Bayesian analysis of matched categorical data

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Matched categorical data

Eye-grade data

(Stuart, 1955)

Right eye	Left eye: Highest	Second	Third	Lowest	Total
Highest	1520	266	124	66	1976
Second	234	1512	432	78	2256
Third	117	362	1772	205	2456
Lowest	36	82	179	492	789
Total	1907	2222	2507	841	7477

Unaided distance vision for women

Hypothesis of interest

`Marginal Homogeneity(MH)`
equality of row and column marginal distribution

$$\pi_{i+} = \pi_{+i}, \quad i = 1, \ldots, s$$

$$\pi_{ij} = Pr(X = i, Y = j)$$

joint probabilities

Binary matched pairs

	Y	
X	π_{11}	π_{12}
	π_{21}	π_{22}

MARGINAL HOMOGENEITY:

$$\pi_{1+} = \pi_{+1}$$

$$\Updownarrow$$

SYMMETRY:

$$\pi_{12} = \pi_{21}$$

- Altham (1971)
- Kateri et al. (2000)
- Irony et al. (2001)
- Broemeling and Gregurich (1996)

Polytomous matched pairs

Square contingency table

Marginal Homogeneity

- no longer equivalent to symmetry
- cannot be tested in terms of log-linear parameters
- can be tested using marginal log-linear parameters

Marginal log-linear parameters

Bergsma and Rudas (2002)
Rudas (2004)

$$\pi \mapsto (\lambda_X^X, \lambda_Y^Y, \lambda_{XY}^{XY})$$

$$\lambda_X^X(i) = \lg \pi_{i+} - \frac{1}{s}\sum_h \lg \pi_{h+}$$

$$\lambda_Y^Y(j) = \lg \pi_{+j} - \frac{1}{s}\sum_k \lg \pi_{+k}$$

$$\lambda_{XY}^{XY}(i,j) = \lg \pi_{ij} - \frac{1}{s}\sum_k \lg \pi_{ik} - \frac{1}{s}\sum_h \lg \pi_{hj} + \frac{1}{s^2}\sum_{hk} \lg \pi_{hk}$$

- parameters satisfy sum-to-zero constraints (effect coding)

$$\beta = A \lg B\pi \qquad (1)$$

vector of non-redundant λ-components, for suitable A and B matrices

- β smooth and variation independent
- inversion of (1) via Newton-Raphson (Glonek and Mc Cullagh, 1995)

Log-linear marginal model

- $\mathcal{H}_0$ linear subspace of $\mathbf{R}^{s^2-1}$

$$\beta \mapsto \begin{pmatrix}\psi \\ \omega\end{pmatrix} = \begin{pmatrix}H_0^t\beta \\ H_1^t\beta\end{pmatrix},$$

- columns of H_0 (H_1) form an orthonormal basis of $\mathcal{H}_0$ ($\mathcal{H}_0^\perp$)
- submodel corresponding to $\mathcal{H}_0$ defined through $\omega = 0$

Marginal Homogeneity identified by

$$\lambda_X^X = \lambda_Y^Y$$

Prior distributions

i) λ_X^X, λ_Y^Y and λ_{XY}^{XY} independent and normally distributed

ii) λ-components exchangeable within each block

i) and ii) imply $\quad \beta \sim \mathcal{N}(0, V)$

Covariance structure

V block-diagonal

$$V_X = v_X\left(\frac{s}{s-1}I_{s-1} - \frac{1}{s-1}J_{s-1}\right)$$

$$V_Y = v_Y\left(\frac{s}{s-1}I_{s-1} - \frac{1}{s-1}J_{s-1}\right)$$

$$V_{XY} = v_{XY}\left(\frac{s}{s-1}I_{s-1} - \frac{1}{s-1}J_{s-1}\right) \otimes \left(\frac{s}{s-1}I_{s-1} - \frac{1}{s-1}J_{s-1}\right)$$

v_X, v_Y, v_{XY} tuning hyper-parameters; for default settings see Dellaportas and Forster (1999)

Prior under MH model

$$p_0(\beta) = p(\beta|\omega = 0)$$

with $p(\cdot|\omega)$ deduced from prior under the unrestricted model

Bayes factor in favor of MH

Savage density ratio

Dickey (1971)
Verdinelli and Wassermann (1995)

$$\mathrm{BF} = \frac{p(\omega = 0|\mathrm{data})}{p(\omega = 0)}$$

- $p(\omega = 0)$ analytically available
- $p(\omega = 0|\mathrm{data})$: use MCMC output to get an approximation
- BF highly sensitive to prior specification; requires careful prior choice
  - set $v_X = v_Y = v_{XY} = v$
  - choose v so that data *perfectly consistent* with MH lead to *decisive* evidence in favor of MH on Jeffreys' scale

Results for Eye-grade data

- BF $\simeq 3.1$, i.e. very mild *positive* evidence in favor MH
- frequentist inference leads to p-value less than 1% (highly significant against MH)
- 95% HPD intervals for $\lambda_X^X(i) - \lambda_Y^Y(i)$ contain zero only for $i = 2, 3$