

(Some) Facets of the Total Matching Polytope

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Edmonds



Edmonds



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Edmonds



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The **maximum cardinality matching** problem can be formulated as

$$\begin{aligned}
 \max \quad & \sum_{e \in E} y_e \\
 & \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V \quad (\text{degree c.}) \\
 & y_e \in \{0, 1\} \quad \forall e \in E.
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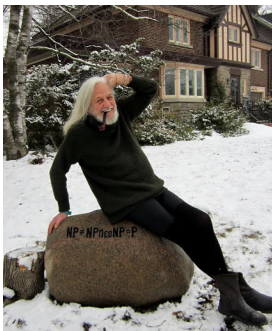
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Theorem 1

Let A_G be the incidence matrix of a graph G . Then, A_G is totally unimodular if and only if G is bipartite.

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Definition 2 (Matching Polytope)

For a given graph G , the **matching polytope** $P_M(G)$ is the convex hull of all the matchings of G :

$$P_M(G) := \text{conv} \left\{ \chi[M] \subseteq \mathbb{R}^{|E|} \mid \chi[M] \text{ is the incidence vector of matching } M \subseteq E \right\}$$

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Theorem 3

The *matching polytope* of $G = (V, E)$ is the set

$$P_M(G) := \left\{ y \in \mathbb{R}^{|E|} \mid y \geq 0, \right. \\ \sum_{e \in \delta(v)} y_e \leq 1, \quad \forall v \in V, \\ \left. \sum_{e \in E[U]} y_e \leq \frac{|U| - 1}{2}, \quad \forall U \subseteq V, |U| \text{ is odd} \right\}$$

(The last are called *blossom inequalities*)

Edmonds and Chvátal



Matchings



Maximum Stable Set

Edmonds and Chvátal: Matchings and Stable Sets

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Given a graph $G = (V, E)$, a **stable set** is a set $S \subseteq V$ of nonadjacent vertices.

The **maximum stable set** problem can be formulated as

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ & x_v + x_w \leq 1 \quad \forall \{v, w\} \in E \\ & x_v \in \{0, 1\} \quad \forall v \in V. \end{aligned}$$



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NOTE: The stable set polytope is not completely characterized.

This Talk: Total Matching



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$P_T(G)$ is **full-dimensional**, that is, $\dim(P_T(G)) = |V| + |E|$.

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Given a graph $G = (V, E)$, a **total matching** is a set $T \subseteq E$ of pairwise disjoint element of $V \cup E$.

The **weighted maximum total matching** problem is formulated as

$$\begin{aligned} \max \quad & \sum_{v \in V} c_v x_v + \sum_{e \in E} c_e y_e \\ & x_v + \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V \\ & x_v + x_w + y_e \leq 1 \quad \forall e = \{v, w\} \in E \\ & x_v \in \{0, 1\}, y_e \in \{0, 1\} \quad \forall v \in V, \forall e \in E. \end{aligned}$$

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This Talk: Motivations

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Total Coloring Problem

A Total Coloring of G is a map $C : V \cup E \rightarrow \{1, \dots, k\}$ such that:

- (a) $C|_V$ is a proper vertex-coloring
- (b) $C|_E$ is a proper edge-coloring
- (c) $C(e) \neq C(v)$, whenever e is incident to v , where $e \in \delta(v)$

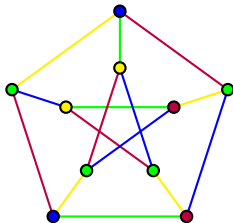


Figure: A 4-total-coloring of the Petersen graph.

Informally: Given a graph $G = (V, E)$, we want to color the edges and the vertices such that adjacent elements receive different colors.

Class of color: A **Total Matching** represents a subset of elements that can take the same color.

Conjecture:

$$\Delta(G) \leq \chi_T(G) \leq \Delta(G) + 2$$

Problem Definition

Let $\mathcal{T}$ be the set of all total matchings. The Total Coloring problem can be formulated with the following **Set Covering model**:

$$\chi_{\mathcal{T}}(G) := \min \sum_{t \in \mathcal{T}} \lambda_t \quad (1)$$

$$\text{s.t.} \quad \sum_{t \in \mathcal{T}} a_{vt} \lambda_t \geq 1 \quad \forall v \in V \quad (2)$$

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QUESTION: Can we find *facet defining inequalities* of the **Total Matching polytope**?

Outline

1 Matching + Stable Set = Total Matching

2 (Some) Facets

3 Computational Results

4 Conclusions

Example of Intersection (Adjoint) Graph

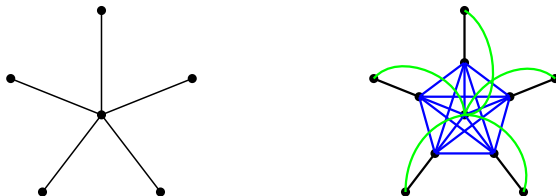


Figure: A star graph and the corresponding **intersection graph**.

Basic facet defining inequalities

Proposition 2.1

Let G be a graph. Then, the inequalities

$$x_v + \sum_{e \in \delta(v)} y_e \leq 1 \quad \forall v \in V \quad (5)$$

$$x_v + x_w + y_e \leq 1 \quad \forall e = \{v, w\} \in E \quad (6)$$

$$x_v, y_e \geq 0 \quad \forall v \in V, \forall e \in E. \quad (7)$$

are **facet-defining** for the **Total Matching Polytope** $P_T(G)$.

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Clique Inequalities

Proposition 2.2

Let G be a graph and let K_h be a **maximal clique** of G . Then, the inequality

$$\sum_{v \in V(K_h)} x_v \leq 1 \quad (8)$$

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- We note that (8) corresponds to **maximal cliques** of cardinality h in H . Hence, they are facet defining for $P_T(G)$.
- In the paper on arxiv, we present a direct proof (without passing by the intersection graph).

Congruent- $2k3$ Cycle Inequalities

Proposition 2.3

Let C_k be an induced (chordless) cycle. Then, if $k \equiv 1 \pmod{3}$ or $k \equiv 2 \pmod{3}$, the **congruent- $2k3$ cycle inequality** defined as

$$\sum_{v \in V(C_k)} x_v + \sum_{e \in E(C_k)} y_e \leq \left\lfloor \frac{2k}{3} \right\rfloor \quad (9)$$

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Remark 2: For $k = 4$, they are facet defining for $P_T(G)$.

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Remark 1: The separation of the congruent- $2k3$ are in P (sequence of shortest path problems on an auxiliary graph)

Remark 2: For $k = 4$, they are facet defining for $P_T(G)$.

Remark 3: They are equivalent to 3-antiweb inequalities on the intersection graph.
(thanks to anonymous reviewer!)

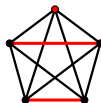
Odd Clique Inequalities

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Let G be a graph and let K_h be a clique of G , where $h \in \mathbb{N}$ is an odd number. Then, the **odd clique inequality** defined as

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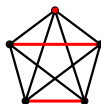
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is valid for the Total Matching Polytope $P_T(G)$, but **it is not facet-defining**.



Remark 1: It is the combination of a clique and a “blossom” inequality:

$$\begin{array}{rcl} \text{(clique)} & \sum_{v \in K_h} x_v & \leq 1 \\ \text{(blossom)} & \sum_{e \in E(K_h)} y_e & \leq \frac{h-1}{2} \\ \hline & \sum_{v \in K_h} x_v + \sum_{e \in E(K_h)} y_e & \leq 1 + \frac{h-1}{2} = \frac{h+1}{2} \end{array}$$

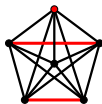
Even Clique Inequalities

Theorem 7

Let G be a graph and let K_h be a clique of G , where $h \in \mathbb{N}$ is an even number. Then, the **even clique inequality** defined as

$$\sum_{v \in V(K_h)} x_v + \sum_{e \in E(K_h)} y_e \leq \frac{h}{2} \quad (11)$$

is **facet-defining** for the **Total Matching Polytope** $P_T(G)$.



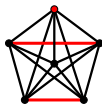
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Remark 1: It is tighter than the combination of a clique and a “blossom” inequality:

$$\begin{array}{rcl}
 \text{(clique)} & \sum_{v \in K_h} x_v & \leq 1 \\
 \text{(blossomlike)} & \sum_{e \in E(K_h)} y_e & \leq \frac{h}{2} \\
 \hline
 & \sum_{v \in K_h} x_v + \sum_{e \in E(K_h)} y_e & \leq 1 + \frac{h}{2} \quad (> \frac{h}{2})
 \end{array}$$

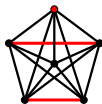
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Remark 2: There is not a simple interpretation on the intersection graph

Separation of Even Clique Inequalities

Given $\hat{x}_v$ and $\hat{y}_e$ solution of an LP relaxation, the separation of even clique inequalities can be solved by the following ILP model:

$$\max \sum_{v \in V} \hat{x}_v x_v + \sum_{e \in E} \hat{y}_e y_e - z \quad (12)$$

$$\text{s.t. } x_v + x_w \leq 1 \quad \forall \{v, w\} \in \bar{E} \quad (13)$$

$$\sum_{v \in V} x_v = 2z \quad (14)$$

$$x_v \geq y_e \quad \forall e = \{u, v\} \in E \quad (15)$$

$$x_u \geq y_e \quad \forall e = \{u, v\} \in E \quad (16)$$

$$x_v + x_u \leq y_e + 1 \quad \forall e = \{u, v\} \in E \quad (17)$$

$$x_v, y_e \in \{0, 1\} \quad \forall v \in V, \forall e \in E \quad (18)$$

$$z \in \mathbb{Z}. \quad (19)$$

If the optimal solution is positive, we get a maximally violated even clique inequality.

Outline

1 Matching + Stable Set = Total Matching

2 (Some) Facets

3 Computational Results

4 Conclusions

Preliminary results on small graphs

Graph Name	n	m	IP	UB_0	Congruent-2k3			Even Clique Ineq.		
					UB_1	cuts	time	UB_2	cuts	time
Complete K_{12}	12	66	6	8.85	8.70	67	2.2	6.00	1	0.0
Chvatal	12	24	8	8.57	8.00	14	0.3	8.57	0	0.2
Flower_snark	20	30	13	13.99	13.71	13	0.0	13.99	0	0.1
Mycielski_5	23	71	16	16.99	16.85	55	5.6	16.99	0	5.6
Tutte	46	69	31	32.19	31.40	25	3.9	32.20	0	3.9
Wheel W_{80}	80	158	53	53.55	53.00	10	2.0	53.55	0	2.0
huck	74	301	49	54.83	54.07	126	62.5	53.30	2	0.1
jean	80	254	58	62.64	62.18	98	59.2	60.30	3	0.1
queen6_6	36	290	21	26.70	26.39	311	34.3	23.50	6	0.4
queen7_7	49	476	28	36.42	36.07	509	84.4	32.37	16	0.6
queen8_8	64	728	36	47.63	47.25	783	214.4	40.25	13	1.1
queen9_9	81	1056	45	60.34	59.93	1137	646.8	53.41	28	2.6
queen10_10	100	1470	55	74.56	74.12	1543	895.1	60.00	18	3.1

Test in Python with Gurobi 9.0 on a commercial laptop. Time is in seconds.

In ongoing work, we are studying the complete description of the Total Matching Polytope for the following classes of graphs:

- Bipartite Graphs
- Line Graphs
- Quasi-Line Graphs

... and we are working on more extensive computational results.

Working paper available on arxiv at
<https://arxiv.org/abs/2105.09827>

Questions?

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$$\left[\begin{array}{c|c} I_{|V|} & A_G \\ \hline A_G^T & I_{|E|} \end{array} \right] \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} \leq \mathbf{1}$$