Recent results in multivalued integration

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   - relations among multi-valued integrals
   - Order-type gauge integrals for multifunctions

Banach lattices

Let $(\mathbf{X}, \|\cdot\|, \leq)$ be a real Banach space, endowed with a compatible ordering $<$. If **X** is stable under finite suprema (and infima) then it is called a **Banach lattice**.

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- The norm $\|\cdot\|$ is **order-continuous**, i.e. $\lim_n \|p_n\| = 0$ as soon as $(p_n)_n$ is a decreasing sequence in X, with infimum 0 ((O)-sequence from now on).

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(o)-convergence

A sequence $(a_n)_n$ in $\mathbf{X}$ is said to be **(o)-convergent** to $a \in \mathbf{X}$ whenever an (o)-sequence $(p_n)_n$ exists, such that $|a_n - a| \leq p_n$ for all n. If this happens, $(p_n)_n$ will be called a *regulating* (o)-sequence for $(a_n)_n$.

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- A Banach lattice M is an M**-space** if there exists a strong unit e in M such that the Minkowski functional ρ relative to e is equivalent to the original norm.

  (Here $\rho(x) = \inf\{r > 0 : |x| \leq re\}$).

  In this situation,

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- A Banach lattice L is an L**-space** if the norm of L satisfies:

  $$\|x + y\| = \|x\| + \|y\|$$

  as soon as x, y are positive elements of L.

Embedding Theorems

Let X be any Banach space, and denote by $cbf(X)$, (resp. $cwk(X)$, $ck(X)$) the family of all non-empty bounded, convex closed (resp. convex weakly compact, convex and compact) subsets of X.
Any of these spaces, when endowed with the **Hausdorff metric** h, is **complete**. The symbol S will denote any of these.

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Definition

The *support function* $s(x^*, A)$ is defined in the dual unit ball of X for any $A \in S$:

$$s(x^*, A) = \sup_{a \in A} x^*(a).$$

In this way, $s(\cdot, A) \in l^\infty(B^*)$.

In case $S = ck(X)$, the support function is w^*-continuous. More generally

Theorem

(See [a] Let S denote any of the spaces $cbf(X)$, $cwk(X)$, $ck(X)$. Then there exist a compact Hausdorff stonian space K and a one-to-one mapping

$$j : S \to C(K)$$

satisfying:

- $$j(\alpha A \oplus \beta B) = \alpha j(A) + \beta j(B)$$
  as soon as A and B are in S, and α, β are non-negative real numbers.
- $$h(A, B) = \|j(A) - j(B)\|_\infty$$
- *$j(S)$ is norm-closed.*

[a] C. C. A. Labuschagne, A. L. Pinchuck, C. J. van Alten, *A vector lattice version of Rådstrőm's embedding theorem*, Quaest. Math. **30** (3) (2007), 285–308

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Use of modulars

(See [1]) Let $(\Omega, \mathcal{A}, \mu)$ be any measure space, and fix any Banach M-space X, with unit e, and endowed with a (compatible) multiplication. Let $\mathcal{L}$ denote the linear space of all *simple* functions, defined in Ω and taking values in X. So, every $f \in \mathcal{L}$ is measurable, has finite range, and vanishes outside a set of finite measure.

Then the *modular*

$$j(f) := \int_{\Omega} |f| d\mu$$

is clearly finite, X-valued, and *positively linear*.

This allows to define integrability also for arbitrary functions $f : \Omega \to X$.

Just the notion of μ-convergence is needed.

[1] A. Boccuto, D. Candeloro, A.R. Sambucini, Vitali-type theorems for filter convergence related to vector lattice-valued modulars and applications to stochastic processes, J. Math. Anal. Appl. **419** (2) (2014), 818–838, AMES prize 2014

Definition

Let $(f_n)_n$ be any sequence of X-valued functions on Ω. The sequence *μ-converges* to a function f if there exist an (O)-sequence $(\sigma_p)_p$ in X, a corresponding (O)-sequence $(\varepsilon_p)_p \in \mathbb{R}$, and a double sequence $(A_n^p)_{p,n}$ in $\mathcal{A}$ such that

- $$A_n^p \supset \{t \in \Omega : |f_n(t) - f(t)| \not\leq \varepsilon_p e\}$$

  for all p and n.
- $\{n \in \mathbb{N} : \mu(A_n^p) \leq \sigma_p\}$ is cofinite.

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Definition

A function $f : \Omega \to X$ is *integrable* if there exist an equi-absolutely continuous sequence of simple functions $(f_n)_n$ μ-converging to f, and a map $I : \mathcal{A} \to X$ such that

$$(O) - \lim_n \sup_{A \in \mathcal{A}} \left| \int_A f_n d\mu - I(A) \right| = 0$$

From now on we consider functions $f : \mathbb{R}^+ \to \mathbf{X}$, and the so-called *moment kernels*, $M_n : \mathbb{R}^+ \to \mathbb{R}$, defined by

$$M_n(w) = w^n \cdot n \cdot 1_{]0,1[}(w), \;\; n \in \mathbb{N}, w \in \mathbb{R}^+.$$

Let $\widehat{L}(\lambda)$ be the set defined by

$$\widehat{L}(\lambda) := \{f \in \mathbf{X}^{\mathbb{R}^+} : \; f \in L(\lambda) \text{ and } M_n\left(\frac{\cdot}{s}\right) f(\cdot) \in L(\mu) \; \forall s \in \mathbb{R}^+, n \in \mathbb{N}\} \tag{1}$$

where $\mu(B) := \displaystyle\int_B \frac{dt}{t}$ for every $B \in \mathcal{B}$. We consider the modular

$$\rho(\cdot) = \int_{\mathbb{R}^+} |\cdot| ds.$$

For every $f \in \widehat{L}(\lambda)$ (see formula (1)) let us consider the operators $T_n f : \mathbb{R}^+ \to \mathbf{X}$ defined by $T_n f(s) = \int_0^{+\infty} M_n\left(\frac{t}{s}\right) f(t) \frac{dt}{t}, \quad n \in \mathbb{N}, \; s \in \mathbb{R}^+$.

Theorem

Let $f \in \widehat{L}(\lambda)$ be a function with compact support $C \subset [a, b]$ with $a > 0$. Then we have the following:

(6.1) *if f is bounded, and $a > 1$, then $T_n f$ are Lipschitz and integrable $\forall n \geq 2$; moreover, if f is uniformly continuous, we have*

$$\int_0^{+\infty} T_n f(s) ds = \frac{n}{n-1} \int_a^b f(s) ds; \tag{2}$$

(6.2) *if f is uniformly continuous, then $(T_n f)_n$ converges uniformly to f;*

(6.3) *if f is uniformly continuous and $a > 1$, then*

(6.3.1) $$(o) \lim_n \int_0^{+\infty} |T_n f(s) - f(s)| ds = 0;$$

(6.3.2) *$\forall n \in \mathbb{N}$, $T_n f$ is uniformly differentiable and its uniform derivative satisfies the following relation:*

$$\frac{s}{n} (T_n f)'(s) + T_n f(s) = s(T_1 f)'(s) + T_1 f(s) = f(s)$$

weak convergence of $T_n f'$

Let $f : [0, +\infty[\to \mathbb{R}$ be any u.c. map, vanishing outside $[a, T]$, with $a > 1$, and fix any C^1 mapping $w : [0, +\infty[\to \mathbb{R}$, with compact support. For each u.c. mapping $g : [0, +\infty[\to \mathbb{R}$, define

$$\rho(g) := \int_0^{+\infty} |g(s)|\,|w'(s)|\,ds.$$

It is easy to see that $\lim_n \rho(T_n f - f) = 0$ thanks to (6.3.1). From this we deduce that

$$\lim_n \int_0^{+\infty} T_n f(s) w'(s) ds = \int_0^{+\infty} f(s) w'(s) ds, \text{ and}$$

$$\lim_n \int_0^{+\infty} w(s)(T_n f)'(s) ds = \int_0^{+\infty} w(s)\, df(s)$$

Though $(T_n f)'$ do not converge in general, we always have

$$\lim_n \int_0^{+\infty} w(s) d(T_n f(s)) = \int_0^{+\infty} w(s) df(s), \forall w.$$

Brownian motion

We shall assume that $\mathbf{B} := (B_t)_{0 \le t < T}$ (with $T < +\infty$) is the standard Brownian Motion defined on a probability space (Ω, Σ, P). Of course, we can consider $\mathbf{B}$ as a mapping from $[0, T]$ into $L_2(\Omega)$. Since L_2 has not a strong unit in general, in order to apply the previous theory we shall suitably restrict it: indeed, thanks to the well-known Maximum Principle, we see that there is a suitable positive element $Z \in L_2$ such that $|B(t)| \le Z$ for all $t \in [0, T]$. Moreover, we remark that[2], there exists a positive random variable W in L_2 such that

$$|B(t+h) - B(t)| \le |h|^{1/4} W$$

holds, as soon as $t, t+h \in [0, T]$. Thus, taking $\mathbf{X}$ as the (complete) subspace of L_2 generated by all elements dominated by some real multiple of $W + Z$, we see that $\mathbf{X}$ has a strong unit (i.e. $W + Z$), that $\mathbf{B}$ is $\mathbf{X}$-valued and is also a uniformly continuous mapping from $[0, T]$ to $\mathbf{X}$, in the sense of our definition

[2] D. NUALART, *Fractional Brownian Motion: Stochastic Calculus and Applications*, Proc. International Congress of Mathematicians, Madrid, Spain, (EMS) Vol. 3, **74** (2006) 1541-1562

Definition

Let $Y : [0, T] \to \mathbf{X}$ be any stochastic process, predictable with respect to $\mathbf{B}$, and with continuous trajectories. If we assume that $\sup_{t \in [0,T]} E(Y(t)^2) = K < +\infty$, then Y is integrable in the Itô's sense, with respect to $\mathbf{B}$. Any process Y of this kind will be called a *regular* process. The Itô integral of Y will be denoted as usual with $(I) \int_0^T Y(t)\, dB(t)$.

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Proposition

If the process Y is regular, and $\sup_{t\in[0,T]} E(Y(t)^2) = K$, then

$$\left\| (I)\int_0^T Y(t)dB(t) \right\|_2^2 \leq KT.$$

Corollary

Let $(Y_n)_n$ be any sequence of regular stochastic processes. Assume that Y_n converges to a regular process Y uniformly in L_2, i.e. for every real $\varepsilon > 0$ an integer N exists, such that $\sup_{t \in [0,T]} \|Y_n(t) - Y(t)\|_2 \leq \varepsilon$ for all $n \geq N$. Then the Itô integrals $(I) \int_0^T Y_n(t) dB(t)$ converge in L_2 to the Itô integral $(I) \int_0^T Y(t) dB(t)$.

We shall apply these results to the process $f : [0, +\infty[\to \mathbf{X}$ defined as follows:

$$f(t) := \begin{cases} 0 & t \notin [a, T] \\ (t - T)(B(t) - B(a)), & t \in [a, T] \end{cases}$$

where a and T are fixed positive numbers, $1 < a < T$. (So f is a process similar to the well-known *Brownian Bridge*).

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Theorem

Let $f(t) = (t - T)(B(t) - B(a))$ be the process defined above. Then f is clearly $\mathbf{X}$-valued and uniformly continuous, and we have

$$\lim_n \int_0^T T_n f(s) dB(s) = (I) \int_0^T f(s) dB(s).$$

(We remark that the integral on the left-hand side is in the Riemann-Stieltjes sense, since the mappings $T_n f$ are more regular than f).

Examples with maple

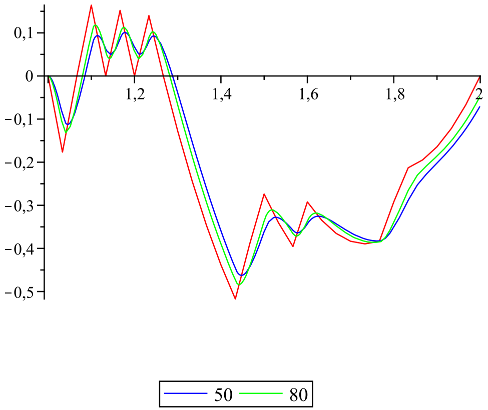

Figure: f (red), $T_{50}f$ (blue), $T_{80}f$ (green)

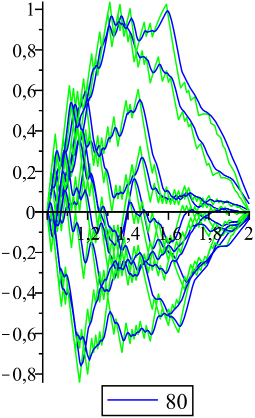

Figure: f (ten trajectories in green) and $T_{80}f$ (blue)

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A *gauge* is any map $\gamma : \Omega \to \mathbb{R}^+$.

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A *partition* Π of Ω is a finite family $\Pi = \{(E_i, t_i) : i = 1, \ldots, k\}$ of pairs such that the E_i are pairwise disjoint sets whose union is Ω and the points t_i are called *tags*.

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If all tags satisfy the condition $t_i \in E_i$ then the partition is said to be of *Henstock* type, or a *Henstock partition*. Otherwise, it is said to be a *free* or *McShane* partition.

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Given a gauge γ, a partition Π is γ-*fine* ($\Pi \prec \gamma$) if $d(w, t_i) < \gamma(t_i)$ for every $w \in E_i$ and $i = 1, \ldots, k$.

Clearly, a gauge γ can also be defined as a mapping associating with each point $t \in \Omega$ an open ball centered at t: this will happen here occasionally.

Gauge integrals

(For the results here see [3]) Let X be any Banach lattice with an order-continuous norm.

[3] D. Candeloro, A.R. Sambucini, *Comparison between some norm and order gauge integrals in Banach lattices*, to appear in PanAmerican Math. Journal (2015), arXiv:1503.04968 [math.FA]

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A function $f : \Omega \to X$ is *norm-integrable* if there exists $J \in X$ such that, for every $\varepsilon > 0$ there is a gauge $\gamma : \Omega \to \mathbb{R}^+$ such that $\|\sigma(f, \Pi) - J\| \leq \varepsilon$ holds, for every γ-fine Henstock partition of Ω, $\Pi = \{(E_i, t_i), i = 1, \ldots, q\}$. (Here, as usual, the symbol $\sigma(f, \Pi)$ means $\sum_{i=1}^{q} f(t_i)\mu(E_i)$). In this case the integral J will be denoted with $H \int f d\mu$.

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Definition

A function $f : \Omega \to X$ is *order-integrable* in the Henstock sense if there exist $J \in X$, an (o)-sequence $(b_n)_n$ in X and a corresponding sequence $(\gamma_n)_n$ of gauges, such that $|\sigma(f, \Pi) - J| \leq b_n$ holds, for every n and every γ_n-fine Henstock partition of Ω, $\Pi = \{(E_i, t_i), i = 1, \ldots, q\}$. In this case the integral J will be denoted with $(oH) \int f$.

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In general, almost equal functions can behave in different ways with respect to the (oH)-integral. An example is given in [4], with a function $f : [0,1] \to c_{00}$ which is a.e. null (with respect to the Lebesgue measure), but **not** (oH)-integrable. However

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Let $f, g : \Omega \to X$ be two bounded maps, such that $f = g$ μ-almost everywhere. Then, f is (oH)-integrable if and only if g is, and the integral is the same.

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Proposition

Let $f, g : \Omega \to X$ be two bounded maps, such that $f = g$ μ-almost everywhere. Then, f is (oH)-integrable if and only if g is, and the integral is the same.

Any (oH)-integrable mapping f satisfies the so-called **Henstock-Saks Lemma** (which is not the case for (H)-integrable mappings in general).

Proposition

Let $f : \Omega \to X$ be any (oH)-integrable function. Then, there exist an (o)-sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gauges, such that $\sum_{E \in \Pi} Ob_n(f, E) \leq b_n$, for every n and every γ_n-fine Henstock partition Π, where

$$Ob_n(E) = \sup_{\Pi'_E, \Pi''_E} \{ | \sum_{F'' \in \Pi''_E} f(\tau_{F''})\mu(F'') - \sum_{F' \in \Pi'_E} f(\tau_{F'})\mu(F') | \}.$$

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Elementary properties

Any (oH)-integrable function f is also integrable in the same sense in every measurable subset A.

Additivity of the integral: namely whenever f is integrable in Ω, and A, B are two disjoint measurable subsets of Ω, then $\int f 1_{A \cup B} d\mu = \int_A f d\mu + \int_B f d\mu$.

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Theorem

Let $f : \Omega \to X$ be any (oH)-integrable function. Then there exist an (o)-sequence $(b_n)_n$ and a corresponding sequence $(\gamma_n)_n$ of gauges, such that:

1) *for every n and every γ_n-fine partition Π one has*

$$\sum_{E\in\Pi} |f(\tau_E)\mu(E) - (\mathrm{oH}) \int_E f d\mu| \leq b_n.$$

2) *for every n and every γ_n-fine partition Π it holds*

$$\sum |f(\tau_E)\mu(E) - f(\tau'_E)\mu(E)| \leq b_n,$$

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Definition

Let $f : \Omega \to X$ be any (oH)-integrable mapping, and set

$$|\mu_f|(A) = \sup\{\sum_{E \in \pi} |(oH) \int_E f d\mu| : \pi \in \Pi(A)\}$$

where $\Pi(A)$ is the family of all finite partitions of A This is the *modulus* of μ_f, denoted by $|\mu_f|$. (This quantity is bounded, see the next theorem).

Consequences

Theorem

If $f : \Omega \to X$ is (oH)*-integrable, then also $|f|$ is.*

Definition

Let $f : \Omega \to X$ be any (oH)-integrable mapping, and set

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Theorem

Assume that $f : \Omega \to X$ is (oH)*-integrable. Then one has*

$$|\mu_f| = \mu_{|f|}.$$

Proposition

Let $f : \Omega \to X$ be any oH-integrable mapping. Then

- *f is also H-integrable, and the two integrals agree;*
- *$|f|$ is H-integrable;*
- *if X is an L-space then f is Bochner integrable.*

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Example

The function $f : [0,1] \to c_{00}$, defined by

$$f(x) = \begin{cases} u_n & \text{if } x = 1/n \\ 0 & \text{otherwise} \end{cases}$$

Example

Define the function $f : [0,1] \longrightarrow X$ in the following way

$$f(t) = \begin{cases} 0, \text{ if } t \in C \text{ or } t = d_i^r, & r \geq 0, 1 \leq i \leq 2^r, \\ 2 \cdot 3^r x_i^r, \text{ if } t \in (a_i^r, d_i^r), & r \geq 0, 1 \leq i \leq 2^r, \\ -2 \cdot 3^r x_i^r, \text{ if } t \in (d_i^r, b_i^r), & r \geq 0, 1 \leq i \leq 2^r. \end{cases}$$

Further consequences

Theorem

Let $f : \Omega \to X$ be (oH)-integrable, and assume that X is an L-space. Then $\|f\|$ is (Lebesgue)-integrable, and f is Bochner (norm)-integrable.

[5] V.A. Skvortsov, A.P. Solodov, *A variational integral for Banach-valued functions*, Real Anal. Exchange **24** (1998-1999), 799-806.

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Remark.
(H)-integrability in general does not imply (oH)-integrability: indeed, if X is any infinite-dimensional Banach space, there exists a McShane (norm)-integrable map $f : [0,1] \to X$ that is not Bochner integrable (see [5]). In particular, when X is an L-space (of infinite dimension), such function f cannot be (oH)-integrable, in view of the previous Theorem.

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Remark
An M-space X admits an equivalent L-norm only if it is of finite dimension. In fact in an M-space X (H)-integrability and (oH)-integrability coincide: so, if an M-space X has an equivalent L-norm then by the previous Remark X must be of finite dimension.

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H-integrable multifunctions

Definition

Given a multifunction $F : \Omega \to cwk(X)$, F is H-*integrable* if there exists an element $J \in cwk(X)$, such that for every $\varepsilon > 0$ there exists a gauge γ such that, for every γ-fine partition Π, the following holds: $d_H(\sum_\Pi F, J) \leq \varepsilon$, where d_H is the Hausdorff distance in $cwk(X)$. In this case we shall write

$$J := (\mathrm{H}) \int_\Omega F d\mu.$$

Also in this case, existence of the integral in Ω implies existence in all measurable subsets E of Ω (which will be denoted by $J_E(F)$).

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Theorem

If $F : \Omega \to cwk(X)$ is H-integrable, then $S^1_{F,H} \neq \emptyset$.

Φ multi-valued integral

Definition

Let $F : \Omega \to 2^X \setminus \emptyset$ be a multifunction, and $E \in \Sigma$. We call $(\| \cdot \|)$-*integral* of F on E the set

$$\begin{aligned}\Phi(F, E) \quad = \quad & \{ z \in X : \forall\, \varepsilon \in \mathbb{R}^+ \exists \gamma : \Omega \to \mathbb{R}^+ : \inf_{c \in \sum_{\Pi_\gamma} F} \|z - c\| \le \varepsilon \\ & \forall\ \gamma\text{-fine partition } \Pi_\gamma := \{(E_i, t_i) : i = 1, \ldots, k\} \text{ of } E.\}\end{aligned}$$

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Alternatively, one can write[6]

$$\Phi(F,E)=\bigcap_n\bigcup_\gamma\bigcap_{P_{\gamma,E}}\left[\Sigma_\Pi F+\frac{B_X}{n}\right], \tag{3}$$

where $P_{\gamma,E}$ is the family of all Henstock γ-fine partitions of E.

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Some relations:

In case F is $cwk(X)$-valued and H-integrable on E then for every $E \in \Sigma$, the following inclusion holds:

1) $J_E(F) \subset \Phi(F, E)$;

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3) Moreover we know that if f is H-integrable for every $E \in \Sigma$, then f is McShane integrable and so Definition 21 is equivalent to the $(\star)$-integral given in (2).
4) If we suppose that X is a separable Banach space and that there exists a countable family $(x'_n)_n$ in X' which separates points of X then the following equalities follow, for any measurable and integrably bounded multifunction F with values in $cwk(X)$:

$$J_E(F) = (\mathrm{AH}) \int_E F d\mu = \Phi(F, E).$$

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[7] A. Boccuto, A. R. Sambucini - A McShane Integral for Multifunctions, J. Concr. Appl. Math. 2 (4) (2004), 307-325.

Definition

Let $F : \Omega \to 2^X$ be a multifunction with non-empty values, and $E \in \Sigma$. We call (o)*-integral* of F on E the set

$$\begin{aligned}\Phi^o(F,E) \;=\; & \{\, z \in X : \exists \text{ an } (o)\text{-sequence } (b_n)_n : \forall n \in \mathbb{N} \; \exists \gamma : T \to \mathbb{R}^+ : \\ & \forall\gamma\text{-fine partition } P_\gamma := \{(E_i, t_i) : i = 1, \ldots, k\} \\ & \text{of } E \; \exists \; c \in \sum_{i \le k} F(t_i)\, \mu(E_i) \text{ with } |z - c| \le b_n \,\}.\end{aligned}$$

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$$\Phi^o(F,E) := \bigcup_{(b_n)_n} \bigcap_n \bigcup_{\gamma_n} \bigcap_{P_{\gamma n}} \mathcal{U}\left(\Sigma_\Pi(F), b_n\right),$$

where $(b_n)_n$ denotes any (o)-sequence, γ_n any *gauge*, P_{γ_n} the family of γ_n-fine partitions of E, and the symbol $\mathcal{U}(C,b)$ (for any set $C \in X$ and any positive element $b \in X$) denotes the set of all elements $z \in X$ such that $|z-y| \le b$ for some $y \in C$

Proposition

If $F : \Omega \to cwk(X)$ *is* oH*-integrable, then*

- *its integral* J_E *is unique;*
- $\Phi^o(F, E) = J_E \in cwk(X)$.

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From now on, we shall assume that $F(t)$ are order-bounded for every $t \in \Omega$.

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Theorem

Let $F : \Omega \to cwk(X)$ be oH-integrable, with integral J, and define

$$g(t) := \sup F(t), \quad S := \sup J.$$

Then, g is oH-integrable, and its integral is S.

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Then, g is oH-integrable, and its integral is S.

Theorem

Let $F : \Omega \to cwk(X)$ be any oH-integrable function, such that $\sup F(t) \in F(t)$ for each $t \in \Omega$. Then F is the sum of an oH-integrable single-valued $g : \Omega \to X$ and an oH-integrable $G : \Omega \to cwk(X)$: $s(x^*, G(t)) \geq 0$ for all $x^* \in X^*$ and $s(x^*, G(t)) = 0$ for all positive elements $x^* \in X^*$.

Open problems

1. Find more general selection theorems for the oH-integral of multifunctions.
2. In the usual space $[a, b]$, with partitions made just with sub-intervals, is it true that a bounded oH-integrable mapping is also McShane integrable? (The answer is yes, if only measurable gauges are allowed).
3. When the space X is not an L-space, are there other cases in which oH-integrability implies Bochner integrability?

THANK YOU!!!

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