A PÉCLET-ROBUST DISCONTINUOUS GALERKIN METHOD FOR NONLINEAR DIFFUSION WITH ADVECTION

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Discontinuous Galerkin (DG) methods were introduced in the 70s [1] and they are nowadays widely regarded as the reference methods for advection-dominated problems. When a polynomial degree $k \geq 1$ is used, classical error estimates for linear diffusion-advection(-reaction) problems show that the error contribution stemming from diffusive terms is $\mathcal{O}(h^k)$ (with h denoting the meshsize), while the one stemming from advective terms is $\mathcal{O}(h^{k+\frac{1}{2}})$.

The present talk, based on [2], aims to show new Péclet-dependent error estimates for a problem with linear advection-reaction and nonlinear p-type diffusion, with Sobolev indices $p \in (1, \infty)$. Convergence analyses for various DG schemes applied to pure p-type diffusion setting can be found, e.g., in [3, 4]. To the authors' knowledge, an investigation of model problems including both advection and nonlinear diffusion is missing in the literature of DG elements. Especially if one aims at developing sharp estimates which respect the local nature of diffusion and convection, the interaction between the (linear) advection and the nonlinear diffusion cannot be accounted for through a simple combination of known techniques, as the estimate of each term becomes dependent on the local regime.

The discretization of the nonlinear diffusion term is based on the full gradient including jump liftings and interior-penalty stabilization while, for the advective contribution, we consider a strengthened version of the classical upwind scheme. The peculiarity of our error estimates is that they track the dependence of the local contributions to the error on local Péclet numbers. In the linear case, corresponding to $p = 2$, local Péclet numbers can be computed based on the sole knowledge of the problem data and the mesh, making it possible to identify a priori advection- and diffusion-dominated elements/faces. We emphasize that our results hold for general polygonal and polyhedral meshes, which we believe is an important asset. The present contribution furthermore sets the stage for future publications developing pressure robust and advection-robust finite elements for time-dependent Navier–Stokes type equations modeling incompressible fluid flows with non-Newtonian rheology.

In the present talk, after presenting the model and the numerical method, we will outline the theoretical results. Finally, a set of numerical tests supporting the theory will be shown.

References

[1] G. A. Baker. Finite element methods for elliptic equations using nonconforming elements. *Math. Comp.* 31.137, pp. 45–49, 1977.

[2] L. Beirão da Veiga, D. A. Di Pietro, and K. B. Haile. A Péclet-robust discontinuous Galerkin method for nonlinear diffusion with advection. Accepted for publication in *Math. Models Methods Appl. Sci.* Preprint available at https://arxiv.org/abs/2402.09814, 2024.

[3] E. Burman, and A. Ern. Discontinuous Galerkin approximation with discrete variational principle for the nonlinear Laplacian. *C. R. Math. Acad. Sci. Paris* 346.17-18, pp. 1013–1016, 2008.

[4] T. Malkmus, M. Ruzicka, S. Eckstein, and I. Toulopoulos. Generalizations of SIP methods to systems with p-structure. IMA *J. Numer. Anal.* 38.3, pp. 1420–1451, 2018.