.

Let

$$V^* \ =\ \prod_{\mathrm{K} \in T_h} H(div; \mathrm{K}), \tag{3.3}$$

with the norm

$$\|\underline{\tau}\|_*^2 \ =\ \|\underline{\tau}\|_0^2 \ +\ \sum_{\mathrm{K}} \|div\ \underline{\tau}\|_{0,\mathrm{K}}^2. \tag{3.4}$$

Introduce the following discrete spaces:

$$V_h \ =\ \{\underline{\tau} \in V^*\ :\ \underline{\tau}_{|\mathrm{K}} \in \Sigma(\mathrm{K}),\ \ \forall \mathrm{K} \in \mathrm{T_h}\ \}, \tag{3.5}$$

$$W_h \ =\ \{\phi \in L^2(\Omega) : \phi_{|\mathrm{K}} \in P(\mathrm{K}),\ \forall \mathrm{K} \in \mathrm{T_h}\ \}, \tag{3.6}$$

$$\Lambda_h \ =\ \{\mu \in L^2(E_h) : \mu_{|e} \in R(e),\ \ \forall e \in E_h;\ \mu = 0\ on\ \Gamma\}, \tag{3.7}$$

with norms in V_h and Λ_h defined by

$$\|\underline{\tau}\|_h^2 \ =\ \|\underline{\tau}\|_0^2 \ +\ \sum_{\mathrm{K}} h_{\mathrm{K}}^2 \|div\ \underline{\tau}\|_{0,\mathrm{K}}^2, \quad \underline{\tau} \in V_h, \tag{3.8}$$

$$|\mu|_{-1/2,h}^2 \ =\ \sum_{e \in E_h} h_e \|\mu\|_{0,e}^2\ , \quad \mu \in \Lambda_h\ . \tag{3.9}$$

Since the polynomials in $\Sigma(\mathrm{K})$ are of degree $\leq N$, the following inverse inequality can be proved:

$$\|div \underline{\tau}\|_{0,\mathrm{K}} \ \leq\ h_{\mathrm{K}}^{-1}\ \|\underline{\tau}\|_{0,\mathrm{K}} \quad for\ \underline{\tau}\ \in\ \Sigma(\mathrm{K}). \tag{3.10}$$

Hence, $\|\underline{\tau}\|_h$ is equivalent to the L^2-norm:

$$\|\underline{\tau}\|_h \ \simeq \ \|\underline{\tau}\|_0 \quad \underline{\tau} \in V_h \ . \tag{3.11}$$

Finally, introduce the bilinear forms

$$b_h(\underline{\tau}, \phi) = \sum_{\mathrm{K}} \int_{\mathrm{K}} div \ \underline{\tau} \ \phi \ dx, \qquad \underline{\tau} \in V_h, \ \phi \in W_h, \tag{3.12}$$

$$c(\mu, \underline{\tau}) = \sum_{\mathrm{K}} \int_{\partial \mathrm{K}} \mu \underline{\tau} \cdot \underline{n} \ ds, \qquad \mu \in \Lambda_h, \ \underline{\tau} \in V_h. \tag{3.13}$$

The discrete formulation of (2.6) is then:

$$\left\{ \begin{array}{l} find \ (\underline{\sigma}_h, \ u_h, \ \lambda_h) \in V_h \times W_h \times \Lambda_h \ such \ that : \\ \qquad a(\underline{\sigma}_h, \underline{\tau}) - b_h(\underline{\tau}, u_h) + c(\lambda_h, \underline{\tau}) \ = \ 0, \qquad \forall \underline{\tau} \in V_h \\ \qquad\qquad b_h(\underline{\sigma}_h, \phi) \ + \ \nu \ (u_h, \phi), \ = \ (f, \phi), \qquad \forall \phi \in W_h \\ \qquad\qquad\qquad\qquad c(\mu, \underline{\sigma}_h) \ = \ 0, \qquad \forall \mu \in \Lambda_h \ . \end{array} \right. \tag{3.14}$$

Let us define the following subspaces:

$$Ker \ B \ = \ \{\underline{\tau} \in V^* \ : \ b_h(\underline{\tau}, \phi) \ = \ 0, \ \ \forall \phi \in W\} \ , \tag{3.15}$$

$$Ker \ B_h \ = \ \{\underline{\tau} \in V_h \ : \ b_h(\underline{\tau}, \phi) \ = \ 0, \ \ \forall \phi \in W_h\} \ , \tag{3.16}$$

$$Ker \ C_h \ = \ \{\underline{\tau} \in V_h \ : \ c(\mu, \underline{\tau}) \ = \ 0, \ \ \forall \mu \in \Lambda_h\}. \tag{3.17}$$

The matrix associated with $\int_{\mathrm{K}} div \ \underline{\tau} \ \phi \ dx$ is, after elimination of the null rows (or columns), a square matrix, due to $(H1)$; moreover, it is nonsingular thanks to (3.1). Then, for $\underline{\tau} \in \Sigma(\mathrm{K})$,

$$\int_{\mathrm{K}} div \ \underline{\tau} \ \phi \ dx \ = \ 0, \quad \forall \phi \in P(\mathrm{K}) \quad \Longrightarrow \quad div \ \underline{\tau}_{|\mathrm{K}} \ = \ 0; \tag{3.18}$$

that is,

$$Ker \ B_h \subset Ker \ B. \tag{3.19}$$

As already mentioned in the introduction, the λ_h's are Lagrange multipliers. Moreover, as proved in [1], they can provide a "good" approximation of u on the interelement

boundaries. Notice that the solution $\underline{\sigma}_h$ of (3.14), sought *a priori* in V_h, actually belongs to $Ker\ C_h$, thanks to the third equation of (3.14). In the known examples of mixed finite elements for second order elliptic problems this is enough to guarantee the continuity of the normal component of $\underline{\sigma}_h$ on the interelement boundaries; then $\underline{\sigma}_h \in V$. However, in the abstract framework we are dealing with here, the inclusion $Ker\ C_h \subset V$ is not true in general (see example 5 in Section 5). Thus, our theory includes nonconforming mixed finite elements.

A projection Π_h from $\widetilde{V}$ to V_h can be defined locally by:

$$\int_e \mu(\underline{\tau} - \Pi_h\underline{\tau}) \cdot \underline{n}\ ds \ = \ 0, \qquad \forall\ edge\ e\ of\ \mathrm{K},\ \ \forall \mu \in R(e), \tag{3.20}$$

$$\int_{\mathrm{K}} (\underline{\tau} - \Pi_h\underline{\tau}) \cdot \underline{\nabla}\phi\ dx \ = \ 0, \qquad \forall \phi \in P(\mathrm{K}), \tag{3.21}$$

$$\int_{\mathrm{K}} (\underline{\tau} - \Pi_h\underline{\tau}) \cdot \underline{q}\ dx \ = \ 0, \qquad \forall \underline{q} \in D, \tag{3.22}$$

with

$$D = \{\underline{\tau} \in \Sigma(\mathrm{K}) : \int_{\mathrm{K}} \underline{\tau}\cdot\underline{\nabla}\phi\ dx + \int_{\partial \mathrm{K}} \mu\underline{\tau}\cdot\underline{n}\ ds\ = 0, \quad \forall \mu \in R(\partial \mathrm{K}), \quad \forall \phi \in P(\mathrm{K})\}. \tag{3.23}$$

It is easy to check that Π_h is well defined, since (3.20) and (3.21) are independent, due to $(H3)$. Hence, we have

$$\|\Pi_h\underline{\tau}\|_h \ \leq \ \gamma \|\underline{\tau}\|_{\widetilde{V}}. \tag{3.24}$$

The constant γ in (3.24) is dependent *a priori* on h, on the decomposition, and on the choice of the finite dimensional spaces. However, in all other examples presented here, and in all other reasonable examples, γ will turn out to be independent of h. Thus, it is not very restrictive to *assume that the constant γ in* (3.24) *is independent of h.*

Remark 3.2. First, note that for $\underline{\tau} \in \widetilde{V}$

$$b(\underline{\tau}, \phi) \ = \ b_h(\underline{\tau}, \phi), \qquad \forall\ \phi \in W_h, \tag{3.25}$$

and

$$c(\mu, \underline{\tau}) \ = \ 0, \qquad \forall \mu \in \Lambda_h. \tag{3.26}$$

Hence, (3.20) and (3.26) imply that for $\underline{\tau} \in \widetilde{V}$

$$\Pi_h \underline{\tau} \in Ker\ C_h. \tag{3.27}$$

Finally, it follows from (3.20), (3.21), and ($H2$) that

$$\int_{\mathrm{K}} div(\underline{\tau} - \Pi_h \underline{\tau})\phi\ dx\ =\ 0, \qquad \forall \phi \in P(\mathrm{K}), \tag{3.28}$$

so that summing over all $\mathrm{K} \in T_h$ gives

$$b_h(\underline{\tau} - \Pi_h \underline{\tau}, \phi)\ =\ 0, \qquad \forall \phi \in W_h. \tag{3.29}$$

A projection P_h from L^2 to W_h can be defined locally by:

$$\int_{\mathrm{K}} div\ \underline{\tau}\ (u - P_h u)\ dx\ =\ 0, \qquad \forall \underline{\tau} \in \Sigma(\mathrm{K}). \tag{3.30}$$

In order to see that P_h is well-defined, thanks to ($H1$) it is sufficient to check that

$$\int_{\mathrm{K}} div\ \underline{\tau} P_h\ u\ dx\ =\ 0, \qquad \forall \underline{\tau} \in \Sigma(\mathrm{K}) \qquad \Longrightarrow \qquad P_h u = 0, \tag{3.31}$$

and this is ensured by (3.1). Summing (3.30) over all $\mathrm{K} \in T_h$, we obtain then

$$b_h(\underline{\tau}, u - P_h u)\ =\ 0, \qquad \forall \underline{\tau} \in V_h\ . \tag{3.32}$$

A projection P_h^e from $L^2(E_h)$ to Λ_h can be defined locally as the usual L^2-projection:

$$\int_e \mu(u - P_h^e u)\ ds\ =\ 0, \qquad \forall\ edges\ e\ of\ \mathrm{K},\ \ \forall \mu \in R(e). \tag{3.33}$$

Lemma 1. *The discrete Inf Sup condition holds:*

$$\exists \beta^*\ >\ 0\ \ :\ \ \forall \phi \in W_h - \{0\},\ \ \exists \underline{\tau} \in Ker\ C_h - \{0\}\ \ :\ \ \frac{b_h(\underline{\tau}, \phi)}{||\underline{\tau}||_h\ ||\phi||_0}\ \geq\ \beta^*. \tag{3.34}$$

Proof. As a consequence of the *Inf Sup* condition (2.11) that holds for the continuous problem, for $\phi \in W_h \subset W$, there exists $\underline{\tau} \in \widetilde{V}$ such that

$$\frac{b(\underline{\tau}, \phi)}{||\underline{\tau}||_{\widetilde{V}}\ ||\phi||_0}\ \geq\ \beta. \tag{3.35}$$

Let $\underline{\tau}^* = \Pi_h \underline{\tau}$, and note that $\Pi_h \underline{\tau} \in Ker\ C_h$, according to (3.27). Due to (3.29), (3.25), and (3.24), from (2.11) we obtain

$$\frac{b_h(\underline{\tau}^*, \phi)}{\|\underline{\tau}^*\|_h\ \|\phi\|_0} \ \geq\ \frac{1}{\gamma}\frac{b(\underline{\tau}, \phi)}{\|\underline{\tau}\|_{\widetilde{V}}\ \|\phi\|_0} \ \geq\ \frac{\beta}{\gamma}. \tag{3.36}$$

Hence, (3.34) holds with $\beta^* \ =\ \beta\gamma^{-1}$.

Remark 3.3. In many cases it will be possible to prove the discrete *Inf Sup* condition with the norm $\|.\|_*$ defined in (3.4) instead of $\|.\|_h$; that is, a stronger version of (3.34). This is the case when, for instance, (3.24) holds with the norm $\|.\|_*$.

Theorem 1. *Problem* (3.14) *has a unique solution.*

Proof. Let $(\underline{\sigma}^*, u^*, \lambda^*) \in V_h \times W_h \times \Lambda_h$ be the solution of the homogeneous discrete problem associated with (3.14). Taking $\underline{\tau} \ =\ \underline{\sigma}^*$ in the first equation of (3.14), and using the second and third equations with $\phi \ =\ u^*$ and $\mu \ =\ \lambda^*$, respectively, we obtain

$$a(\underline{\sigma}^*, \underline{\sigma}^*) \ +\ \nu\ (u^*, u^*) \ =\ 0. \tag{3.37}$$

This implies that $\underline{\sigma}^* = 0$, and, if $\nu > 0$, $u^* = 0$. Instead, if $\nu = 0$, use the first equation of (3.14) with $\underline{\tau} \in Ker\ C_h$, to obtain

$$b_h(\underline{\tau}, u^*) \ =\ 0, \qquad \forall \underline{\tau} \in Ker\ C_h. \tag{3.38}$$

Thanks to (3.34), this implies $u^* \ =\ 0$. Hence, we are left with the equation

$$c(\lambda^*, \underline{\tau}) \ =\ 0, \qquad \forall \underline{\tau} \in V_h, \tag{3.39}$$

implying that $\lambda^* = 0$, due to (3.2).

■

4. Error estimates

In order to give error estimates, let us first note that the solution $(\underline{\sigma}, u)$ of (2.6) verifies

$$\begin{cases} a(\underline{\sigma}, \underline{\tau}) \ -\ b_h(\underline{\tau}, u) \ +\ c(u, \underline{\tau}) \ =\ 0, & \underline{\tau} \in V_h, \\ b_h(\underline{\sigma},\ \phi) \ +\ \nu\ (u, \phi) \ =\ (f, \phi), & \phi \in W_h, \\ c(\mu, \underline{\sigma}) \ =\ 0, & \mu \in \Lambda_h. \end{cases} \tag{4.1}$$

By subtracting (3.14) from (4.1) and using (3.32), (3.29), and (3.20), we obtain the error equations

$$(4.2)\qquad \begin{cases} a(\underline{\sigma}-\underline{\sigma}_h,\underline{\tau}) \;-\; b_h(\underline{\tau},P_hu-u_h) \;+\; c(u-\lambda_h,\underline{\tau}) \;=\; 0, & \underline{\tau}\in V_h,\\ b_h(\Pi_h\underline{\sigma}-\underline{\sigma}_h,\;\phi) \;+\; \nu\;(\;u-u_h\;,\;\phi\;) \;=\; 0, & \phi\in W_h,\\ c(\mu,\Pi_h\underline{\sigma}-\underline{\sigma}_h) \;=\; 0, & \mu\in\Lambda_h. \end{cases}$$

Notice that the third equation of (4.2) implies that

$$(4.3)\qquad \Pi_h\underline{\sigma}-\underline{\sigma}_h\in Ker\;C_h.$$

For any discrete space $Z\subseteq V_h$ let us now define

$$(4.4)\qquad N_h(Z,v)\;=\;\sup_{\underline{\tau}\in Z}\frac{c(v,\underline{\tau})}{||\underline{\tau}||_h},\quad N_h^*(Z,v)\;=\;\sup_{\underline{\tau}\in Z}\frac{c(v,\underline{\tau})}{||\underline{\tau}||_*},$$

where v is a function defined on E_h. Notice that if $Ker\;C_h\subset V$, then $N_h(KerC_h,v)$ and $N_h^*(KerC_h,v)$ are zero. In a way, $N_h(Z,v)$ ($N_h^*(Z,v)$) can be considered as a measure of "non-conformity".

We can prove the following result.

Theorem 2. *If* $(\underline{\sigma},u)$ *is the solution of* (2.6) *and* $(\underline{\sigma}_h,u_h,\lambda_h)$ *that of* (3.14), *the following estimate holds*:

$$(4.5)\qquad \begin{aligned} \frac{\alpha}{2}||\underline{\sigma}-\underline{\sigma}_h||_0^2+\;\frac{\nu}{2}\;||u-u_h||_0^2 \;&\le\; C\;(\;||\underline{\sigma}\;-\;\Pi_h\underline{\sigma}||_0^2\\ &+\;\frac{\nu}{2}\;||u-P_hu||_0^2\;+\;N_h(Ker\;C_h,u)^2). \end{aligned}$$

Proof. We have

$$(4.6)\qquad \begin{aligned} \alpha||\underline{\sigma}-\underline{\sigma}_h||_0^2\;&\le\;a(\underline{\sigma}\;-\;\underline{\sigma}_h,\underline{\sigma}\;-\;\underline{\sigma}_h)\\ &=\;a(\underline{\sigma}\;-\;\underline{\sigma}_h,\underline{\sigma}\;-\;\Pi_h\underline{\sigma})\;+\;a(\underline{\sigma}\;-\;\underline{\sigma}_h,\Pi_h\underline{\sigma}\;-\;\underline{\sigma}_h). \end{aligned}$$

From the first equation of (4.2) with $\underline{\tau}=\Pi_h\underline{\sigma}-\underline{\sigma}_h$ and the third equation of (4.2) with $\mu=\lambda_h$, we deduce that

$$(4.7)\qquad \begin{aligned} a(\underline{\sigma}-\underline{\sigma}_h,\Pi_h\underline{\sigma}-\underline{\sigma}_h)\;&=\;b_h(\Pi_h\underline{\sigma}\;-\;\underline{\sigma}_h,P_hu\;-\;u_h)\;-\;c(u-\lambda_h,\Pi_h\underline{\sigma}\;-\;\underline{\sigma}_h)\\ &=\;b_h(\Pi_h\underline{\sigma}\;-\;\underline{\sigma}_h,P_hu\;-\;u_h)\;-\;c(u\;,\Pi_h\underline{\sigma}\;-\;\underline{\sigma}_h). \end{aligned}$$

From the second equation of (4.2) we have

$$
\begin{aligned}
(4.8)\quad b_h(\Pi_h\underline{\sigma} - \underline{\sigma}_h, P_h u - u_h) &= -\nu(u-u_h, P_h u - u_h) \\
&= -\nu \, \|u-u_h\|_0^2 - \nu \, (u-u_h, P_h u - u) \\
&\leq -\nu \, \|u-u_h\|_0^2 + \nu \, \|u-u_h\|_0 \, \|P_h u - u\|_0 \\
&\leq -\frac{\nu}{2} \, \|u-u_h\|_0^2 + \frac{\nu}{2} \, \|P_h u - u\|_0^2 .
\end{aligned}
$$

From (4.6), (4.7), and (4.8) we see that

$$
\begin{aligned}
(4.9)\quad \alpha\|\underline{\sigma}-\underline{\sigma}_h\|_0^2 + \frac{\nu}{2} \, \|u-u_h\|_0^2 &\leq C \, \|\underline{\sigma} - \underline{\sigma}_h\|_0 \, \|\underline{\sigma} - \Pi_h\underline{\sigma}\|_0 \\
&\quad + \frac{\nu}{2} \, \|u - P_h u\|_0^2 - c(u \, , \Pi_h\underline{\sigma} - \underline{\sigma}_h).
\end{aligned}
$$

By (4.3), the last term in (4.9) can be bounded using (4.4) with $Z = KerC_h$. Recalling then (3.11), we can write

$$
\begin{aligned}
(4.10)\quad c(u, \Pi_h\underline{\sigma} - \underline{\sigma}_h) &\leq N_h(Ker \; C_h, u) \, \|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|_h \\
&\leq N_h(Ker \; C_h, u) \, \|\Pi_h\underline{\sigma} - \underline{\sigma}\|_0 + N_h(Ker \; C_h, u) \, \|\underline{\sigma} - \underline{\sigma}_h\|_0 .
\end{aligned}
$$

Finally, (4.5) easily follows from (4.9) and (4.10).

■

Remark 4.1. As we shall see in the next section, in some applications the term

$$
(4.11)\qquad b_h(\Pi_h\underline{\sigma} - \underline{\sigma}_h, P_h u - u_h) = -\nu(u-u_h, P_h u - u_h)
$$

is nonpositive. (This, for instance, will always be the case if P_h is a true L^2-projection). Therefore, a final estimate can be derived directly from (4.6), (4.7) and (4.11):

$$
(4.12)\qquad \alpha\|\underline{\sigma}-\underline{\sigma}_h\|_0^2 \leq C \, \|\underline{\sigma} - \underline{\sigma}_h\|_0 \, \|\underline{\sigma} - \Pi_h\underline{\sigma}\|_0 - c(u, \Pi_h\underline{\sigma} - \underline{\sigma}_h),
$$

and no dependence on the error for u appears.

Notice that, for $\nu \neq 0$, (4.5) gives at the same time estimates for the vector and scalar variables. On the contrary, if $\nu = 0$, only the error for the vector variable is provided. In such a case, an estimate on u can be derived directly via the *Inf Sup* condition as in Theorem 3 below. We recall that in some cases [3] the interpolation

estimate for $\underline{\sigma}$ is more accurate than that for u, so that a separation of the errors is actually desirable.

Theorem 3. *If* $(\underline{\sigma}, u)$ *that of* (2.6) *and* $(\underline{\sigma}_h, u_h, \lambda_h)$ *that of* (3.14), *the following estimate holds:*

$$\|u - u_h\|_0 \ \leq \ \|P_h u - u\|_0 \ + \ \frac{1}{\beta^*}(\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ N_h(Ker\ C_h, u)). \tag{4.13}$$

Proof. The error estimate for u follows from the *Inf Sup* condition (3.34) and the first equation of (4.2):

$$\begin{aligned} \beta^*\ \|P_h u - u_h\|_0 &\leq \sup_{\underline{\tau} \in Ker C_h} \frac{b_h(\underline{\tau}, P_h u - u_h)}{\|\underline{\tau}\|_h} \ = \ \sup_{\underline{\tau}} \frac{a(\underline{\sigma} - \underline{\sigma}_h, \underline{\tau})}{\|\underline{\tau}\|_h} \ + \ \sup_{\underline{\tau}} \frac{c(u - \lambda_h, \underline{\tau})}{\|\underline{\tau}\|_h} \\ &\leq \ C\ \|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ N_h(Ker\ C_h, u). \end{aligned} \tag{4.14}$$

Hence, (4.13) follows.

■

Remark 4.2. If the *Inf Sup* condition holds with the norm (3.4), then $N_h^*(Ker\ C_h, u)$ can be substituted for $N_h(Ker\ C_h, u)$ in (4.13) -(4.14), and

$$\|u - u_h\|_0 \leq \|P_h u - u\|_0 \ + \ \frac{1}{\beta^*}(\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ N_h^*(Ker\ C_h, u)). \tag{4.15}$$

$$\beta^*\ \|P_h u - u_h\|_0 \leq \ C\ \|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ N_h^*(Ker\ C_h, u). \tag{4.16}$$

Let us give now an error estimate for the Lagrange multipliers.

Theorem 4. *If* $(\underline{\sigma}, u)$ *is the solution of* (2.6) *and* $(\underline{\sigma}_h, u_h, \lambda_h)$ *that of* (3.14), *the following estimate holds*:

$$\begin{aligned} |\lambda_h - P_h^e u|_{-1/2,h} \ &\leq \ C\ (\ h\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ \|P_h u - u_h\|_0 \\ &+ \ h\ N_h(V_h, u - P_h^e u)), \end{aligned} \tag{4.17}$$

where the norm $|.|_{-1/2,h}$ *is defined in* (3.9).

Proof. Let us consider an edge $e \in E_h$ and let K=K(e) be an element of T_h having e as an edge. From (3.20)-(3.22) it follows that there exists a unique $\underline{\tau}^* \in \Sigma(\mathrm{K})$ such that

$$
(4.18) \qquad \begin{cases} \displaystyle\int_e \underline{\tau}^* \cdot \underline{n}\ \mu\ ds \ = \ \int_e (\lambda_h - P_h^e u)\ \mu\ ds, & \mu \in R(e), \\ \displaystyle\int_{\partial \mathrm{K} \backslash e} \underline{\tau}^* \cdot \underline{n}\ \mu\ ds \ = \ 0, & \mu \in R(\partial \mathrm{K}), \\ \displaystyle\int_{\mathrm{K}} \underline{\tau}^* \cdot \underline{\nabla} \phi\ dx \ = \ 0, & \phi \in P(\mathrm{K}), \\ \displaystyle\int_{\mathrm{K}} \underline{\tau}^* \cdot \underline{q}\ dx \ = \ 0, & \underline{q} \in D. \end{cases}
$$

Then, define $\underline{\tau}^e \in V_h$ to be such that

$$
(4.19) \qquad \underline{\tau}^e \in V_h \quad : \quad \underline{\tau}^e \ = \ \underline{\tau}^* \quad on\ \mathrm{K}; \quad \underline{\tau}^e \ = \ 0 \quad on\ \Omega \backslash \mathrm{K}.
$$

Using a scaling argument, we have

$$
(4.20) \qquad \|\underline{\tau}^e\|_{0,\mathrm{K}} \ + \ h_{\mathrm{K}} \|div\ \underline{\tau}^e\|_{0,\mathrm{K}} \ \leq \ C\ h_{\mathrm{K}}^{1/2} \|\underline{\tau}^e \cdot \underline{n}\|_{0,e}.
$$

The first equation of (4.18) implies that $P_h^e \underline{\tau}^e \cdot \underline{n} = \lambda_h - P_h^e u$. Moreover, due to (4.18), $\|P_h^e \underline{\tau}^e \cdot \underline{n}\|_{0,e}$ is a norm of $\underline{\tau}^e$. Hence, we have

$$
(4.21) \qquad \|\underline{\tau}^e \cdot \underline{n}\|_{0,e} \ \leq \ C\ \|\lambda_h - P_h^e u\|_{0,e}.
$$

From (4.21) we deduce in (4.20) that

$$
(4.22) \qquad \|\underline{\tau}^e\|_{0,\mathrm{K}} \ + \ h_{\mathrm{K}} \|div\ \underline{\tau}^e\|_{0,\mathrm{K}} \ \leq \ C\ h_{\mathrm{K}}^{1/2} \|\lambda_h - P_h^e u\|_{0,e}.
$$

With the choice (4.19) of $\underline{\tau}^e$, the first equation of (3.14) becomes

$$
(4.23) \qquad \int_{\mathrm{K}} a^{-1} \underline{\sigma}_h \cdot \underline{\tau}^e\ dx \ - \ \int_{\mathrm{K}} div\ \underline{\tau}^e\ u_h\ dx \ + \ \int_e \lambda_h\ \underline{\tau}^e \cdot \underline{n}\ ds \ = \ 0.
$$

Moreover, since $\underline{\sigma} \ = \ -a \underline{\nabla} u$, then

$$
(4.24) \qquad \int_{\mathrm{K}} a^{-1} \underline{\sigma} \cdot \underline{\tau}^e\ dx \ - \ \int_{\mathrm{K}} div\ \underline{\tau}^e\ u\ dx \ + \ \int_e u\ \underline{\tau}^e \cdot \underline{n}\ ds \ = \ 0.
$$

By subtracting (4.24) from (4.23) and using the definition (3.30) of the projection P_h, we obtain

$$\int_e (\lambda_h - u)\underline{\tau}^e \cdot \underline{n}\, ds \;=\; \int_{\mathrm{K}} a^{-1}(\underline{\sigma} - \underline{\sigma}_h)\cdot \underline{\tau}^e \, dx \;-\; \int_{\mathrm{K}} div\ \underline{\tau}^e\ (P_h u - u_h)\, dx. \tag{4.25}$$

Then, (4.25) and the definition (4.19) of $\underline{\tau}^e$ imply that

$$\begin{aligned}
\|\lambda_h - P_h^e u\|_{0,e}^2 \;&=\; \int_e (\lambda_h - P_h^e u)\underline{\tau}^e \cdot \underline{n}\, ds \;=\; \int_e (\lambda_h - u)\underline{\tau}^e \cdot \underline{n}\, ds \\
&\quad + \int_e (u - P_h^e u)\underline{\tau}^e \cdot \underline{n}\, ds \\
&= \int_{\mathrm{K}} a^{-1}(\underline{\sigma} - \underline{\sigma}_h)\cdot \underline{\tau}^e \, dx \;-\; \int_{\mathrm{K}} div\ \underline{\tau}^e (P_h u - u_h)\, dx \\
&\quad + \int_e (u - P_h^e u)\underline{\tau}^e \cdot \underline{n}\, ds \qquad (4.26) \\
&\le C\ \|\underline{\sigma} - \underline{\sigma}_h\|_{0,\mathrm{K}}\ \|\underline{\tau}^e\|_{0,\mathrm{K}} \;+\; \|P_h u - u_h\|_{0,\mathrm{K}}\ \|div\ \underline{\tau}^e\|_{0,\mathrm{K}} \\
&\quad + \int_e (u - P_h^e u)\underline{\tau}^e \cdot \underline{n}\, ds.
\end{aligned}$$

Using now (4.22) and the inequality $ab \le \frac{\epsilon}{2}a^2 + \frac{1}{2\epsilon}b^2$, we have from (4.26)

$$\begin{aligned}
\|\lambda_h - P_h^e u\|_{0,e}^2 \;&\le C\ (\|\underline{\sigma} - \underline{\sigma}_h\|_{0,\mathrm{K}}^2\ h_{\mathrm{K}} \;+\; \|P_h u - u_h\|_{0,\mathrm{K}}^2\ h_{\mathrm{K}}^{-1} \\
&\quad + \int_e (u - P_h^e u)\underline{\tau}^e \cdot \underline{n}\, ds\ ). \qquad (4.27)
\end{aligned}$$

Now set $\tilde{\underline{\tau}} = \sum_{e\in E_h} \underline{\tau}^e h_e$. Since the decomposition is regular, $h_e \sim h_{\mathrm{K}(e)}$, so that from (4.22) we easily have

$$\|\tilde{\underline{\tau}}\|_h \;\le\; Ch|\lambda_h - P_h^e u|_{-1/2,h}. \tag{4.28}$$

Multiplying each term of (4.27) by h_e and summing over all e in E_h, we have

$$\begin{aligned}
|\lambda_h - P_h^e u|_{-1/2,h}^2 \;&\le\; C(h^2\|\underline{\sigma} - \underline{\sigma}_h\|_0^2 \;+\; \|u_h - P_h u\|_0^2) \\
&\quad + \sum_e \int_e (u - P_h^e u)\tilde{\underline{\tau}} \cdot \underline{n} ds. \qquad (4.29)
\end{aligned}$$

Note that in (4.29) we also used the (obvious) fact that each element K of T_h relates to at most four edges $e \in E_h$. From (4.28), (4.29), and (4.4) we have then (4.17).

■

Remark 4.3. If the assumption $(H1)$ is dropped, (3.19) no longer holds and P_h cannot be defined as in (3.30). This does not affect the uniqueness result but, in general, a loss of accuracy in the error estimates should be expected (and for lower order elements this could mean no convergence at all). However, in some particular cases, even if $Ker\ B_h \not\subset Ker\ B$, the elements of $Ker\ B_h$ still exhibit some special property (like being orthogonal to some space bigger than W_h) which could possibly help in the estimates. We refer to [13] for an example of mixed finite elements for fourth order elliptic problems and to [11] for an example in this context.

5. Examples

We shall give examples of finite elements that fulfill the abstract framework of sections 3 and 4. The first four are known families of mixed finite elements, while the last two are new.

Example 1. The Raviart Thomas elements [12] over a triangular decomposition of Ω. For any integer $K \geq 0$, let

$$\Sigma(\mathrm{K}) = \underline{P}_k(\mathrm{K}) + \underline{x}P_k(\mathrm{K}), \tag{5.1}$$

$$P(\mathrm{K}) = P_k(\mathrm{K}), \tag{5.2}$$

$$R(\partial\mathrm{K}) = \prod_e P_k(e), \tag{5.3}$$

where, as usual, $P_k(\mathrm{K})$ ($P_k(e)$) denotes the set of polynomials of degree $\leq k$ in K (in e).

Example 2. The Raviart Thomas elements [12] over a rectangular decomposition of Ω. For any integer $k \geq 0$, let

$$\Sigma(\mathrm{K}) = (Q_k(\mathrm{K}) + x_1 Q_k(\mathrm{K})) \times (Q_k(\mathrm{K}) + x_2 Q_k(\mathrm{K})), \tag{5.4}$$

$$P(\mathrm{K}) = Q_k(\mathrm{K}), \tag{5.5}$$

$$R(\partial\mathrm{K}) = \prod_e Q_k(e), \tag{5.6}$$

where $Q_k(\mathrm{K})$ denotes the polynomials of degree $\leq k$ in each variable x_1 and x_2, and $Q_k(e) \equiv P_k(e)$.

Example 3. The BDM elements [3] over a triangular decomposition of Ω.
For any integer $k \geq 1$, let

$$\Sigma(\mathrm{K}) = \underline{P}_k(\mathrm{K}), \tag{5.7}$$

$$P(\mathrm{K}) = P_{k-1}(\mathrm{K}), \tag{5.8}$$

$$R(\partial \mathrm{K}) = \prod_e P_k(e). \tag{5.9}$$

Example 4. The BDM elements [3] over a rectangular decomposition of Ω.
For any integer $k \geq 1$, let

$$\Sigma(\mathrm{K}) = P_k(\mathrm{K}) \oplus \{\underline{rot}(x_1, x_2^{k+1})\} \oplus \{\underline{rot}(x_2, x_1^{k+1})\}, \tag{5.10}$$

$$P(\mathrm{K}) = P_{k-1}(\mathrm{K}), \tag{5.11}$$

$$R(\partial \mathrm{K}) = \prod_e P_k(e). \tag{5.12}$$

It is easily seen that hypotheses $(H1)$ and $(H2)$ are verified in all four examples, and

$$div(\Sigma(\mathrm{K})) = P(\mathrm{K}). \tag{5.13}$$

Moreover, it is known [3], [12] that the vectors $\underline{\tau}$ in $\Sigma(\mathrm{K})$ are determined by a choice of degrees of freedom which contains

$$\begin{cases} \displaystyle\int_{\mathrm{K}} \underline{\tau} \cdot \underline{\nabla}\phi \, dx, & \phi \in P(\mathrm{K}); \\ \displaystyle\int_{\partial \mathrm{K}} \mu \underline{\tau} \cdot \underline{n} \, ds, & \mu \in R(\partial \mathrm{K}). \end{cases} \tag{5.14}$$

Hence, hypothesis $(H3)$ is fulfilled, too. Then, the abstract theory applies. These examples however exhibit some further features which allow us to simplify (and somewhat improve) the abstract results. In particular, (5.13) implies that the projection P_h from W to W_h can be taken as the usual L^2-projection:

$$\int_{\mathrm{K}} (u - P_h u)\phi \, dx = 0, \qquad \forall \phi \in P(\mathrm{K}). \tag{5.15}$$

Thus, using (3.28) and (5.15), we have, for $\underline{\tau} \in \widetilde{V}$,

(5.16)
$$\sum_{\mathrm{K}} \int_{\mathrm{K}} div\ \Pi_h \underline{\tau}\ \phi\ dx\ =\ \sum_{\mathrm{K}} \int_{\mathrm{K}} div\ \underline{\tau}\ \phi\ dx\ =\ \sum_{\mathrm{K}} \int_{\mathrm{K}} P_h div\ \underline{\tau}\ \phi\ dx, \qquad \forall \phi \in W_h.$$

The results can be summarized in the following commuting diagram [9]:

(5.17)
$$\begin{array}{ccccc} \widetilde{V} & \stackrel{div}{\longrightarrow} & W & & \\ \downarrow \Pi_h & & \downarrow P_h & & \\ Ker\ C_h & \stackrel{div}{\longrightarrow} & W_h & \longrightarrow & 0 \end{array}$$

Property (5.16) allows the use of the norm (3.4) in V_h, since

(5.18)
$$(\sum_{\mathrm{K}} ||div\ (\Pi_h \underline{\tau} - \underline{\tau})||^2_{0,\mathrm{K}})^{1/2}\ =\ ||P_h div\ \underline{\tau} - div \underline{\tau}||_{0,\Omega}\ \leq\ ||\underline{\tau}||_{\widetilde{V}}.$$

Hence, (3.24) holds with $||.||_*$:

(5.19)
$$||\Pi_h \underline{\tau}||_*\ \leq\ \gamma\ ||\underline{\tau}||_{\widetilde{V}}\ ,$$

so that the *Inf Sup* condition (3.34) also holds with $||.||_*$. Due to (5.15), we can substitute $P_h u$ for u in (4.11), giving

(5.20)
$$b_h(\Pi_h \underline{\sigma} - \underline{\sigma}_h, P_h u - u_h)\ =\ -\nu ||P_h u - u_h||^2_0\ \leq\ 0\ .$$

Therefore, the abstract estimate for $\underline{\sigma} - \underline{\sigma}_h$ reduces to (4.12). Another simplifying property is that, for all the examples, the choice of the spaces implies that

(5.21)
$$\underline{\tau} \cdot \underline{n} \in R(\partial \mathrm{K}) \qquad \text{for } \underline{\tau} \in \Sigma(\mathrm{K}).$$

In particular, $[\underline{\tau} \cdot \underline{n}]|_e \in R(e)$, where $[\underline{\tau} \cdot \underline{n}]|_e$ denotes the jump in $\underline{\tau} \cdot \underline{n}$ across e. Then, according to (3.17), $\underline{\tau} \in Ker\ C_h$ implies that $[\underline{\tau} \cdot \underline{n}]|_e = 0, \quad \forall e \in E_h$; i.e.,

(5.22)
$$\underline{\tau} \in Ker\ C_h\ \Longrightarrow \underline{\tau} \cdot \underline{n}\ \textit{continuous across the interelement boundaries.}$$

Accordingly, for $\underline{\tau} \in Ker\ C_h$ we have $c(\phi, \underline{\tau}) = 0, \quad \forall \phi \in H^1(\Omega)$. Thus,

(5.23)
$$N_h(Ker\ C_h, u) = N_h^*(Ker\ C_h, u) = 0.$$

Using (5.23), estimates (4.12) and (4.13) give

$$\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ \leq \ C\|\underline{\sigma} - \Pi_h\underline{\sigma}\|_0, \tag{5.24}$$

$$\|u - u_h\|_0 \ \leq \ \|P_h u - u\|_0 \ + \ C\|\underline{\sigma} - \Pi_h\underline{\sigma}\|_0. \tag{5.25}$$

Estimate (4.17) also simplifies. Actually, due to (5.21) and to definition (3.33), we have

$$N_h(V_h, u - P_h^e u) \ = \ 0. \tag{5.26}$$

Hence, (4.17) reads

$$|\lambda_h - P_h^e u|_{-1/2,h} \ \leq \ C\ (h\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ \|P_h u - u_h\|_0\ ). \tag{5.27}$$

Note that, in the present cases, the last integral in (4.26) vanishes identically, so that from (4.26) and (4.22) we also obtain

$$\|\lambda_h - P_h^e u\|_{0,e} \ \leq \ C(h_{\mathrm{K}}^{1/2}\|\underline{\sigma} - \underline{\sigma}_h\|_{0,\mathrm{K}} \ + \ h_{\mathrm{K}}^{-1/2}\|P_h u - u_h\|_{0,\mathrm{K}}). \tag{5.28}$$

Remark 5.1 . We point out that superconvergence results hold for $\|P_h u - u_h\|_0$, as proved in [8],[3] via duality arguments, for all the above families. Accordingly, a superconvergence result also follows for $|\lambda_h - P_h^e u|_{-1/2,h}$.

In these four examples the final estimates are derived from the abstract results using the commuting diagram property (5.17) and the regularity of $KerC_h$ ($KerC_h \subset V$ due to (5.22)). As already pointed out, assumptions $(H1) - (H3)$ themselves do not imply (5.17) and (5.22). The two key properties which added to $(H1) - (H3)$ give (5.17) and (5.22) are (5.13) and (5.21). In the next two examples we shall see one case where (5.13) is verified but (5.21) is not fulfilled and one case where the opposite holds. Example 5 provides a non-conforming element with a "good" divergence, while Example 6 shows a conforming element with "bad" divergence, in the sense that the "local" divergence is not in $P(\mathrm{K})$, but $div(\Sigma(\mathrm{K})) \ \supset \ P(\mathrm{K})$.

Example 5.

We introduce now a new mixed finite element over a triangular decomposition of Ω. For each triangle $\mathrm{K} \in T_h$, we choose the finite-dimensional polynomial sets as follows:

$$\Sigma(\mathrm{K}) = span\ \{\underline{\tau}^1, \underline{\tau}^2, \underline{\tau}^3\}, \tag{5.29}$$

$$P(\mathrm{K}) = P_0(\mathrm{K}), \tag{5.30}$$

$$R(\partial\mathrm{K}) = \prod_e P_0(e), \tag{5.31}$$

where

$$\underline{\tau}^1 = (1,0)\ ,\ \underline{\tau}^2 = (0,1)\ ,\ \underline{\tau}^3 = (\psi_1, \psi_2). \tag{5.32}$$

The first choice for ψ_1 and ψ_2 is

$$\psi_1\ ,\ \psi_2 \in P_1(\mathrm{K}), \tag{5.33}$$

and, for a chosen edge e of K,

$$\left\{ \begin{array}{lll} \displaystyle\int_e \underline{\tau}^3 \cdot \underline{n}\ ds & = & 1, \\ \displaystyle\int_{\tilde{e}} \underline{\tau}^3 \cdot \underline{n}\ ds & = & 0, \qquad \tilde{e} \neq e, \\ \displaystyle\int_{\mathrm{K}} \psi_1\ dx = \int_{\mathrm{K}} \psi_2\ dx & = & 0. \end{array} \right. \tag{5.34}$$

Conditions (5.33)-(5.34) determine a one-dimensional manifold; $\underline{\tau}^3$ can be chosen arbitrarily as an element of this manifold. For instance, to fix ideas, $\underline{\tau}^3$ can be chosen as the element of minimum norm, but many other choices are possible. Note that (5.34) implies that

$$\int_{\mathrm{K}} div\ \underline{\tau}^3\ dx = 1 \quad , \quad div\ \underline{\tau}^3{}_{|\mathrm{K}} = 1/meas(\mathrm{K}). \tag{5.35}$$

Proposition 1. *If* $\Sigma(\mathrm{K})$, $P(\mathrm{K})$, *and* $R(\partial\mathrm{K})$ *are defined by* (5.29)-(5.34), *then hypotheses* $(H1)$-$(H3)$ *are satisfied.*

Proof. It can be easily seen that, with the choice (5.33), $div(\Sigma(\mathrm{K})) = P(\mathrm{K})$, so that $(H1)$ is verified; $(H2)$ holds trivially. Hypothesis $(H3)$ reads in this case

$$\forall \mu \in R(\partial\mathrm{K}),\ \ \mu \neq 0\ \ ,\ \ \exists \underline{\tau} \in \Sigma(\mathrm{K})\ \ :\ \ \int_{\partial\mathrm{K}} \mu \underline{\tau} \cdot \underline{n}\ ds \neq 0. \tag{5.36}$$

To prove (5.36), choose in $R(\partial \mathrm{K})$ the usual basis given by $\mu_i = 1$ on the edge e_i and $\mu_i = 0$ on the other two edges $(i = 1, 2, 3)$. It is now immediate to check that the matrix

$$\int_{\partial \mathrm{K}} \mu_i \underline{\tau}^j \cdot \underline{n}\, ds \tag{5.37}$$

is non-singular. Hence, $(H3)$ is fulfilled.

Thanks to $(H1)$-$(H3)$, the abstract theory applies. In particular, we can define the projections $\Pi_h : \widetilde{V} \longrightarrow V_h, \quad P_h : W \longrightarrow W_h, \quad P_h^e : L^2(E_h) \longrightarrow \Lambda_h$.
In this case, (3.20)-(3.22) reduce to

$$\int_e (\underline{\tau} - \Pi_h \underline{\tau}) \cdot \underline{n}\, ds = 0, \quad \forall e \textit{ edge of } \mathrm{K}. \tag{5.38}$$

Moreover, since (5.13) holds, then $P_h : W \longrightarrow W_h$ defined in (3.30) is the orthogonal L^2-projection. Then, using the same arguments as in (5.16), (5.17)-(5.19) hold. In particular, the *Inf Sup* condition (3.34) holds with the norm (3.4). Moreover, we have the following interpolation estimates:

$$\|\phi - P_h \phi\|_0 \leq Ch \|\phi\|_1, \tag{5.39}$$

$$\|\underline{\tau} - \Pi_h \underline{\tau}\|_0 \leq Ch \|\underline{\tau}\|_1, \tag{5.40}$$

$$\sum_{\mathrm{K}} \|div(\underline{\tau} - \Pi_h \underline{\tau})\|_{0,\mathrm{K}}^2 \leq Ch^2 \|div \underline{\tau}\|_1^2 . \tag{5.41}$$

Before giving the error estimates, let us prove the following lemma.

Lemma 1. *If* $\Sigma(\mathrm{K})$, $P(\mathrm{K})$, *and* $R(\partial \mathrm{K})$ *are defined by* (5.29)-(5.34), *then*

$$N_h^*(Ker\ C_h, u) \leq Ch|u|_1. \tag{5.42}$$

Proof. By definition (3.17) we have, for $\underline{\tau} \in Ker\ C_h$,

$$c(u, \underline{\tau}) = c(u - P_h^e u, \underline{\tau}). \tag{5.43}$$

Notice that, due to the choice of $\Sigma(\mathrm{K})$, $div\ \underline{\tau} = 0$ implies that $\underline{\tau} =$ constant. In particular, $\underline{\tau} \cdot \underline{n}$ is constant, and therefore $\underline{\tau} \cdot \underline{n} \in R(\partial \mathrm{K})$. For $\underline{\tau} \in Ker\ C_h$, $\underline{\tau} = \underline{\tau}' + \underline{\tau}''$,

with $\underline{\tau}' \in span\{\underline{\tau}^1, \underline{\tau}^2\}$ and $\underline{\tau}'' \in span\{\underline{\tau}^3\}$ in K, $\forall$K $\in T_h$. By definition (3.33) and (5.43), we have

$$c(u, \underline{\tau}) \;=\; c(u - P_h^e u, \underline{\tau}''). \tag{5.44}$$

Since $div\ \underline{\tau}'' = 0$ in K $\iff$ $\underline{\tau}'' = 0$ in K, then $||div\ \underline{\tau}''||_{0,\mathrm{K}}$ is a norm. A simple scaling argument gives

$$\begin{aligned} c(u - P_h^e u, \underline{\tau}'') \;&\leq\; C \sum_{\mathrm{K}} ||u - P_h^e u||_{0,\partial\mathrm{K}} ||div\ \underline{\tau}''||_{0,\mathrm{K}} h_{\mathrm{K}}^{1/2} \\ &\leq\; C \sum_{\mathrm{K}} h_{\mathrm{K}} |u|_{1,\mathrm{K}}\ ||\underline{\tau}||_*. \end{aligned} \tag{5.45}$$

Hence, from definition (4.4) and estimate (5.45) the result (5.42) follows.

■

Theorem 5. *If* $(\underline{\sigma}, u)$ *is the solution of* (2.6) *and* $(\underline{\sigma}_h, u_h, \lambda_h)$ *that of* (3.14) *corresponding to the discretization* (5.29)-(5.34), *the following hold:*

$$||\underline{\sigma} - \underline{\sigma}_h||_0 \;\leq\; Ch\ ||u||_2, \tag{5.46}$$

$$||u - u_h||_0 \;\leq\; Ch\ ||u||_2. \tag{5.47}$$

Proof. Notice that, P_h being the L^2-projection on W_h, we have

$$(u - u_h, P_h u - u_h) \;=\; ||P_h u - u_h||_0^2. \tag{5.48}$$

We are in the situation of Remark 4.1 since the term (4.11) is nonpositive. Estimate (4.12) gives

$$\alpha||\underline{\sigma} - \underline{\sigma}_h||_0^2 \;\leq\; C||\underline{\sigma} - \underline{\sigma}_h||_0\ ||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 \;+\; N_h^*(Ker\ C_h, u)||\underline{\sigma}_h - \Pi_h\underline{\sigma}||_*. \tag{5.49}$$

Note that, by the second equation of (4.2) and (5.48), we deduce that

$$div\ (\underline{\sigma}_h - \Pi_h\underline{\sigma}) \;=\; \nu(P_h u - u_h). \tag{5.50}$$

Using (5.50) in (5.49), we obtain

$$\begin{aligned} \alpha||\underline{\sigma} - \underline{\sigma}_h||_0^2 \;&\leq C||\underline{\sigma} - \underline{\sigma}_h||_0\ ||\underline{\sigma} - \Pi_h\underline{\sigma}||_0 \\ &\quad +\; N_h^*(Ker\ C_h, u)(||\underline{\sigma}_h - \Pi_h\underline{\sigma}||_0 \;+\; \nu||P_h u - u_h||_0). \end{aligned} \tag{5.51}$$

On the other hand, (4.16) and (5.42) give

$$\|P_h u - u_h\|_0 \ \leq \ C(\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ h|u|_1). \tag{5.52}$$

Finally, using (5.52) and (5.42) in (5.51) and (4.15) and recalling the interpolation estimates (5.39) and (5.40), we obtain the result.

■

To estimate the Lagrange multipliers we have the following

Theorem 6. *If* $(\underline{\sigma}, u)$ *is the solution of* (2.6) *and* $(\underline{\sigma}_h, u_h, \lambda_h)$ *that of* (3.14) *corresponding to the discretization* (5.29)-(5.34), *the following holds:*

$$|\lambda_h - P_h^e u|_{-1/2,h} \ \leq \ Ch\|u\|_2, \tag{5.53}$$

where the norm $|.|_{-1/2,h}$ *is defined in* (3.9).

Proof. Consider the abstract estimate (4.17). Unlike the previous examples, the last term in (4.17) does not vanish. Recalling (4.4), we can use a scaling argument for each $e \in E_h$, and we obtain

$$\int_e (u - P_h^e u)\underline{\tau} \cdot \underline{n} ds \ \leq \ C \ h_{\mathrm{K}}^{1/2} |u|_{1,\mathrm{K}} \|\underline{\tau} \cdot \underline{n}\|_{0,e} \ \leq \ C \ |u|_{1,\mathrm{K}} \|\underline{\tau}\|_{0,\mathrm{K}} \quad \textit{for } \underline{\tau} \in V_h. \tag{5.54}$$

Hence, from (4.4) and (5.54) we have

$$N_h(V_h, u - P_h^e u) \ \leq \ C|u|_1. \tag{5.55}$$

Inserting (5.55) in (4.17) yields

$$|\lambda_h - P_h^e u|_{-1/2,h} \ \leq \ C \ (h\|\underline{\sigma} - \underline{\sigma}_h\|_0 \ + \ \|P_h u - u_h\|_0 \ + \ h). \tag{5.56}$$

Finally, by using (5.46) and (5.52) in (5.56), the result (5.53) follows.

■

Example 6.

Another new mixed finite element over a triangular decomposition of Ω can be introduced starting from the choice (5.29)-(5.32) of the spaces $\Sigma(\mathrm{K})$, $P(\mathrm{K})$, and $R(\partial \mathrm{K})$. Instead of (5.33), we take now

$$\psi_1 \ , \ \psi_2 \ \in \ P_2(\mathrm{K}), \tag{5.57}$$

determined as follows. Having chosen an edge e of K, we define $\underline{\tau}^3 = (\psi_1, \psi_2)$ through the following degrees of freedom:

$$
(5.58) \qquad \begin{cases} \underline{\tau}^3 \cdot \underline{n}_{|e} & = 1, \\ \underline{\tau}^3 \cdot \underline{n}_{|\partial \mathrm{K} \setminus e} & = 0, \\ \displaystyle\int_{\mathrm{K}} \psi_1 \, dx \; = \; \int_{\mathrm{K}} \psi_2 \, dx & = 0, \\ \displaystyle\int_{\mathrm{K}} \underline{\tau}^3 \cdot \underline{rot} \, (\lambda_1 \, \lambda_2 \, \lambda_3) \, dx & = 0, \qquad \lambda_i \textit{ the } i^{th} \textit{ barycentric coordinate.} \end{cases}
$$

Proposition 2 . *If* $\Sigma(\mathrm{K})$, $P(\mathrm{K})$, *and* $R(\partial \mathrm{K})$ *are defined by* (5.29)-(5.32), (5.57), and (5.58), *then hypotheses* $(H1)$-$(H3)$ *are satisfied.*

Proof. Notice that $dim(div(\Sigma(\mathrm{K})) = 1$, so that $(H1)$ is satisfied; $(H2)$ holds trivially. In order to check $(H3)$, the same argument as in Proposition 1 applies.

■

Thanks to Proposition 2, the abstract theory applies. In particular, we can define the projections Π_h, P_h, and P_h^e. As before, definition (3.20)-(3.22) of Π_h reduces to (5.38), and the interpolation estimate (5.40) still holds true. However, in this case property (5.13) is not satisfied and the projection P_h from W to W_h defined in (3.30) is no longer the L^2-projection. The interpolation error (5.39) still holds true, but now (5.18) is violated. Consequently, estimate (5.41) cannot be derived. However, thanks to (3.11), (3.24) holds. Consequently, the *Inf Sup* condition (3.34) holds with the norm (3.3).

The choice (5.58) implies $\underline{\tau} \cdot \underline{n} \in R(\partial \mathrm{K})$. Hence, following the same line of argument as in (5.21)-(5.23) and (5.26), we obtain

$$
(5.59) \qquad N_h(Ker \; C_h, u) \; = \; 0,
$$

$$
(5.60) \qquad N_h(V_h, u - P_h^e u) \; = \; 0.
$$

We can collect now the error estimates in the following theorem.

Theorem 7 . *If* $(\underline{\sigma}, u)$ *is the solution of* (2.6) *and* $(\underline{\sigma}_h, u_h, \lambda_h)$ *that of* (3.14) *corresponding to the discretization* (5.29)-(5.32), (5.57)-(5.58), *the following hold:*

$$
(5.61) \qquad ||\underline{\sigma} - \underline{\sigma}_h||_0 \; \leq \; Ch \; ||u||_2 \; ,
$$

$$\|u - u_h\|_0 \;\leq\; Ch\,\|u\|_2\;, \tag{5.62}$$

$$|\lambda_h - P_h^e u|_{-1/2,h} \;\leq\; Ch\,\|u\|_2\;. \tag{5.63}$$

Proof. Taking into account the interpolation error (5.39)-(5.40) and the result (5.59), the abstract estimate (4.5) gives (5.61) and, if $\nu \neq 0$, also (5.62). If $\nu = 0$, in the same manner, (5.62) follows from (4.13). Finally, (5.63) follows from (4.17) applying (5.61), (4.14) and (5.59)-(5.60).

■

Remark 5.2 . We conclude this section by emphasizing an interesting property of our new finite elements. Due to the discontinuity of the elements of V_h and W_h, the unknowns $\underline{\sigma}_h$ and u_h in (3.14) can be eliminated, element by element, by static condensation. The system (3.14) reduces then to one in the unknown λ_h only, whose matrix is symmetric and positive-definite. Moreover, with the elements introduced in Examples 5 and 6, this matrix is also an M-matrix (if the triangulation is of weakly acute type) for all $\nu \geq 0$. This property can be of great interest in some applications, such as in semiconductor device simulation or, in general, in problems where a discrete maximum principle is desirable. We remark that, for $\nu \neq 0$, none of the elements of the four families described in Examples 1-4 produces an M-matrix. For $\nu = 0$ only the lowest-order element of Example 1 gives an M-matrix, as proved in [4], [5].

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