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Long Time Behavior of the Cahn-Hilliard Equation with Irregular Potentials and Dynamic Boundary Conditions

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(*Dedicated to Professor Roger Temam on the Occasion of his* 70*th Birthday*)

Abstract The Cahn-Hilliard equation with irregular potentials and dynamic boundary conditions is considered. The existence of the global attractor is proved and the long time behavior of the trajectories, namely, the convergence to steady states, is studied.

Keywords Cahn-Hilliard equation, Dynamic boundary conditions, Irregular potentials, Global attractor, ω-limit sets, Convergence to steady states
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1 Introduction

We consider in this paper the following Cahn-Hilliard system with dynamic boundary conditions:

$$\partial_t u - \Delta w = 0, \quad \text{in } \Omega \quad \text{and} \quad \partial_n w = 0, \quad \text{on } \Gamma, \tag{1.1}$$

$$w = -\Delta u + f_0(u) + \lambda u - h, \quad \text{in } \Omega, \tag{1.2}$$

$$v = u|_\Gamma \quad \text{and} \quad \partial_t v + (\partial_n u)|_\Gamma - \Delta_\Gamma v + f_\Gamma(v) + \lambda_\Gamma v = h_\Gamma, \quad \text{on } \Gamma, \tag{1.3}$$

$$u|_{t=0} = u_0. \tag{1.4}$$

In the above equations, $\Omega \subset \mathbb{R}^3$ is the domain occupied by the material, Γ is its boundary, and Δ_Γ and ∂_n are the Laplace-Beltrami operator on Γ and the outward normal derivative, respectively. Moreover, f_0 is a function on $(-1,1)$ which is smooth and monotone, but becomes infinite at its end points. This forces the function u to take values in $(-1,1)$. Furthermore, f_Γ is an everywhere defined smooth function and λ, λ_Γ are real constants. Finally, h and h_Γ are given source terms and u_0 is a prescribed initial datum.

In the applications, the sum $f_0(u)+\lambda u$ has the form $F'(u)$, where F is a double-well potential, and a thermodynamically relevant case is given by the so-called logarithmic potential, obtained

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by choosing

$$f_0(u) = c \ln \frac{1+u}{1-u} \quad \text{and} \quad F(u) = c \int_0^u f_0(r) \mathrm{d}r + \frac{\lambda}{2} u^2 \quad \text{for } u \in (-1,1), \tag{1.5}$$

where c is a positive constant. In such a case, F actually presents a double-well if $\lambda < -2c$.

Dynamic boundary conditions have recently been proposed by physicists in order to account for the interactions with the walls in confined systems (see [7–9] and the references therein, see also [10, 11]).

The Cahn-Hilliard system, endowed with these boundary conditions, has been studied in [6, 18, 23, 24, 26, 27] (see also [4–6, 12–15, 17] for similar boundary conditions for the Caginalp phase-field system).

Now, while the problem is well-understood for regular nonlinear terms f_0 and f_Γ, in the sense that we have rather complete and satisfactory results concerning the well-posedness, the regularity of the solutions and the asymptotic behavior of the system (namely, the existence of finite-dimensional attractors and the convergence of trajectories to steady states), the situation is less clear for an irregular nonlinear bulk term f_0, and, in particular, for the above logarithmic function. The first existence result was obtained in [18], under sign conditions on the surface nonlinear term f_Γ close to the singular points of f_0 (see also [5] for similar results for the Caginalp system); roughly speaking, these conditions force the order parameter to stay away from the pure states on the boundary. Furthermore, it was proved in [24] that, when these sign conditions are not satisfied, then one can expect nonexistence of classical (i.e., in the sense of distributions) solutions. A weaker notion of a solution, based on a variational inequality, was then proposed in [24] (see also [19] for a different, yet related, approach, based on duality, for the Caginalp system). Furthermore, it was proved that the variational solutions are classical ones when the sign conditions are satisfied. Finally, finite-dimensional attractors for the dynamical system based on these variational solutions were constructed.

Our aim in this paper is to study the asymptotic behavior of (1.1)–(1.4) and, contrary to [24], we only consider classical solutions and thus assume that proper sign conditions hold. We first prove the existence of global attractors. Our main results then concern the study of the ω-limit sets of single trajectories and the proof, based on the Simon-Łojasiewicz method, of the convergence of trajectories to steady states.

The paper is organized as follows. In the next section, we carefully describe the problem and state our results. The remaining sections are then devoted to the proofs of these results.

2 Main Results

As mentioned in the introduction, Ω is the body where the evolution is considered and $\Gamma := \partial\Omega$. We assume $\Omega \subset \mathbb{R}^3$ to be open, bounded, connected, and smooth (say, of class C^2), and write $|\Omega|$ for its Lebesgue measure. Similarly, $|\Gamma|$ denotes the 2-dimensional measure of Γ. Now, we introduce our assumptions on the structure of system (1.1)–(1.4). We give two functions and two constants satisfying the conditions listed below,

$$f_0 : (-1,1) \to \mathbb{R} \text{ is a } C^1\text{-function with } f_0(0) = 0 \quad \text{and} \quad f_0' \geq 0, \tag{2.1}$$

$$\lim_{r \to \pm 1} f_0(r) = \pm\infty \quad \text{and} \quad \lim_{r \to \pm 1} f_0'(r) = +\infty, \tag{2.2}$$

$$f_\Gamma : \mathbb{R} \to \mathbb{R} \text{ is a } C^1\text{-function} \quad \text{and} \quad f_\Gamma \text{ and } f'_\Gamma \text{ are bounded,} \tag{2.3}$$

$$\lambda, \lambda_\Gamma \in \mathbb{R} \quad \text{and} \quad \lambda_\Gamma > 0. \tag{2.4}$$

As far as the source terms are concerned, we assume

$$h \in L^\infty(\Omega) \quad \text{and} \quad h_\Gamma \in L^\infty(\Gamma). \tag{2.5}$$

Moreover, we require that there exist $r_0 \in (0,1)$ and $\eta \in (0,1)$ such that

$$f_\Gamma(r) + \lambda_\Gamma r - h_\Gamma(x) \le -\eta \quad \text{for every } r \in (-1, -r_0] \text{ and a.e. } x \in \Gamma, \tag{2.6}$$

$$f_\Gamma(r) + \lambda_\Gamma r - h_\Gamma(x) \ge \eta \quad \text{for every } r \in [r_0, 1) \text{ and a.e. } x \in \Gamma. \tag{2.7}$$

We further set, for $r \in (-1,1)$,

$$f(r) := f_0(r) + \lambda r, \quad F_0(r) := \int_0^r f_0(s)\mathrm{d}s \quad \text{and} \quad F(r) := \int_0^r f(s)\mathrm{d}s + C_0, \tag{2.8}$$

where the constant C_0 is chosen such that (this is possible, since $h \in L^\infty(\Omega)$ by (2.5))

$$F(r) - h(x)r \ge 0 \quad \text{for every } r \in (-1,1) \text{ and for a.e. } x \in \Omega. \tag{2.9}$$

Notice that F_0 is a convex function of class C^2 such that $\min F_0 = F_0(0) = 0$.

Remark 2.1 The notation used in the case (1.5) of a logarithmic potential agrees with (2.8). Moreover, we observe that F_0 is bounded in that case.

Now, we introduce the phase space which depends on a real parameter m. We set

$$\Phi_m := \{(u,v) \in H^1(\Omega) \times H^1(\Gamma) : v = u|_\Gamma,\ F_0(u) \in L^1(\Omega),\ \langle u \rangle_\Omega = m\} \tag{2.10}$$

with the notation

$$\langle u \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega u \quad \text{for } u \in L^1(\Omega) \tag{2.11}$$

for the mean value. If Φ_m is not empty, then $F(u) \in L^1(\Omega)$ for some $u \in H^1(\Omega)$. It follows that $|u| \le 1$ a.e. in Ω, whence $|m| \le 1$. However, it is easy to see that problem (1.1)–(1.4) (see its precise formulation below) does not have any classical solution if $m = \pm 1$, due to the singularities of f_0 at ± 1. Therefore, we assume $|m| < 1$. In such a case, Φ_m is nonempty and it is a complete metric space with respect to the metric d defined on $\Phi_m \times \Phi_m$ by the formula

$$d((u_1,v_1),(u_2,v_2)) := \|u_1 - u_2\|_{H^1(\Omega)} + \|v_1 - v_2\|_{H^1(\Gamma)} + \|F_0(u_1) - F_0(u_2)\|_{L^1(\Omega)}, \tag{2.12}$$

where the norms involved are the standard ones. For instance, $\|v\|^2_{H^1(\Gamma)} = \int_\Gamma (|v|^2 + |\nabla_\Gamma v|^2)\mathrm{d}\sigma$, where ∇_Γ is the surface gradient. A similar self-explaining notation is used in what follows for the norms that we have to consider. For the sake of simplicity, the symbol $\|\cdot\|_X$ also denotes the norm of any power of X. In order not to use a heavy notation in writing the precise formulation of problem (1.1)–(1.4), we set

$$V := H^1(\Omega), \quad H := L^2(\Omega), \quad V_\Gamma := H^1(\Gamma), \quad H_\Gamma := L^2(\Gamma), \tag{2.13}$$

$$\mathcal{V} := \{(u,v) \in V \times V_\Gamma : v = u|_\Gamma\}. \tag{2.14}$$

Moreover, V^* denotes the dual space of V and $\langle \cdot\,, \cdot \rangle$ stands for the duality pairing between V^* and V. It is understood that H is embedded into V^* in the usual way, i.e., so that $\langle u_*, u\rangle = (u_*, u)_H$, the standard inner product in H, whenever $u_* \in H$ and $u \in V$.

At this point, we can give our precise formulation of the problem that we want to deal with. Given $m \in (-1,1)$ and $(u_0, v_0) \in \Phi_m$, we look for a triplet (u,v,w) of real functions on $[0,+\infty)$ satisfying

$$u \in L^\infty(0,T;V) \cap H^1(0,T;V^*) \quad \text{and} \quad f_0(u) \in L^2(0,T;H), \tag{2.15}$$

$$v \in L^\infty(0,T;V_\Gamma) \cap H^1(0,T;H_\Gamma) \quad \text{and} \quad w \in L^2(0,T;V) \tag{2.16}$$

for every $T \in (0,+\infty)$,

$$(u(t), v(t)) \in \Phi_m, \tag{2.17}$$

$$\langle \partial_t u(t), y\rangle + \int_\Omega \nabla w(t)\cdot\nabla y \mathrm{d}x = 0, \tag{2.18}$$

$$\begin{aligned}\int_\Omega w(t) y \mathrm{d}x = \int_\Omega \nabla u(t)\cdot\nabla y\mathrm{d}x + \int_\Omega (f(u(t)) - h) y \mathrm{d}x + \int_\Gamma \partial_t v(t) z \mathrm{d}\sigma \\ + \int_\Gamma \nabla_\Gamma v(t)\cdot\nabla_\Gamma z\mathrm{d}\sigma + \int_\Gamma (f_\Gamma(v(t)) + \lambda_\Gamma v(t) - h_\Gamma) z \mathrm{d}\sigma\end{aligned} \tag{2.19}$$

for a.e. $t > 0$, and

$$u(0) = u_0, \tag{2.20}$$

where (2.18) and (2.19) hold for every $y \in V$ and every $(y,z) \in \mathcal{V}$, respectively.

Remark 2.2 We note that condition (2.17) that we require in the definition of a solution says, in particular, that u is a conserved parameter, i.e., its mean value remains constant during the evolution. Now, such a property immediately follows from (2.18). Indeed, taking $y = \frac{1}{|\Omega|}$ as a test function, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle u\rangle_\Omega = \left\langle \partial_t u, \frac{1}{|\Omega|}\right\rangle = 0, \quad \text{whence } \langle u(t)\rangle_\Omega = \langle u_0\rangle_\Omega = m \text{ for every } t \geq 0. \tag{2.21}$$

The following well-posedness result holds, where we take $m \in (-1,1)$ only. Exactly because of (2.21), we have to exclude $m = \pm 1$, indeed, since the extreme cases cannot be compatible with any summability property for $f(u)$.

Theorem 2.1 *Assume* (2.1)–(2.7). *Then, for every* $m \in (-1,1)$ *and* $(u_0, v_0) \in \Phi_m$, *problem* (2.15)–(2.20) *has a unique global solution.*

Remark 2.3 We do not prove the above theorem in detail and will just give some comments below. A similar and very general result is obtained in [18]. Even though the compatibility condition given by (2.6)–(2.7) does not fit the assumpions of the quoted paper exactly, the same ideas can be used in the present case, in particular, regarding uniqueness. As far as existence is concerned, let us describe very shortly the outline of the proof and refer to the next section for further details. First, an approximating problem depending on $\varepsilon \in (0,1)$ is considered.

More precisely, the irregular functions f_0 and F_0 are replaced by everywhere defined smoother functions $f_{0\varepsilon}$ and $F_{0\varepsilon}$. As in [18] (see [2, p. 28]),

$$f_{0\varepsilon} \text{ is the Yosida regularization of } f_0 \quad \text{and} \quad F_{0\varepsilon}(r) := \int_0^r f_{0\varepsilon}(s)\mathrm{d}s \quad \text{for } r \in \mathbb{R}. \tag{2.22}$$

Accordingly, we define f_ε and F_ε by setting

$$f_\varepsilon(r) = f_{0\varepsilon}(r) + \lambda r \quad \text{and} \quad F_\varepsilon(r) = \int_0^r f_\varepsilon(s)\mathrm{d}s + C_0 \quad \text{for } r \in \mathbb{R}, \tag{2.23}$$

where C_0 is the same as in (2.8). It turns out that $f_{0\varepsilon}$ is a C^1 function such that $0 \leq f'_{0\varepsilon} \leq \frac{1}{\varepsilon}$. More precisely, we see f_0 as a maximal monotone operator in $\mathbb{R} \times \mathbb{R}$ in order to use (2.22) as a definition, according to the general theory (namely, f_0 is identified with the subdifferential of the natural convex l.s.c. extension of F_0 to the whole real line). Then, the approximating problem is stated as follows. We look for a triplet satisfying all the requirements (2.15)–(2.20) in which we read

$$f_{0\varepsilon} \text{ and } w - \varepsilon\partial_t u \text{ instead of } f_0 \text{ and of } w, \text{ respectively, in (2.19).} \tag{2.24}$$

Moreover, we replace the initial datum u_0 by a suitable regularization as well. The variational problem that we obtain is much more regular than the original one, since no singular nonlinearity appears and some parabolicity has been added. Therefore, by arguing as in [18] and using a Galerkin procedure, one shows that the approximating problem has a unique global solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$. Moreover, such a solution is smoother than expected. For instance, the time derivative $\partial_t u_\varepsilon$ belongs at least to $L^2(0,T;H)$ for every finite T, while $\partial_t u$ is expected to exist just in $L^2(0,T;V^*)$. At this point, one can perform several a priori estimates on $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ and use compactness and monotonicity methods to let ε tend to 0, as in the case that we have quoted. However, one important modification is needed in a precise point, as we explain in the next Remark 2.4.

Remark 2.4 As [18] only deals with well-posedness on a finite time interval, the authors did not care about minimizing the assumptions on the initial datum. On the contrary, here, we want to construct a semigroup acting on Φ_m and are forced to assume u_0 to belong to Φ_m, only. Therefore, one of the a priori estimates needs some modification. More precisely, this is the case when we differentiate. By the way, it is not clear that (2.18)–(2.19) can be differentiated, because of the singular term $f(u)$ and of the bad regularity of $\partial_t u$. On the contrary, the approximating variational equations solved by $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ can actually be differentiated with respect to t. The nonlinear functions that are involved are everywhere defined and smooth, indeed. Moreover, the approximating initial datum can be chosen in order to satisfy the compatibility conditions that are needed to have higher time regularity for the approximating solution. More precisely, such a regularity could be proved in a rigorous way by acting on the Galerkin discretization. Once one can differentiate the approximating problem with respect to time, one can perform new a priori estimates, e.g., by testing by the time derivative of some component of the solution. However, this generally leads to assume a stronger regularity for the initial datum. In order to avoid this, one should use weighted test functions. Also on that point, we say some more words in the next section (see Remark 3.2 below).

Our next result asserts the existence of a global attractor. We refer to [1, 29] for the main definitions and properties regarding the notion of a global attractor.

Theorem 2.2 *Assume* (2.1)–(2.7), $m \in (-1, 1)$, *and*

$$F_0 \text{ is bounded.} \tag{2.25}$$

Then, problem (2.15)–(2.20) *defines a continuous semigroup on the phase space* Φ_m *endowed with the weaker metric* d_w *defined by*

$$d_w((u_1, v_1), (u_2, v_2)) := \|u_1 - u_2\|_{V^*} + \|v_1 - v_2\|_{L^2(\Gamma)} \tag{2.26}$$

and this semigroup possesses a global attractor $\mathcal{A}_m$ *which is compact in* $H^1(\Omega) \times H^1(\Gamma)$ *and bounded in* $H^{\frac{3}{2}}(\Omega) \times H^{\frac{3}{2}}(\Gamma)$.

Even though condition (2.25) holds for the important case of a logarithmic potential (see Remark 2.1), one can wonder whether it can be avoided. Actually, we are able to allow a more singular potential, provided that its singularities have a finite order (see the forthcoming Remark 3.3).

As a next step, we analyze single solution trajectories and characterize their ω-limits. Due to Theorem 2.1, for every $m \in (-1, 1)$ and $(u_0, v_0) \in \Phi_m$, we can consider the trajectory (u, v), where (u, v, w) is the global solution to problem (2.15)–(2.20), and define its ω-limit set. It must be pointed out that several topologies could be considered in doing that (see the forthcoming Remark 5.1). We make a choice among others and set

$$\omega(u_0, v_0) = \{(u_\omega, v_\omega) = \lim (u(t_n), v(t_n)) \text{ strongly in} \mathcal{V} : t_n \uparrow +\infty\}. \tag{2.27}$$

The above concise definition obviously means that a point $(u_\omega, v_\omega) \in \mathcal{V}$ belongs to the ω-limit set $\omega(u_0, v_0)$ if and only if there exists an increasing diverging sequence $\{t_n\}$ in $(0, +\infty)$ such that the sequence $\{(u(t_n), v(t_n))\}$ converges to (u_ω, v_ω) strongly in $\mathcal{V}$.

On the other hand, we can consider the steady states of (2.15)–(2.20), i.e., the solutions to the correponding stationary problem, whose variational formulation is the following. A steady state is a pair (u_s, v_s) with

$$(u_s, v_s) \in \Phi_m \quad \text{and} \quad f(u_s) \in L^2(\Omega) \tag{2.28}$$

such that there exists a real constant w_s satisfying

$$\begin{aligned} &\int_\Omega \nabla u_s \cdot \nabla y \mathrm{d}x + \int_\Omega (f(u_s) - h - w_s) y \mathrm{d}x + \int_\Gamma \nabla_\Gamma v_s \cdot \nabla_\Gamma z \mathrm{d}\sigma \\ &+ \int_\Gamma (f_\Gamma(v_s) + \lambda_\Gamma v_s - h_\Gamma) z \mathrm{d}\sigma = 0 \quad \text{for every } (y, z) \in \mathcal{V}. \end{aligned} \tag{2.29}$$

Notice that, for a given (u_s, v_s), the constant w_s is unique, as easily seen by taking $y = 1$ and $z = 1$ in (2.29). The following result holds.

Theorem 2.3 *Assume* (2.1)–(2.7), $m \in (-1, 1)$, (2.25), *and* $(u_0, v_0) \in \Phi_m$. *Then, the* ω*-limit set* (2.27) *is non-empty, compact, and connected in the strong topology of* $\mathcal{V}$. *Moreover, every* $(u_\omega, v_\omega) \in \omega(u_0, v_0)$ *is a solution* (u_s, v_s) *to problem* (2.28)–(2.29). *In particular, for every* $m \in (-1, 1)$, *such a stationary problem has at least one solution.*

As a final step, we apply the so-called Simon-Łojasiewicz method (see, e.g., [6]) to prove that, under additional assumptions on the nonlinear terms, the ω-limit set of any trajectory consists of a single point.

Theorem 2.4 *Assume* (2.1)–(2.7)*,* $m \in (-1,1)$*,* (2.25)*, and* $(u_0, v_0) \in \Phi_m$*. Additionally, assume*

$$h \in H^1(\Omega) \quad \textit{and} \quad h_\Gamma \in H^1(\Gamma), \tag{2.30}$$

$$\exists \zeta \geq 0 \quad \textit{such that } f(r)\operatorname{sign} r + \zeta r^2 \textit{ is convex} \tag{2.31}$$

and, finally,

$$f, \ f_\Gamma|_{[-1,1]} \textit{ are analytic functions.} \tag{2.32}$$

Then, the ω*-limit set of any weak solution consists of a unique stationary point* $(\overline{u}, \overline{v})$*. More precisely,*

$$(u(t), v(t)) \to (\overline{u}, \overline{v}), \quad \text{weakly in } H^3(\Omega) \times H^3(\Gamma). \tag{2.33}$$

Remark 2.5 It will be clear from the proof that the H^3-convergence could in fact be improved in a way limited only by the regularity of h, h_Γ, and Ω.

The rest of the paper is devoted to the proofs of such results and is organized as follows. Section 3 is devoted to the derivation of several a priori estimates which are preliminary to the proofs of the above theorems. In the same section, we give some more details on the proof of the existence of a solution in the sense of Theorem 2.1. In Sections 4 and 5, we prove Theorems 2.2 and 2.3, respectively. Finally, the proof of Theorem 2.4 is given in Section 6.

3 A priori Estimates

In this section, we prepare some auxiliary materials which will be used in the next sections to prove Theorems 2.2 and 2.3. By the way, what we do here gives the main ideas that are needed to prove Theorem 2.1 as well. However, most of the estimate that we derive are formal, since the test functions that we choose often do not satisfy the required regularity. A completely rigorous procedure could be obtained by performing the same estimates on the approximating problem mentioned in Remark 2.3. In particular, Remark 2.4 should be taken into account as well. On the other hand, using a completely correct argument would lead to an unnecessarily heavy paper. So, we just try to make the formal procedure as close as possible to the rigorous one. In particular, we do not use the a priori bound $|u| < 1$ which follows from the definition of a solution. Such a bound is not satisfied by the approximating u_ε, indeed.

However, before starting, we recall some facts. First, as Ω is bounded and smooth, for every $u \in V$, the following inequalities hold:

$$\|u\|^2_{H^1(\Omega)} \leq c_\Omega(\|\nabla u\|^2_{L^2(\Omega)} + \|u|_\Gamma\|^2_{L^2(\Gamma)}) \quad \text{and} \quad \|u\|^2_{H^1(\Omega)} \leq c_\Omega(\|\nabla u\|^2_{L^2(\Omega)} + |\langle u\rangle_\Omega|^2), \tag{3.1}$$

where c_Ω depends on Ω, only. Moreover, the same assumptions on Ω imply that V and H are compactly embedded into H and V^*, respectively. In particular, the following inequality holds:

$$\|u\|^2_{L^2(\Omega)} \leq \delta\|\nabla u\|^2_{L^2(\Omega)} + c_\delta\|u\|^2_{V^*} \quad \text{for every } u \in V, \tag{3.2}$$

where $\delta > 0$ is arbitrary and c_δ depends on Ω and δ, only. Next, we define

$$\operatorname{dom}\mathcal{N} := \{u_* \in V^* : \langle u_*, 1\rangle = 0\} \quad \text{and} \quad \mathcal{N} : \operatorname{dom}\mathcal{N} \to \{u \in V : \langle u\rangle_\Omega = 0\} \tag{3.3}$$

by setting, for $u_* \in \operatorname{dom}\mathcal{N}$,

$$\mathcal{N}u_* \in V, \quad \langle \mathcal{N}u_*\rangle_\Omega = 0 \quad \text{and} \quad \int_\Omega \nabla\mathcal{N}u_* \cdot \nabla y = \langle u_*, y\rangle \quad \text{for every } y \in V, \tag{3.4}$$

i.e., $\mathcal{N}u_*$ is the solution u to the generalized Neumann problem for $-\Delta$ with datum u_* which satisfies $\langle u\rangle_\Omega = 0$. As Ω is bounded, smooth, and connected, it turns out that (3.4) yields a well-defined isomorphism which satisfies

$$\langle u_*, \mathcal{N}v_*\rangle = \langle v_*, \mathcal{N}u_*\rangle = \int_\Omega (\nabla\mathcal{N}u_*) \cdot (\nabla\mathcal{N}v_*) \quad \text{for } u_*, v_* \in \operatorname{dom}\mathcal{N}. \tag{3.5}$$

Moreover, if we define $\|\cdot\|_* : V^* \to [0, +\infty)$ by the formula

$$\|u_*\|_*^2 := \|\nabla\mathcal{N}(u_* - \langle u_*\rangle_\Omega)\|_{L^2(\Omega)}^2 + |\langle u_*\rangle_\Omega|^2 \ \text{ for } u_* \in V^*, \quad \text{where } \langle u_*\rangle_\Omega := \frac{1}{|\Omega|}\langle u_*, 1\rangle, \tag{3.6}$$

it is straightforward to prove that $\|\cdot\|_*$ is a norm which makes V^* a Hilbert space. Therefore, the following inequalities hold:

$$\frac{1}{c_\Omega}\|u_*\|_{V^*} \le \|u_*\|_* \le c_\Omega\|u_*\|_{V^*} \quad \text{for } u_* \in V^*, \tag{3.7}$$

where c_Ω depends on Ω, only. Indeed, as the latter (trivially) holds, the former follows from the open mapping theorem, provided that we replace c_Ω by a larger constant, if necessary. In particular, we can replace $\|\cdot\|_{V^*}$ by $\|\cdot\|_*$ in (3.2). Note that

$$\langle u_*, \mathcal{N}u_*\rangle = \|u_*\|_*^2 \quad \text{for every } u_* \in \operatorname{dom}\mathcal{N} \tag{3.8}$$

by (3.5)–(3.6). Furthermore, owing to (3.5) once more, we see that

$$2\langle \partial_t u_*(t), \mathcal{N}u_*(t)\rangle = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla\mathcal{N}u_*(t)|^2 = \frac{\mathrm{d}}{\mathrm{d}t}\|u_*(t)\|_*^2 \quad \text{for a.e. } t \in (0, +\infty) \tag{3.9}$$

for every $u_* \in H^1_{\mathrm{loc}}(0, +\infty; V^*)$ satisfying $\langle u_*(t)\rangle_\Omega = 0$ for every $t \ge 0$. Finally, we stress the following consequence of (2.21):

$$\partial_t u(t) \text{ and } u(t) - m \text{ belong to } \operatorname{dom}\mathcal{N} \text{ for } t > 0, \tag{3.10}$$

whenever (u, v, w) is a solution to problem (2.15)–(2.20). Moreover, the same property holds for the component u_ε of the solution to the approximating problem mentioned in Remark 2.3.

Remark 3.1 In the rest of the section, in order to simplify the notation, we use the same symbol c (small c) for constants which can be different from each other (even in the same chain of inequalities) and depend on Ω, f_0, f_Γ, λ, λ_Γ, on the L^∞-norms of h and h_Γ, and on m, only. Symbols such as c_δ or $c(M)$ allow the constants to depend on the positive parameter δ or M, in addition. On the contrary, symbols such as C, C_1, etc., with a capital letter, are used to denote precise constants (e.g., precise values of the above constants c's), in order to be able to refer to them, if necessary. Symbols such as $C_i(M)$ denote a dependence on the parameter M, in addition. This is done in estimating from above, mainly. Similarly, α, α_1, etc., are used for estimates from below and t_1, t_2, etc., stand for values of t which depend on the quantity specified above, only, while symbols such as $t_i(R)$ allow a further dependence on the parameter R, in addition. We stress that none of such constants and values c, C_i, α_i, t_i depends on the initial datum u_0.

In agreement with the introduction to the present section, we state and prove some properties of the approximating functions defined in (2.22)–(2.23). These are ε-versions of the inequalities (see [22, Appendix, Proposition A.1] for some of them) which will be used below when performing our formal estimates, namely,

$$f(r)(r-m) \geq M(r-m)^2 - C(M) \quad \text{for } |r|<1, \tag{3.11}$$

$$f(r)(r-m) \geq \alpha(F(r) + \|h\|_\infty |r|) - C \quad \text{for } |r|<1, \tag{3.12}$$

$$f(r)(r-m) \geq \alpha|f(r)| - C \quad \text{for } |r|<1, \tag{3.13}$$

whose proofs would be simpler or even trivial. In (3.11), $M>0$ is arbitrary and $C(M)$ exists accordingly. In (3.12)–(3.13), both $\alpha>0$ and C are suitably chosen. For the sake of convenience, we summarize some properties of the Yosida regularization $f_{0\varepsilon}$ of f_0 and of its primitive $F_{0\varepsilon}$ (see, e.g., [2, p. 28, p. 39]). For every $\varepsilon>0$ and $r\in\mathbb{R}$, $f_{0\varepsilon}(r)$ is the unique $s\in\mathbb{R}$ satisfying $r-\varepsilon s\in(-1,1)$ and $f_0(r-\varepsilon s)=s$, by definition. Thus,

$$r-\varepsilon f_{0\varepsilon}(r) \in (-1,1) \quad \text{and} \quad f_0(r-\varepsilon f_{0\varepsilon}(r)) = f_{0\varepsilon}(r) \quad \text{for every } r\in\mathbb{R}. \tag{3.14}$$

As f_0 is a C^1 function, the same holds for $f_{0\varepsilon}$. Moreover, $0 \leq f'_{0\varepsilon}(r) \leq \frac{1}{\varepsilon}$ for every $r\in\mathbb{R}$. Finally, we have

$$|f_{0\varepsilon}(r)| \leq |f_0(r)| \qquad \text{and} \quad 0 \leq F_{0\varepsilon}(r) \leq F_0(r) \quad \text{for every } r\in(-1,1), \tag{3.15}$$

$$\lim_{\varepsilon\to 0} f_{0\varepsilon}(r) = f_0(r) \quad \text{and} \quad \lim_{\varepsilon\to 0} F_{0\varepsilon}(r) = F_0(r) \quad \text{for every } r\in(-1,1). \tag{3.16}$$

In particular, both $f_{0\varepsilon}$ and $F_{0\varepsilon}$ are uniformly bounded on every compact subset of $(-1,1)$.

Lemma 3.1 *For every $M>0$, we have*

$$f'_{0\varepsilon}(r) \geq M \quad \textit{for } |r|\geq r_* \textit{ and } \varepsilon\in(0,\varepsilon_*) \tag{3.17}$$

for suitable $r_*=r_*(M)$ and $\varepsilon_*=\varepsilon_*(M)$ belonging to $(0,1)$.

Proof We fix $M>0$ and prove (3.17). We deal with $r\geq 0$, only, since a similar argument holds for $r\leq 0$. Recalling (2.2), we choose $r_0\in(0,1)$ such that

$$f'_0(r) \geq 2M \quad \text{for every } r\in(r_0,1),$$

and fix $r_1\in(r_0,1)$. As $f_{0\varepsilon}(r_1)$ tends to $f_0(r_1)$ as $\varepsilon\to 0$, we can fix $\varepsilon_0\in(0,1)$ in order to fulfil the inequality $r_1-\varepsilon f_{0\varepsilon}(r_1) > r_0$ for $\varepsilon\in(0,\varepsilon_0)$. On the other hand, the function $r\mapsto r-\varepsilon f_{0\varepsilon}(r)$ is nondecreasing on $\mathbb{R}$, since $f'_{0\varepsilon}\leq\frac{1}{\varepsilon}$. We deduce that

$$f'_0(r-\varepsilon f_{0\varepsilon}(r)) \geq 2M \quad \text{for } r\geq r_1 \text{ and } \varepsilon<\varepsilon_0.$$

After differentiating (3.14), the above inequality yields

$$f_{0\varepsilon}'(r) = \frac{f_0'(r-\varepsilon f_{0\varepsilon}(r))}{1+\varepsilon f_0'(r-\varepsilon f_{0\varepsilon}(r))} \geq \frac{2M}{1+2M\varepsilon}$$

since the function $s \mapsto \frac{s}{1+\varepsilon s}$ is increasing on $[0,+\infty)$. Therefore, we deduce that (3.17) holds for every $r \geq r_1$ and ε small enough.

Lemma 3.2 *For every $M > 0$, there exists $C(M)$ such that*

$$f_{0\varepsilon}(r)(r-m) \geq M(r-m)^2 - C(M) \quad \textit{and} \quad f_\varepsilon(r)(r-m) \geq M(r-m)^2 - C(M) \tag{3.18}$$

for every $r \in \mathbb{R}$ and $\varepsilon > 0$ small enough. Moreover, $\alpha > 0$ and $C > 0$ exist such that

$$f_\varepsilon(r)(r-m) \geq \alpha(F_\varepsilon(r) + \|h\|_\infty |r|) - C \quad \textit{and} \quad f_\varepsilon(r)(r-m) \geq \alpha|f_\varepsilon(r)| - C \tag{3.19}$$

for every $r \in \mathbb{R}$ and $\varepsilon > 0$ small enough.

Proof Given $M > 0$, we prove the first inequality (3.18). We argue, e.g., for $r \geq 0$. By applying Lemma 3.1, we find r_* and ε_* such that $f_{0\varepsilon}'(r) \geq 2M$ for $r \geq r_*$ and $\varepsilon \leq \varepsilon_*$. Clearly, we can assume $r_* \geq m$. Then, we have, for $r \geq r_*$,

$$f_{0\varepsilon}(r)(r-m) \geq f_{0\varepsilon}(r_*)(r-m) + 2M(r-r_*)(r-m) \quad \text{for } \varepsilon \in (0,\varepsilon_*).$$

As $f_{0\varepsilon}(r_*)$ converges to $f_0(r_*)$ as $\varepsilon \to 0$, by assuming ε small enough, we have, for every $\delta > 0$,

$$\begin{aligned} f_{0\varepsilon}(r)(r-m) &\geq -\delta(r-m)^2 - c_{\delta,M} + 2M((r-m)^2 + (m-r_*)(r-m)) \\ &\geq -\delta(r-m)^2 - c_{\delta,M} + 2M(r-m)^2 - 2M\delta(r-m)^2 - 2Mc_{\delta,M} \end{aligned}$$

and the desired inequality follows for $r \geq r_*$ by choosing δ small enough. On the other hand, everything is bounded for $r \in [0, r_*]$. In order to prove the second (3.18), we apply the first one with $2M + |\lambda|$ in place of M and have

$$\begin{aligned} f_\varepsilon(r)(r-m) &\geq (2M+|\lambda|)(r-m)^2 - c(M) + \lambda((r-m)^2 + m(r-m)) \\ &\geq 2M(r-m)^2 - c(M) - |\lambda|\ |m|\ |r-m|. \end{aligned}$$

Then, we conclude in an obvious way. Now, let us come to inequalities (3.19). As $m \in (-1,1)$, we can fix $m_* \in (|m|, 1)$. We first show an estimate from below of the form

$$f_{0\varepsilon}(r)(r-m) \geq \alpha_1 F_{0\varepsilon}(r) - c. \tag{3.20}$$

We recall that $F_{0\varepsilon}$ is convex, $F_{0\varepsilon}(0) = 0$, and $f_{0\varepsilon} = F_{0\varepsilon}'$. Moreover, noting that $f_{0\varepsilon}(0) = 0$, since $f_0(0) = 0$, we have $f_{0\varepsilon}(r)(r-m) \geq 0$ if $|r| \geq m_*$. Therefore, if we choose $\alpha_1 \in (0,1)$ such that

$$\frac{r}{r-m} \leq \frac{1}{\alpha_1} \quad \text{for } |r| \geq m_*,$$

we have, for such values of r,

$$F_{0\varepsilon}(r) \leq r f_{0\varepsilon}(r) = \frac{r}{r-m} f_{0\varepsilon}(r)(r-m) \leq \frac{1}{\alpha_1} f_{0\varepsilon}(r)(r-m)$$

and (3.20) holds with any $c \geq 0$. Now, assume $|r| \leq m_*$. Then, we have

$$\alpha_1 F_{0\varepsilon}(r) - f_{0\varepsilon}(r)(r-m) \leq \alpha_1 F_0(r) + 2m_*|f_0(r)|.$$

As the right-hand side is bounded on $[-m_*, m_*]$, (3.20) is established. At this point, we can set $\alpha := \frac{\alpha_1}{2}$. It follows that (see (2.22)–(2.23))

$$\begin{aligned} f_\varepsilon(r)(r-m) &\geq \alpha F_{0\varepsilon}(r) - c + \Big(\frac{1}{2} f_{0\varepsilon}(r) + \lambda r\Big)(r-m) \\ &= \alpha(F_\varepsilon(r) + \|h\|_\infty |r|) - \alpha \|h\|_\infty |r| + \frac{1}{2} f_{0\varepsilon}(r)(r-m) + \Big(1 - \frac{\alpha}{2}\Big)\lambda r^2 - \lambda m r - c \\ &\geq \alpha(F_\varepsilon(r) + \|h\|_\infty |r|) + \frac{1}{2} f_{0\varepsilon}(r)(r-m) - c r^2 - c|r| - c \\ &\geq \alpha(F_\varepsilon(r) + \|h\|_\infty |r|) + \frac{1}{2} f_{0\varepsilon}(r)(r-m) - c(r-m)^2 - c \end{aligned}$$

for every $r \in \mathbb{R}$ and the first inequality (3.19) easily follows by applying the first (3.18). Finally, we prove the second (3.19). We argue, e.g., for $r \geq 0$. For $r \geq m_*$, we have

$$\begin{aligned} f_\varepsilon(r)(r-m) &= \frac{1}{2}|f_{0\varepsilon}(r)|(r-m) + \frac{1}{2} f_{0\varepsilon}(r)(r-m) + \lambda r(r-m) \\ &\geq \frac{1}{2}(|f_\varepsilon(r)| - |\lambda| r)(r-m) + \frac{1}{2} f_{0\varepsilon}(r)(r-m) - |\lambda| r(r-m) \\ &\geq \frac{m_* - m}{2}|f_\varepsilon(r)| + \frac{1}{2} f_{0\varepsilon}(r)(r-m) - c(r-m)^2 - c \end{aligned}$$

and the desired inequality follows with $\alpha = \frac{m_* - m}{2}$ by applying the first (3.18) with $M = 2c$. On the other hand, everything is uniformly bounded on $[0, m_*]$ and we conclude.

Now, we can start proving our formal estimates and we refer to inequalities (3.11)–(3.13) in doing that. On the other hand, it is clear that corresponding rigorous estimates could be performed on the approximating solution by using the previous lemmas.

First a priori estimate

Owing to (3.10), we formally choose $y = \mathcal{N}(\partial_t u + u - m)$ in (2.18). At the same time, we take $y = -(\partial_t u + u - m)$ and $z = -(\partial_t v + v - m)$ in (2.19). Then, we sum the equalities that we obtain. We have (at any positive time)

$$\begin{aligned} &\langle \partial_t u, \mathcal{N}(\partial_t u + u - m)\rangle + \int_\Omega \nabla w \cdot \nabla \mathcal{N}(\partial_t u + u - m) \mathrm{d}x \\ &- \int_\Omega w(\partial_t u + u - m)\mathrm{d}x + \int_\Omega \nabla u \cdot \nabla(\partial_t u + u - m)\mathrm{d}x + \int_\Omega (f(u) - h)(\partial_t u + u - m)\mathrm{d}x \\ &+ \int_\Gamma \partial_t v(\partial_t v + v - m)\mathrm{d}\sigma + \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma(\partial_t v + v - m)\mathrm{d}\sigma \\ &+ \int_\Gamma (f_\Gamma(v) + \lambda_\Gamma v - h_\Gamma)(\partial_t v + v - m)\mathrm{d}\sigma = 0. \end{aligned}$$

The third integral on the left-hand side cancels the second one, thanks to (3.4). Then, owing to the other properties of $\mathcal{N}$ just mentioned and adding $\|u - m\|_*^2$ to both sides for convenience, we easily deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u - m\|_*^2 + \|\nabla u\|_{L^2(\Omega)}^2 + 2\int_\Omega (F(u) - hu)\mathrm{d}x$$

$$
\begin{aligned}
&+ \|v-m\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \lambda_\Gamma \|v\|^2_{L^2(\Gamma)}\Big) \\
&+ \|\partial_t u\|^2_* + \|u-m\|^2_* + \|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega f(u)(u-m)\mathrm{d}x \\
&+ \|\partial_t v\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \lambda_\Gamma \int_\Gamma v(v-m)\mathrm{d}\sigma \\
= {}& \|u-m\|^2_* + \int_\Omega h(u-m)\mathrm{d}x + \int_\Gamma (h_\Gamma - f_\Gamma(v))(\partial_t v + v - m)\mathrm{d}\sigma.
\end{aligned}
$$

We only have to deal with terms without a definite sign. We recall that $F(u) - hu \geq 0$ by (2.8). So, as far as the terms on the left-hand side are concerned, we have

$$
\begin{aligned}
&\int_\Omega f(u)(u-m)\mathrm{d}x \\
&\geq \frac{\alpha}{3}\int_\Omega (F(u) + \|h\|_\infty |u|)\mathrm{d}x + \frac{\alpha}{3}\int_\Omega |f(u)|\mathrm{d}x + \frac{1}{3}\int_\Omega f(u)(u-m)\mathrm{d}x - c \\
&\geq \frac{\alpha}{3}\int_\Omega (F(u) - hu)\mathrm{d}x + \frac{\alpha}{3}\|f(u)\|_{L^1(\Omega)} + M\|u-m\|^2_{L^2(\Omega)} - c(M)
\end{aligned}
\tag{3.21}
$$

for every $M > 0$, thanks to (3.11)–(3.13). On the other hand, we trivially have

$$
\int_\Gamma v(v-m)\mathrm{d}\sigma = \frac{1}{2}\|v\|^2_{L^2(\Gamma)} + \frac{1}{2}\|v-m\|^2_{L^2(\Gamma)} - \frac{m^2}{2}|\Gamma|.
$$

The first two terms on the right-hand side can be treated as follows:

$$
\|u-m\|^2_* + \int_\Omega h(u-m)\mathrm{d}x \leq c\|u-m\|^2_{L^2(\Omega)} + c\|u-m\|_{L^2(\Omega)} \leq c\|u-m\|^2_{L^2(\Omega)} + c
$$

and can be compensated with (3.21) by choosing M large enough there. Finally, we have

$$
\int_\Gamma (h_\Gamma - f_\Gamma(v))(\partial_t v + v - m)\mathrm{d}\sigma \leq \delta\|\partial_t v\|^2_{L^2(\Gamma)} + \delta\|v-m\|^2_{L^2(\Gamma)} + c_\delta
$$

for every $\delta > 0$. Collecting all the inequalities that we have obtained and choosing δ small enough, we deduce that the following holds for suitable $\alpha_1 > 0$ and $C_1 > 0$:

$$
\frac{\mathrm{d}}{\mathrm{d}t}E + \alpha_1 E + \|f(u)\|_{L^1(\Omega)} + \|\partial_t u\|^2_* + \|\partial_t v\|^2_{L^2(\Gamma)} \leq C_1, \quad \text{a.e. in } (0,+\infty), \tag{3.22}
$$

where the (nonnegative) energy E is defined by

$$
\begin{aligned}
E := {}& \|u-m\|^2_* + \|\nabla u\|^2_{L^2(\Omega)} + 2\int_\Omega (F(u) - hu)\mathrm{d}x \\
&+ \|v-m\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \lambda_\Gamma\|v\|^2_{L^2(\Gamma)}.
\end{aligned}
\tag{3.23}
$$

We note that (the first (3.1) must be used)

$$
E \leq c(\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)} + \|F_0(u)\|_{L^1(\Omega)} + 1), \tag{3.24}
$$

$$
\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Gamma)} + \|F_0(u)\|_{L^1(\Omega)} \leq c(E+1) \tag{3.25}
$$

at any time $t \geq 0$. From (3.22), we deduce that

$$
E(t) \leq E(0)\mathrm{e}^{-\alpha_1 t} + C_2 \quad \text{for every } t \geq 0, \tag{3.26}
$$

where $C_2 := \frac{C_1}{\alpha_1}$.

Consequences

We now fix a bounded subset $\mathcal{B}$ of Φ_m and assume $(u_0, v_0) \in \mathcal{B}$. This corresponds to assuming (besides $\langle u_0 \rangle_\Omega = m$, of course)

$$\|u_0\|_{H^1(\Omega)} + \|v_0\|_{H^1(\Gamma)} + \|F_0(u_0)\|_{L^1(\Omega)} \le R, \tag{3.27}$$

where R is a fixed positive number. Then, $E(0) \le c(R)$ by (3.24). Hence, accounting for (3.25), we deduce that

$$\|u(t)\|_{H^1(\Omega)} + \|v(t)\|_{H^1(\Gamma)} + \|F_0(u(t))\|_{L^1(\Omega)} \le c(R)\mathrm{e}^{-\alpha_1 t} + C \quad \text{for every } t \ge 0, \tag{3.28}$$

$$\int_t^{t+1} (\|f(u)\|_{L^1(\Omega)} + \|\partial_t u\|_*^2 + \|\partial_t v\|_{L^2(\Gamma)}^2)\mathrm{d}\tau \le c(R)\mathrm{e}^{-\alpha_1 t} + C \quad \text{for every } t \ge 0. \tag{3.29}$$

We stress that, in the above relations (as well as in other ones of the same type below), C is not allowed to depend on R.

Estimate of $\boldsymbol{w}$

By (2.18) with $y = w(t)$ and (2.21), we deduce that

$$\int_\Omega |\nabla w|^2 \mathrm{d}x = -\langle \partial_t u, w - \langle w \rangle_\Omega \rangle \le c\|\partial_t u\|_*^2$$

for all $t \ge 0$. Then, by (3.29),

$$\int_t^{t+1} \|\nabla w\|_{L^2(\Omega)}^2 \mathrm{d}\tau \le c(R)\mathrm{e}^{-\alpha_1 t} + C \quad \text{for every } t \ge 0. \tag{3.30}$$

Second a priori estimate

We test (2.19) by $y = u - m$ and $z = v - m$. We have

$$\begin{aligned} &\|\nabla u\|_{L^2(\Omega)}^2 + \int_\Omega f(u)(u-m)\mathrm{d}x + \|\nabla_\Gamma v\|_{L^2(\Gamma)}^2 + \lambda_\Gamma \int_\Gamma v(v-m)\mathrm{d}\sigma \\ &= \int_\Omega (w+h)(u-m)\mathrm{d}x + \int_\Gamma (h_\Gamma - f_\Gamma - \partial_t v)(v-m)\mathrm{d}\sigma. \end{aligned} \tag{3.31}$$

We estimate the first integral from below by using (3.11) and (3.13) as follows:

$$\begin{aligned} \int_\Omega f(u)(u-m)\mathrm{d}x &= \frac{1}{2}\int_\Omega f(u)(u-m)\mathrm{d}x + \frac{1}{2}\int_\Omega f(u)(u-m)\mathrm{d}x \\ &\ge \alpha\|f(u)\|_{L^1(\Omega)} - c + \|u-m\|_{L^2(\Omega)}^2 - c. \end{aligned} \tag{3.32}$$

Moreover, we simply write $2v(v-m) = v^2 + (v-m)^2 - m^2$ in the last term on the left-hand side. As far as the right-hand side is concerned, we have

$$\begin{aligned} \int_\Omega (w+h)(u-m)\mathrm{d}x &= \int_\Omega (w - \langle w \rangle_\Omega + h)(u-m)\mathrm{d}x \\ &\le (\|w - \langle w \rangle_\Omega\|_{L^2(\Omega)} + \|h\|_{L^2(\Omega)})\|u-m\|_{L^2(\Omega)} \\ &\le c(\|\nabla w\|_{L^2(\Omega)} + 1)\|\nabla u\|_{L^2(\Omega)} \end{aligned} \tag{3.33}$$

thanks to the second (3.1). The boundary integral in (3.31) can be treated as follows:

$$\int_\Gamma (h_\Gamma - f_\Gamma - \partial_t v)(v-m)\mathrm{d}\sigma \le \|h_\Gamma - f_\Gamma - \partial_t v\|_{L^2(\Gamma)}\|v-m\|_{L^2(\Gamma)} \le c(\|\partial_t v\|_{L^2(\Gamma)}+1). \quad (3.34)$$

Thus, squaring (3.31) and using (3.32)–(3.34), we end up with (here and below, the value of α_1 may vary)

$$\begin{aligned} &\|\nabla u\|^4_{L^2(\Omega)} + \alpha\|f(u)\|^2_{L^1(\Omega)} + \|\nabla_\Gamma v\|^4_{L^2(\Gamma)} \\ &\le c(\|\nabla w\|^2_{L^2(\Omega)}+1)\|\nabla u\|^2_{L^2(\Omega)} + c(\|\partial_t v\|^2_{L^2(\Gamma)}+1) \\ &\le (\|\nabla w\|^2_{L^2(\Omega)}+1)(c(R)\mathrm{e}^{-\alpha_1 t}+C) + c(\|\partial_t v\|^2_{L^2(\Gamma)}+1), \end{aligned} \quad (3.35)$$

where (3.28) has been used to deduce the last inequality. Then, integrating over $(t,t+1)$ for a generic $t \ge 0$ and using (3.29) and (3.30), we find

$$\int_t^{t+1} \|f(u)\|^2_{L^1(\Omega)}\mathrm{d}\tau \le c(R)\mathrm{e}^{-\alpha_1 t} + C \quad \text{for all } t \ge 0. \quad (3.36)$$

By simply taking $y = \frac{1}{|\Omega|}$ and $z = \frac{1}{|\Omega|}$ in (2.19), we also deduce that

$$\langle w\rangle_\Omega = \langle f(u)\rangle_\Omega - \langle h\rangle_\Omega + \frac{1}{|\Omega|}\int_\Gamma (\partial_t v + f_\Gamma(v) + \lambda_\Gamma v - h_\Gamma)\mathrm{d}\sigma,$$

whence immediately an estimate for $|\langle w\rangle_\Omega|$, thanks to (3.36) and the bounds already proved. Using (3.30) and the second (3.1) once more, we conclude that

$$\int_t^{t+1} \|w\|^2_{H^1(\Omega)}\mathrm{d}\tau \le c(R)\mathrm{e}^{-\alpha_1 t} + C \quad \text{for all } t \ge 0. \quad (3.37)$$

Remark 3.2 We can give the outline of the proof of the existence of a solution, as stated in Theorem 2.1. As we deal with a fixed initial datum, we choose R satisfying equality in (3.27), or something connected with such a value, in order for the same bound to be satisfied by the approximating initial datum uniformly with respect to ε. By going through the proofs of our previous estimates, one clearly sees that bounds depending on R and some final time for several norms can be found. In particular, proceeding as in the proof of (3.26), one can see that

$$\begin{aligned} &\|u\|^2_{L^\infty(0,T;V)} + \|\partial_t u\|^2_{L^2(0,T;V^*)} + \|v\|^2_{L^\infty(0,T;V_\Gamma)} \\ &+ \|\partial_t v\|^2_{L^2(0,T;H_\Gamma)} + \|w\|^2_{L^2(0,T;V)} \le C(T,R) \end{aligned} \quad (3.38)$$

for every $T \in (0,+\infty)$, where $C(T,R)$ depends on R and on the final time T. Moreover, if we use the same procedure on the solution $(u_\varepsilon, v_\varepsilon, w_\varepsilon)$ to the approximating problem described in Remark 2.3, an estimate similar to (3.38) is found with a constant $C(R,T)$ which does not depend on ε. At this point, one can let ε tend to 0 and see that the weak limits given by weak compactness actually provide a solution to a suitable formulation of problem (2.15)–(2.20). However, the term $f(u)$ in the limit problem should be understood in a weaker sense (see, e.g., [19, 24] for more details), since no L^2-bound for it has been proved yet (the best that we have up to now is (3.36)).

In order to obtain a solution in a stronger sense, we have to find a direct L^2-estimate of $f(u)$ and just note here that a similar argument holds for the corresponding term $f_\varepsilon(u_\varepsilon)$ of the approximating problem. This will require the compatibility conditions (2.6)–(2.7) not yet used. We first give a lemma.

Lemma 3.3 *For every $n \in \mathbb{N}$, define $f_{0,n}, F_{0,n}, \psi_{0,n}, \Psi_{0,n} : (-1,1) \to \mathbb{R}$ by*

$$f_{0,n}(r) := \min\{n, \max\{-n, f_0(r)\}\}, \quad F_{0,n}(r) := \int_0^r f_{0,n}(s)\mathrm{d}s, \tag{3.39}$$

$$\psi_{0,n}(r) := \min\{n, \max\{-n, (F_0(r))^{\frac{1}{2}} \operatorname{sign} r\}\} \quad \text{and} \quad \Psi_{0,n}(r) := \int_0^r \psi_{0,n}(s)\mathrm{d}s \tag{3.40}$$

for $r \in (-1,1)$. Then, $f_{0,n}$ and $\psi_{0,n}$ are monotone and Lipschitz continuous. Moreover, constants $\alpha > 0$ and $C > 0$ exist such that

$$(f_\Gamma(r) + \lambda_\Gamma r - h_\Gamma(x))f_{0,n}(r) \geq \alpha(F_{0,n}(r) + |f_{0,n}(r)|) - C, \tag{3.41}$$

$$(f_\Gamma(r) + \lambda_\Gamma r - h_\Gamma(x))\psi_{0,n}(r) \geq -C \tag{3.42}$$

for every $r \in (-1,1)$, a.e. $x \in \Gamma$, and every $n \in \mathbb{N}$.

Proof Clearly, $f_{0,n}$ is monotone and Lipschitz continuous. As $f_{0,n}(0) = 0$, $F_{0,n}$ is a convex function with minimum at 0. Therefore, it is nonnegative, decreasing in $(-1,0)$, and increasing in $(0,1)$. It follows that $\psi_{0,n}$ is monotone. In order to show the Lipschitz continuity, it suffices to prove that the function $\psi : (-1,1) \to \mathbb{R}$ given by $\psi(r) = (F_0(r))^{\frac{1}{2}} \operatorname{sign} r$ is piecewise C^1. Clearly, only points where F_0 vanishes could lead to some trouble. We recall that F_0 is convex. Moreover, it is nonnegative everywhere and strictly positive near ± 1, since f_0 is unbounded there. As $F_0(0) = 0$, we can find $r_\pm \in (-1,1)$ such that $\pm r_\pm \geq 0$, $F_0(r) = 0$ for $r \in [r_-, r_+]$, and $F_0(r) > 0$ elsewhere. The following formulas hold:

$$\lim_{r \uparrow r_-} \psi'(r) = \Big(\frac{f_0'(r_-)}{2}\Big)^{\frac{1}{2}} \quad \text{and} \quad \lim_{r \downarrow r_+} \psi'(r) = \Big(\frac{f_0'(r_+)}{2}\Big)^{\frac{1}{2}} \tag{3.43}$$

and we prove, e.g., the second one. By using l'Hôpital's rule, we have

$$\lim_{r \downarrow r_+} (\psi'(r))^2 = \frac{1}{4} \lim_{r \downarrow r_+} \frac{f_0^2(r)}{F_0(r)} = \frac{1}{4} \lim_{r \downarrow r_+} \frac{2 f_0(r) f_0'(r)}{f_0(r)} = \frac{1}{2} f_0'(r_+)$$

and the desired formula follows. As ψ is continuous, we deduce that $\psi'_\pm(r_\pm) = (\frac{f_0'(r_\pm)}{2})^{\frac{1}{2}}$, where $\psi'_+(r)$ and $\psi'_-(r)$ denote the right and left derivatives. It follows that ψ'_+ is right-continuous at r_+ and that ψ'_- is left-continuous at r_-. Hence, if $r_\pm = 0$, ψ is piecewise C^1. On the other hand, the same conclusion holds if either $r_\pm$ is nonzero, since f_0 vanishes in $[r_-, r_+]$ in such a case. Let us come to (3.41)–(3.42). We only prove (3.41), since the proof of (3.42) is similar and easier. We set, for convenience, $\sigma(r) := f_\Gamma(r) + \lambda_\Gamma r - h_\Gamma$ for $r \in (-1,1)$, without stressing the dependence on x in the notation. We recall (2.6)–(2.7) and assume $r_0 \leq r < 1$ (a similar argument holds for $-1 < r \leq -r_0$). We notice that $F_{0,n}$ is convex, since $f_{0,n}$ is monotone. Then, we have

$$\sigma(r) f_{0,n}(r) \geq \eta f_{0,n}(r) \geq \frac{\eta}{2} r f_{0,n}(r) + \frac{\eta}{2}|f_{0,n}(r)| \geq \frac{\eta}{2} F_{0,n}(r) + \frac{\eta}{2}|f_{0,n}(r)|$$

and (3.41) holds with $\alpha = \frac{\eta}{2}$ and any $C \geq 0$. Now, assume $|r| < r_0$. By recalling that $h_\Gamma \in L^\infty(\Gamma)$, we have

$$|\sigma(r) f_{0,n}(r)| + \Big(\frac{\eta}{2}\Big)(F_{0,n}(r) + |f_{0,n}(r)|) \leq |\sigma(r) f_0(r)| + \Big(\frac{\eta}{2}\Big)(F_0(r) + |f_0(r)|) \leq c.$$

Therefore, $(\frac{\eta}{2})(F_{0,n}(r)+|f_{0,n}(r)|)-\sigma(r)f_{0,n}(r)\le c$ and we conclude.

Third a priori estimate

Our aim is to estimate $F_0(v)$ in $L^1(\Gamma)$. A bound is obviously obtained by using our boundedness assumption (2.25) on F_0. Indeed, we immediately have

$$F_0(v(t))\in L^\infty(\Gamma) \quad \text{and} \quad \|F_0(v(t))\|_{L^\infty(\Gamma)}\le|\Gamma|\sup_{|r|<1}|F_0(r)|=c \quad \text{for } t\ge 0. \tag{3.44}$$

We point out that this is the first time that we account for (2.25). However, as we would like to extend our result to functionals which violate such an assumption (as explained in the forthcoming Remark 3.3), we do not use (3.44) and argue in a more complicated way. We test (2.19) by $y=f_{0,n}(u)$ and $z=f_{0,n}(v)$ (see (3.39)) and notice that such an estimate is rigorous, since $f_{0,n}$ is a Lipschitz continuous function. We obtain

$$\begin{aligned}&\int_\Omega f_{0,n}'(u)|\nabla u|^2\mathrm{d}x+\int_\Omega f_0(u)f_{0,n}(u)\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma F_{0,n}(v)\mathrm{d}\sigma\\&\quad+\int_\Gamma f_{0,n}'(v)|\nabla_\Gamma v|^2\mathrm{d}\sigma+\int_\Gamma(f_\Gamma(v)+\lambda_\Gamma v-h_\Gamma)f_{0,n}(v)\mathrm{d}\sigma\\&=\int_\Omega(h+w-\lambda u)f_{0,n}(u)\mathrm{d}x\le\frac{1}{2}\|f_{0,n}(u)\|^2_{L^2(\Omega)}+\frac{1}{2}\|h+w-\lambda u\|^2_{L^2(\Omega)}.\end{aligned} \tag{3.45}$$

On the other hand, the second integral on the left-hand side bounds $\|f_{0,n}(u)\|^2_{L^2(\Omega)}$ from above. So, if we treat the last boundary integral on the left-hand side accounting for (3.41) as follows:

$$\int_\Gamma(f_\Gamma(v)+\lambda_\Gamma v-h_\Gamma)f_{0,n}(v)\mathrm{d}\sigma\ge\alpha\int_\Gamma(F_{0,n}(v)+|f_{0,n}(v)|)\mathrm{d}\sigma-c,$$

we deduce from (3.45) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma F_{0,n}(v)\mathrm{d}\sigma+\alpha_2(\|F_{0,n}(v)\|_{L^1(\Gamma)}+\|f_{0,n}(v)\|_{L^1(\Gamma)}+\|f_{0,n}(u)\|^2_{L^2(\Omega)})\\&\le C_3(\|u\|^2_{L^2(\Omega)}+\|w\|^2_{L^2(\Omega)}+1) \quad \text{for } t\ge 0.\end{aligned} \tag{3.46}$$

Such an inequality provides an estimate of $F_{0,n}(v(t))$, thus of $F_0(v(t))$ by letting $n\to+\infty$, in $L^1(\Gamma)$ at some time t, provided that we have an estimate of the same norm at some earlier time. So, let us start by assuming

$$F_0(v(t_*))\in L^1(\Gamma) \quad \text{and} \quad \|F_0(v(t_*))\|_{L^1(\Gamma)}\le R_* \quad \text{for some } t_*\ge 0 \tag{3.47}$$

and control how the next estimates depend on R_* in what follows. Estimating the right-hand side of (3.46) by means of (3.28) and (3.37), we end up with a differential inequality of the form

$$Y'+\alpha_2(Y+\|f_{0,n}(u)\|^2_{L^2(\Omega)})\le M+c \quad \text{for all } t\ge 0 \tag{3.48}$$

where $Y(t)=\|F_{0,n}(v(t))\|_{L^1(\Gamma)}$ and

$$\int_t^{t+1}M(\tau)\mathrm{d}\tau\le c(R)\mathrm{e}^{-\alpha_1 t}+C \quad \text{for all } t\ge 0. \tag{3.49}$$

Integrating (3.48) and then letting n tend to $+\infty$, it is not difficult to see that there exists $\alpha_3 > 0$ such that

$$\|F_0(v(t))\|_{L^1(\Gamma)} \leq c(R_*)\mathrm{e}^{-\alpha_3(t-t_*)} + c(R)\mathrm{e}^{-\alpha_3 t} + C \quad \text{for all } t \geq t_*. \tag{3.50}$$

Once such an estimate is established, by integrating (3.46) over $(t, t+1)$ for $t \geq t_*$, we also find

$$\int_t^{t+1} (\|f_0(v)\|_{L^1(\Gamma)} + \|f_0(u)\|^2_{L^2(\Omega)})\mathrm{d}\tau \leq c(R_*)\mathrm{e}^{-\alpha_3(t-t_*)} + c(R)\mathrm{e}^{-\alpha_3 t} + C. \tag{3.51}$$

Remark 3.3 It is absolutely obvious that the dependence on R_* of the constants that we found can be replaced by a dependence on R, only, whenever R_* can be estimated in terms of R. The simplest case is given by assuming that F_0 is bounded. In such a situation, we do not have any dependence on R at all. Here, we want to show how (3.47) can be verified for some R_* and t_* depending on R in some cases of an unbounded F_0, namely, for a functional f_0 such that

$$f_0(r) \sim \pm|r \mp 1|^{-\gamma}, \qquad \text{as } r \to \pm 1 \tag{3.52}$$

with $\gamma \geq 1$. However, we proceed very formally.

As a first step, let us define, for every positive integer i,

$$\varphi_i(r) := \frac{1}{(1-r)^{\frac{2i-1}{2}}} - \frac{1}{(1+r)^{\frac{2i-1}{2}}}, \quad \Phi_i(r) := \int_0^r \varphi_i(s)\mathrm{d}s, \quad r \in (-1,1). \tag{3.53}$$

Notice that φ_i is smooth and monotone. Moreover, $\varphi_i(0) = \Phi_i(0) = 0$ for all $i \in \mathbb{N}$. Observe also that Φ_1 is bounded. Then, take $y = \varphi_1(u)$ and $z = \varphi_1(v)$ in (2.19). This leads to

$$\begin{aligned} &\int_\Omega \varphi_1'(u)|\nabla u|^2\mathrm{d}x + \int_\Omega f_0(u)\varphi_1(u)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \Phi_1(v)\mathrm{d}\sigma + \int_\Gamma \varphi_1'(v)|\nabla_\Gamma v|^2\mathrm{d}\sigma \\ &= \int_\Omega (w - \lambda u + h)\varphi_1(u)\mathrm{d}x + \int_\Gamma (h_\Gamma - f_\Gamma(v) - \lambda_\Gamma v)\varphi_1(v)\mathrm{d}\sigma, \end{aligned} \tag{3.54}$$

and we have to estimate several terms. First, we notice that

$$\int_\Omega (w - \lambda u + h)\varphi_1(u)\mathrm{d}x \leq \|\varphi_1(u)\|^2_{L^2(\Omega)} + c(\|w\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + 1). \tag{3.55}$$

Now, (3.52) and (3.53) (with $i = 1$) entail that φ_1 has a growth rate as $|r| \to 1$ which is strictly slower than the one of f_0. In particular, this entails

$$\|\varphi_1(u)\|^2_{L^2(\Omega)} \leq \frac{1}{2}\int_\Omega f_0(u)\varphi_1(u)\mathrm{d}x + c. \tag{3.56}$$

Moreover, again by (3.52) and (3.53), there exists $p_1 \in (1,2)$ depending on γ such that

$$\frac{1}{2}\int_\Omega f_0(u)\varphi_1(u)\mathrm{d}x \geq \alpha\|f_0(u)\|^{p_1}_{L^{p_1}(\Omega)} - c. \tag{3.57}$$

Next, regarding the boundary term in (3.54), relations (2.6)–(2.7) entail

$$\int_\Gamma [h_\Gamma - f_\Gamma(v) - \lambda_\Gamma v]\varphi_1(v)\mathrm{d}\sigma \leq c. \tag{3.58}$$

Indeed, the quantity in square brackets is strictly positive for $v \sim 1$ and strictly negative for $v \sim -1$, while φ_1 is locally bounded in $(-1,1)$.

Collecting (3.55)–(3.58) and recalling estimates (3.26) and (3.37), we then find that, for any $T > 0$, there exists a constant $C(T,R)$ (where, as in (3.27), R is a measure of the "energy" of the initial datum, i.e., of their "magnitude" w.r.t. the metric d (see (2.12)) such that

$$\int_0^T \varphi_1'(u)|\nabla u|^2 \mathrm{d}t + \int_0^T f_0(u)\varphi_1(u)\mathrm{d}t + \int_0^T \|f_0(u)\|_{L^{p_1}(\Omega)}^{p_1} \mathrm{d}t \le C(T,R). \tag{3.59}$$

In particular, we can take $T = 1$ and set, correspondingly, $C_1(R) := C(1,R)$.

Now, we aim to proceed by (finitely many) iteration steps. More precisely, let us assume that the following analogue of (3.59) holds:

$$\begin{aligned} &\int_{T_{i-1}}^{T_{i-1}+1} \int_\Omega \varphi_{i-1}'(u)|\nabla u|^2 \mathrm{d}x\mathrm{d}t + \int_{T_{i-1}}^{T_{i-1}+1} \int_\Omega f_0(u)\varphi_{i-1}(u)\mathrm{d}x\mathrm{d}t \\ &\quad + \int_{T_{i-1}}^{T_{i-1}+1} \|f_0(u)\|_{L^{p_{i-1}}(\Omega)}^{p_{i-1}} \mathrm{d}t \le C_{i-1}(R), \end{aligned} \tag{3.60}$$

where T_{i-1} is a suitable "initial" time belonging to the interval $(0, i-1)$ and $p_{i-1} \in (1,2)$.

Then, as a consequence, we can find $T_i \in (T_{i-1}, T_{i-1}+1)$ (so that, in particular, $T_i \in (0,i)$) such that

$$\int_\Omega \varphi_{i-1}'(u(T_i))|\nabla u(T_i)|^2 \mathrm{d}x + \int_\Omega f_0(u(T_i))\varphi_{i-1}(u(T_i))\mathrm{d}x \le C_{i-1}(R). \tag{3.61}$$

Let us now observe that Gagliardo's trace theorem in $W^{1,1}(\Omega)$ (see, e.g., [16]) entails

$$\begin{aligned} &\|\varphi_{i-1}(v(T_i))\|_{L^1(\Gamma)} \\ &\le c_\Omega \int_\Omega |\varphi_{i-1}(u(T_i))|\mathrm{d}x + c_\Omega \int_\Omega \varphi_{i-1}'(u(T_i))|\nabla u(T_i)|\mathrm{d}x \\ &\le c_\Omega \int_\Omega |\varphi_{i-1}(u(T_i))|\mathrm{d}x + c\int_\Omega \varphi_{i-1}'(u(T_i))\mathrm{d}x + c\int_\Omega \varphi_{i-1}'(u(T_i))|\nabla u(T_i)|^2 \mathrm{d}x \\ &\le c\int_\Omega f_0(u(T_i))\varphi_{i-1}(u(T_i))\mathrm{d}x + c + c\int_\Omega \varphi_{i-1}'(u(T_i))|\nabla u(T_i)|^2 \mathrm{d}x \\ &\le cC_{i-1}(R) + c, \end{aligned} \tag{3.62}$$

where we have also used (3.61) and the obvious fact that both $\varphi_{i-1}(r)$ and $\varphi_{i-1}'(r)$ do not grow faster than $f_0(r)\varphi_{i-1}(r)$ as $|r| \sim 1$ (see (3.52) and note that $\gamma \ge 1$).

Formula (3.62) allows to go on with the next iteration step. More precisely, we would like to take $y = \varphi_i(u)$ and $z = \varphi_i(v)$ in (2.19). However, we can do this only as long as φ_i grows slower than f_0. In case φ_i grows as fast as f_0 or faster (namely, if γ in (3.52) is less than or equal to $\frac{2i-1}{2}$ in (3.53)), then the iteration argument stops. That said, we notice that (3.54) obviously is modified as

$$\begin{aligned} &\int_\Omega \varphi_i'(u)|\nabla u|^2 \mathrm{d}x + \int_\Omega f_0(u)\varphi_i(u)\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \Phi_i(v)\mathrm{d}\sigma + \int_\Gamma \varphi_i'(v)|\nabla_\Gamma v|^2 \mathrm{d}\sigma \\ &= \int_\Omega (w - \lambda u + h)\varphi_i(u)\mathrm{d}x + \int_\Gamma (h_\Gamma - f_\Gamma(v) - \lambda_\Gamma v)\varphi_i(v)\mathrm{d}\sigma \end{aligned} \tag{3.63}$$

and estimates (3.56)–(3.58) can be repeated with obvious modifications. Notice that, since $\frac{2i-1}{2} < \gamma$, we still have $p_i < 2$ in the analogue of (3.57). Thus, we can integrate (3.63) over the interval $(T_i, T_i + 1)$. Noticing that

$$\|\Phi_i(v(T_i))\|_{L^1(\Gamma)} \le c(1 + \|\varphi_{i-1}(v(T_i))\|_{L^1(\Gamma)}) \le c(1 + C_{i-1}(R)) \tag{3.64}$$

by (3.53) and (3.62), we then obtain the i-analogue of (3.60), namely,

$$\begin{aligned} &\int_{T_i}^{T_i+1}\int_\Omega \varphi_i'(u)|\nabla u|^2 \mathrm{d}x\mathrm{d}t + \int_{T_i}^{T_i+1}\int_\Omega f_0(u)\varphi_i(u)\mathrm{d}x\mathrm{d}t \\ &+ \int_{T_i}^{T_i+1} \|f_0(u)\|^{p_i}_{L^{p_i}(\Omega)}\mathrm{d}t \le C_i(R). \end{aligned} \tag{3.65}$$

Finally, let us assume that we have reached the end of the iteration, i.e., we have (3.65), where i is such that

$$\frac{2i-1}{2} < \gamma \le \frac{2(i+1)-1}{2}. \tag{3.66}$$

By (3.65), we see that $t_* \in (T_i, T_i + 1)$ exists such that

$$\int_\Omega \varphi_i'(u(t_*))|\nabla u(t_*)|^2 \mathrm{d}x + \int_\Omega f_0(u(t_*))\varphi_i(u(t_*))\mathrm{d}x \le C_i(R). \tag{3.67}$$

Hence, proceeding as in (3.62) and finally using (3.67), we have

$$\begin{aligned} \|F_0(v(t_*))\|_{L^1(\Gamma)} &\le c_\Omega \int_\Omega |F_0(u(t_*))|\mathrm{d}x + c_\Omega \int_\Omega f_0(u(t_*))|\nabla u(t_*)|\mathrm{d}x \\ &\le c_\Omega \int_\Omega |F_0(u(t_*))|\mathrm{d}x + c\int_\Omega f_0(u(t_*))\mathrm{d}x + c\int_\Omega f_0(u(t_*))|\nabla u(t_*)|^2\mathrm{d}x \\ &\le c\int_\Omega f_0(u(t_*))\varphi_i(u(t_*))\mathrm{d}x + c\int_\Omega \varphi_i'(u(t_*))|\nabla u(t_*)|^2\mathrm{d}x + c \\ &\le cC_i(R) + c, \end{aligned} \tag{3.68}$$

where the second last inequality follows from (3.52)–(3.53) and (3.66). Then, if $R_* = C(R)$ denotes the last constant of (3.68), we obtain (3.47), as desired.

Remark 3.4 As detailed at the beginning, the above argument is just formal, due to insufficient regularity of test functions. Nevertheless, to make it rigorous, it would be sufficient to proceed as in the proof of Lemma 3.3. More precisely, one should use at each i-step suitable truncations $\varphi_{i,n}$ of the functions φ_i. Then, due to the Lipschitz continuity of $\varphi_{i,n}$, $\varphi_{i,n}(u)$ and $\varphi_{i,n}(v)$ would be admissible test functions and could be used in place of $\varphi_i(u)$ and $\varphi_i(v)$. Finally, to perform the iteration argument rigorously, one should pass to the limit $n \nearrow +\infty$ before performing the subsequent $(i+1)$-step.

Therefore, by accounting either for (2.25) or for the previous remark, we can replace (3.50) and (3.51) by the estimate

$$\begin{aligned} &\|F_0(v(t))\|_{L^1(\Gamma)} + \int_t^{t+1} (\|f_0(v)\|_{L^1(\Gamma)} + \|f_0(u)\|^2_{L^2(\Omega)})\mathrm{d}\tau \\ &\le c(R)\mathrm{e}^{-\alpha_4 t} + C \quad \text{for all } t \ge t_* \end{aligned} \tag{3.69}$$

for a suitable $\alpha_4 > 0$. We point out that t_* is independent of the "magnitude" R of the initial data and, in fact, it only depends on the exponent γ in (3.52).

Remark 3.5 Thanks to (3.69), one could at this point improve the existence result sketched in Remark 3.2. Indeed, one could now prove the existence of a global weak solution satisfying, in addition, $f_0(u) \in L^2(0,T;H)$. We stress once more that, in particular, this regularity relies on the compatibility assumption (2.6)–(2.7).

Our next aim is to prove parabolic regularization properties of the solution.

Fourth a priori estimate

We set

$$\varphi := \partial_t u, \quad \psi := \partial_t w \quad \text{and} \quad \vartheta := \partial_t v \tag{3.70}$$

and formally differentiate (2.18)–(2.19) with respect to time. We obtain

$$\langle \partial_t \varphi, y \rangle + \int_\Omega \nabla\psi \cdot \nabla y \mathrm{d}x = 0, \tag{3.71}$$

$$\begin{aligned} \int_\Omega \psi y \mathrm{d}x &= \int_\Omega \nabla\varphi \cdot \nabla y \mathrm{d}x + \int_\Omega f'(u)\varphi y \mathrm{d}x + \int_\Gamma \partial_t \vartheta z \mathrm{d}\sigma \\ &\quad + \int_\Gamma \nabla_\Gamma \vartheta \cdot \nabla_\Gamma z \mathrm{d}\sigma + \int_\Gamma (f'_\Gamma(v)\vartheta + \lambda_\Gamma \vartheta) z \mathrm{d}\sigma \end{aligned} \tag{3.72}$$

at any time $t > 0$, where (3.71) and (3.72) hold for every $y \in V$ and every $(y,z) \in \mathcal{V}$, respectively. Now, we formally choose $y = \mathcal{N}\varphi$ in the former and $(y,z) = (-\varphi, -\vartheta)$ in the latter and sum the equalities that we obtain. We have

$$\begin{aligned} &\langle \partial_t \varphi, \mathcal{N}\varphi \rangle + \int_\Omega \nabla\psi \cdot \nabla\mathcal{N}\varphi \mathrm{d}x - \int_\Omega \psi\varphi \mathrm{d}x + \int_\Omega |\nabla\varphi|^2 \mathrm{d}x + \int_\Omega f'(u)\varphi^2 \mathrm{d}x \\ &+ \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Gamma \vartheta^2 \mathrm{d}\sigma + \int_\Gamma |\nabla_\Gamma \vartheta|^2 \mathrm{d}\sigma + \int_\Gamma (f'_\Gamma(v) + \lambda_\Gamma)\vartheta^2 \mathrm{d}\sigma = 0. \end{aligned}$$

Owing to (3.9) and (3.4), we deduce

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2) + \|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\nabla_\Gamma \vartheta\|_{L^2(\Gamma)}^2 \\ &= -\int_\Gamma (f'_\Gamma(v) + \lambda_\Gamma)\vartheta^2 \mathrm{d}\sigma - \int_\Omega f'(u)\varphi^2 \mathrm{d}x \le c\|\vartheta\|_{L^2(\Gamma)}^2 - \int_\Omega f'(u)\varphi^2 \mathrm{d}x, \end{aligned} \tag{3.73}$$

the last inequality is due to (2.3). We easily estimate the last term as follows (when dealing with the approximating problem, Lemma 3.1 should be used here). Owing to (2.2), we find $r_* \in (0,1)$ such that $f'(r) \ge 0$ for $|r| \ge r_*$. Then, if we denote by Ω_* the (time dependent) set on which $|u| < r_*$, we have

$$-\int_\Omega f'(u)\varphi^2 \mathrm{d}x \le -\int_{\Omega_*} f'(u)\varphi^2 \mathrm{d}x \le \sup_{|r|\le r_*} |f'(r)| \int_{\Omega_*} \varphi^2 \mathrm{d}x \le c\|\varphi\|_{L^2(\Omega)}^2.$$

On the other hand, (3.2) yields, for every $\delta > 0$,

$$\|\varphi\|_{L^2(\Omega)}^2 \le \delta\|\nabla\varphi\|_{L^2(\Omega)}^2 + c_\delta\|\varphi\|_*^2. \tag{3.74}$$

Collecting (3.73) and the last inequalities and choosing δ small enough, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2) + \|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\nabla_\Gamma\vartheta\|_{L^2(\Gamma)}^2 \le c(\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2).$$

In particular, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2) \le c(\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2)$$

and we are allowed to apply the uniform Gronwall lemma (see, e.g., [29, Lemma I.1.1]) in view of (3.29). Therefore, we have the existence of a time t_1 depending on R and of a constant c which is independent of R such that

$$\|\varphi\|_*^2 + \|\vartheta\|_{L^2(\Gamma)}^2 \le c \quad \text{for } t \ge t_1(R). \tag{3.75}$$

By integrating (3.73) over $(t, t+1)$, we also have

$$\int_t^{t+1} (\|\nabla\varphi\|_{L^2(\Omega)}^2 + \|\nabla_\Gamma\vartheta\|_{L^2(\Gamma)}^2)\mathrm{d}\tau \le c \quad \text{for } t \ge t_1(R).$$

By accounting for (3.29) and (3.74) once more, we find a similar estimate for the full norms of φ and ϑ in $H^1(\Omega)$ and $H^1(\Gamma)$, respectively. By adding (3.75) to this, we conclude that

$$\|\partial_t u(t)\|_*^2 + \|\partial_t v(t)\|_{L^2(\Gamma)}^2 + \int_t^{t+1} (\|\partial_t u\|_{H^1(\Omega)}^2 + \|\partial_t v\|_{H^1(\Gamma)}^2)\mathrm{d}\tau \le C_4 \quad \text{for } t \ge t_1(R), \tag{3.76}$$

where C_4 does not depend on R.

Finally, using the improved regularity of time derivatives following from (3.76) and coming back to the second estimate, it is not difficult to see that (3.37) can be improved up to

$$\|w(t)\|_{H^1(\Omega)}^2 \le C \quad \text{for all } t \ge t_1(R). \tag{3.77}$$

Consequences

The solution (u, v, w) satisfies the elliptic problem

$$-\Delta u = h_1 := w - f(u) + h, \quad \text{in } \Omega, \tag{3.78}$$

$$-\Delta_\Gamma v = h_2 := h_\Gamma - f_\Gamma(v) - \lambda_\Gamma v - \partial_t v - \partial_n u|_\Gamma, \quad \text{on } \Gamma, \tag{3.79}$$

in a generalized sense, in principle, where t is just seen as a parameter (and does not appear in the notation). To make the meaning of (3.78)–(3.79) precise, we use a bootstrap argument. First, (3.78) surely holds in the sense of distributions (take $y \in C_c^\infty(\Omega)$ in (2.19)) and a comparison shows that Δu is a function in $L^2(\Omega)$ (rather than a distribution) at every time. As $v = u|_\Gamma \in H^1(\Gamma)$, we deduce that $u \in H^{\frac{3}{2}}(\Omega)$ (by the elliptic theory in Ω) and that $(\partial_n u)|_\Gamma$ makes sense and belongs to $H^{-\frac{1}{4}}(\Gamma)$ (actually, it belongs to $H^s(\Gamma)$ for every $s < 0$). In particular, (3.79) has a precise meaning and $h_2 \in H^{-\frac{1}{4}}(\Gamma)$. By applying the boundary version of [21, Theorem 7.5, p. 204], we deduce that $v \in H^{2-\frac{1}{4}}(\Gamma) \subset H^{\frac{3}{2}}(\Gamma)$, whence also $u \in H^2(\Omega)$ by the elliptic theory in Ω once more. Thus, we can improve the regularity of $(\partial_n u)|_\Gamma$. More precisely, $(\partial_n u)|_\Gamma \in H^{\frac{1}{2}}(\Gamma) \subset L^2(\Gamma)$. As $\partial_t v \in L^2(\Gamma)$, we infer that $h_2 \in L^2(\Gamma)$, i.e., $\Delta_\Gamma v \in L^2(\Gamma)$. Therefore, we conclude that $v \in H^2(\Gamma)$ by the regularity theory on the boundary just mentioned.

Moreover, each step of the above reasoning is complemented by a corresponding estimate. So, we have, at any time,

$$\begin{aligned}
&\|u\|_{H^2(\Omega)} + \|v\|_{H^2(\Gamma)} \\
&\leq c(\|u\|_{H^2(\Omega)} + \|\Delta_\Gamma v\|_{L^2(\Gamma)} + \|v\|_{L^2(\Gamma)}) \\
&\leq c(\|u\|_{H^2(\Omega)} + \|v\|_{L^2(\Gamma)} + \|\partial_t v\|_{L^2(\Gamma)} + \|\partial_n u|_\Gamma\|_{L^2(\Gamma)} + 1) \\
&\leq c(\|u\|_{H^2(\Omega)} + \|v\|_{L^2(\Gamma)} + \|\partial_t v\|_{L^2(\Gamma)} + 1) \\
&\leq c(\|\Delta u\|_{L^2(\Omega)} + \|v\|_{H^{\frac{3}{2}}(\Gamma)} + \|\partial_t v\|_{L^2(\Gamma)} + 1) \\
&\leq c(\|\Delta u\|_{L^2(\Omega)} + \|\Delta_\Gamma v\|_{H^{-\frac{1}{4}}(\Gamma)} + \|v\|_{L^2(\Gamma)} + \|\partial_t v\|_{L^2(\Gamma)} + 1) \\
&\leq c(\|\Delta u\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma)} + \|\partial_t v\|_{L^2(\Gamma)} + \|\partial_n u|_\Gamma\|_{H^{-\frac{1}{4}}(\Gamma)} + 1) \\
&\leq c(\|\Delta u\|_{L^2(\Omega)} + \|v\|_{L^2(\Gamma)} + \|\partial_t v\|_{L^2(\Omega)} + \|u\|_{H^{\frac{3}{2}}(\Gamma)} + 1) \\
&\leq c(\|\Delta u\|_{L^2(\Omega)} + \|v\|_{H^1(\Gamma)} + \|\partial_t v\|_{L^2(\Omega)} + 1) \\
&\leq c(\|w\|_{L^2(\Omega)} + \|f(u)\|_{L^2(\Omega)} + \|v\|_{H^1(\Gamma)} + \|\partial_t v\|_{L^2(\Omega)} + 1).
\end{aligned} \tag{3.80}$$

By combining this with the previous estimates, we deduce that

$$\int_t^{t+1} (\|u\|^2_{H^2(\Omega)} + \|v\|^2_{H^2(\Gamma)}) \mathrm{d}\tau \leq c \quad \text{for } t \geq t_1(R). \tag{3.81}$$

On the other hand, (3.76) holds and the classical interpolation theory yields

$$\begin{aligned}
&\|y\|_{C^0(I;H^{\frac{3}{2}}(\Omega))} \leq c_I(\|y\|_{L^2(I;H^2(\Omega))} + \|\partial_t y\|_{L^2(I;H^1(\Omega))}), \\
&\|z\|_{C^0(I;H^{\frac{3}{2}}(\Gamma))} \leq c_I(\|z\|_{L^2(I;H^2(\Gamma))} + \|\partial_t z\|_{L^2(I;H^1(\Gamma))})
\end{aligned}$$

for every y and z belonging to the spaces related to the corresponding right-hand sides and where I is an arbitrary compact interval. As c_I depends on I just through its length, we conclude that

$$\|u(t)\|_{H^{\frac{3}{2}}(\Omega)} + \|v(t)\|_{H^{\frac{3}{2}}(\Gamma)} \leq C_5 \quad \text{for } t \geq t_1(R), \tag{3.82}$$

that is, $(u(t), v(t))$ belongs to a fixed compact subset of $H^1(\Omega) \times H^1(\Gamma)$ for t large enough.

Fifth a priori estimate

We recall (3.40) and choose $y = \psi_{0,n}(u)$ and $z = \psi_{0,n}(v)$ in (2.19). We obtain

$$\begin{aligned}
&\int_\Omega \psi'_{0,n}(u)|\nabla u|^2 \mathrm{d}x + \int_\Omega |f_0(u)|\ |F_{0,n}(u)|^{\frac{1}{2}} \mathrm{d}x + \int_\Gamma \psi'_{0,n}(v)|\nabla_\Gamma v|^2 \mathrm{d}\sigma \\
&\quad + \int_\Gamma (f_\Gamma(v) + \lambda_\Gamma v - h_\Gamma)\psi_{0,n}(v) \mathrm{d}\sigma \\
&= \int_\Omega (w + h - \lambda u)\psi_{0,n}(u) \mathrm{d}x - \int_\Gamma \partial_t v \psi_{0,n}(v) \mathrm{d}\sigma.
\end{aligned}$$

Now, we observe that

$$|f_0(r)| \geq |f_{0,n}(r)| \geq r f_{0,n}(r) \geq F_{0,n}(r) \quad \text{and} \quad |\psi_{0,n}(r)| \leq (F_0(r))^{\frac{1}{2}} \quad \text{for } r \in (-1,1),$$

the former since $\operatorname{sign} f_{0,n}(r) = \operatorname{sign} r$ and $F_{0,n}$ is convex. Thus, by owing first to (3.42) and then to (3.28), we deduce that

$$\begin{aligned}
&\int_\Omega |F_{0,n}(u)|^{\frac{3}{2}}\mathrm{d}x \\
&\le c(\|w\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|(F_0(u))^{\frac{1}{2}}\|^2_{L^2(\Omega)} + \|\partial_t v\|^2_{L^2(\Gamma)} + \|(F_0(v))^{\frac{1}{2}}\|^2_{L^2(\Gamma)} + 1) \\
&\le c(\|w\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|F_0(u)\|_{L^1(\Omega)} + \|\partial_t v\|^2_{L^2(\Gamma)} + \|F_0(v)\|_{L^1(\Gamma)} + 1).
\end{aligned}$$

Then, letting n tend to infinity, it is not difficult to obtain

$$\int_\Omega |F_0(u)|^{\frac{3}{2}}\mathrm{d}x \le c(\|w\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} + \|\partial_t v\|^2_{L^2(\Gamma)} + \|F_0(v)\|_{L^1(\Gamma)} + 1).$$

By accounting for (3.77), (3.28), (3.75) (with the notation (3.70)), and (3.69), we conclude that

$$\int_\Omega |F_0(u)|^{\frac{3}{2}}\mathrm{d}x \le C_6 \quad \text{for } t \ge t_1(R). \tag{3.83}$$

Sixth a priori estimate

Such an estimate will be used to prove Theorem 2.3. We choose $y = \mathcal{N}\partial_t u$ in (2.18) and $(y, z) = (-\partial_t u, -\partial_t v)$ in (2.19). Then, we sum the equalities that we obtain. We have (at any positive time)

$$\begin{aligned}
&\langle \partial_t u, \mathcal{N}\partial_t u\rangle + \int_\Omega \nabla w \cdot \nabla \mathcal{N}\partial_t u \mathrm{d}x - \int_\Omega w\partial_t u \mathrm{d}x + \int_\Omega \nabla u \cdot \nabla \partial_t u \mathrm{d}x + \int_\Omega (f(u) - h)\partial_t u \mathrm{d}x \\
&+ \int_\Gamma \partial_t v \partial_t v \mathrm{d}\sigma + \int_\Gamma \nabla_\Gamma v \cdot \nabla_\Gamma \partial_t v \mathrm{d}\sigma + \int_\Gamma (f_\Gamma(v) + \lambda_\Gamma v - h_\Gamma)\partial_t v \mathrm{d}\sigma = 0.
\end{aligned}$$

By arguing as we did for our first a priori estimate (here, it is even simpler), we find

$$\begin{aligned}
&\|\partial_t u\|^2_* + \|\partial_t v\|^2_{L^2(\Gamma)} + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|\nabla u\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + 2\int_\Omega (F(u) - hu)\mathrm{d}x + \lambda_\Gamma \|v\|^2_{L^2(\Gamma)}\Big) \\
&= \int_\Gamma (h_\Gamma - f_\Gamma(v))\partial_t v \mathrm{d}\sigma.
\end{aligned}$$

Now, we integrate over $(0, t)$, where $t > 0$ is arbitrary. By forgetting some positive terms on the left-hand side, we obtain

$$\begin{aligned}
&\int_0^t (\|\partial_t u\|^2_* + \|\partial_t v\|^2_{L^2(\Gamma)})\mathrm{d}\tau + \frac{\lambda_\Gamma}{2}\|v(t)\|^2_{L^2(\Gamma)} \\
&\le \frac{1}{2}\Big(\|\nabla u_0\|^2_{L^2(\Omega)} + \|\nabla_\Gamma v_0\|^2_{L^2(\Gamma)} + 2\int_\Omega (F(u_0) - hu_0)\mathrm{d}x + \lambda_\Gamma \|v_0\|^2_{L^2(\Gamma)}\Big) \\
&\quad + \int_\Gamma (h_\Gamma v(t) - F_\Gamma(v(t)) - h_\Gamma v_0 + F_\Gamma(v_0))\mathrm{d}\sigma,
\end{aligned}$$

where F_Γ is the primitive of f_Γ vanishing, e.g., at the origin. As the last integral is bounded by $c\|v(t)\|_{L^2(\Gamma)} + c(v_0)$ and $t > 0$ is arbitrary, we immediately find

$$\partial_t u \in L^2(0, +\infty; V^*) \quad \text{and} \quad \partial_t v \in L^2(0, +\infty; H_\Gamma). \tag{3.84}$$

4 Proof of Theorem 2.2

First, the continuity of the semigroup $S(t)$ with respect to the d_w-metric is standard (see, e.g., the proof of uniqueness in [18]). This implies, in particular, that $S(t)$ is *closed* in the sense of Pata-Zelik [25]. Therefore, the existence of the global attractor with the desired regularity follows from classical attractors' existence results (see, e.g., [1, 29]), once we have proved the dissipativity and the asymptotic compactness of $S(t)$.

To do so, we fix a bounded set $\mathcal{B}$ of initial data. More precisely, we assume that (3.27) holds for any $(u_0, v_0) \in \mathcal{B}$ and a suitable $R > 0$. Then, the a priori estimates of Section 3 hold. In particular, thanks to (3.28), (3.82) and (3.83), $(u(t), v(t))$ belongs to the set $\mathcal{K}_m$ defined by

$$\begin{aligned} \mathcal{K}_m := \{(y, z) \in \Phi_m \cap (H^{\frac{3}{2}}(\Omega) \times H^{\frac{3}{2}}(\Gamma)) : \\ \|y\|_{H^{\frac{3}{2}}(\Omega)} + \|z\|_{H^{\frac{3}{2}}(\Gamma)} \le C_5,\ \|F_0(y)\|_{L^{\frac{3}{2}}(\Omega)} \le C_6\} \end{aligned} \tag{4.1}$$

for all $t \ge t_1(R)$. In other words, $\mathcal{K}_m$ is an absorbing set for $S(t)$. Moreover, we easily see that it is also compact in Φ_m. Indeed, it is of course compact in $\mathcal{V}$. Moreover, the $L^{\frac{3}{2}}$-bound of $F_0(u)$ following from (3.83) and Lebesgue's theorem clearly imply the compactness with respect to the metric (2.12), hence the asymptotic compactness of $S(t)$. This finishes the proof.

5 Proof of Theorem 2.3

As we deal with a precise initial datum, we can choose R satisfying equality in (3.27). However, we will not stress the dependence of the constants on R in the notation here. As above, $(u(t), v(t))$ belongs to the set $\mathcal{K}_m$ defined in (4.1), at least for t large enough, and (u, v) is a weakly continuous $\mathcal{V}$-valued function. As $\mathcal{K}_m$ is a compact subset of $\mathcal{V}$ and we take the datum (u_0, v_0) as a starting point, the first part of the statement follows from general results (see, e.g., [20, p. 12]). Now, we prove our characterization of the ω-limit. Thus, we pick $(u_\omega, v_\omega) \in \omega(u_0, v_0)$ and a corresponding sequence $t_n \uparrow +\infty$ such that $(u_\omega, v_\omega) = \lim(u(t_n), v(t_n))$ strongly in $\mathcal{V}$. Moreover, as we prefer to consider functions on a finite time interval rather than pointwise (in time) values, we fix $T \in (0, +\infty)$ once and for all and set

$$u_n(t) := u(t + t_n), \quad v_n(t) := v(t + t_n) \quad \text{and} \quad w_n(t) := w(t + t_n) \quad \text{for } t \ge 0. \tag{5.1}$$

Then, the triplet (u_n, v_n, w_n) clearly satisfies (2.18)–(2.19) (besides (2.17)). Hence, it also satisfies an integrated version of them, namely,

$$\int_0^T \langle \partial_t u_n, y\rangle \mathrm{d}t + \int_0^T \int_\Omega \nabla w_n \cdot \nabla y \mathrm{d}x\mathrm{d}t = 0, \tag{5.2}$$

$$\begin{aligned} \int_0^T \int_\Omega w_n y \mathrm{d}x\mathrm{d}t = &\int_0^T \int_\Omega \nabla u_n \cdot \nabla y \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega (f(u_n) - h) y \mathrm{d}x\mathrm{d}t \\ &+ \int_0^T \int_\Gamma \partial_t v_n z \mathrm{d}\sigma\mathrm{d}t + \int_0^T \int_\Gamma \nabla_\Gamma v_n \cdot \nabla_\Gamma z \mathrm{d}\sigma\mathrm{d}t \\ &+ \int_0^T \int_\Gamma (f_\Gamma(v_n) + \lambda_\Gamma v_n - h_\Gamma) z \mathrm{d}\sigma\mathrm{d}t \end{aligned} \tag{5.3}$$

for every $y \in L^2(0, T; V)$ and $(y, z) \in L^2(0, T; \mathcal{V})$, respectively. Our aim is to take the limits of such variational equations as n tends to infinity by using compactness methods. To do so, we account for our a priori estimates to derive corresponding estimates for the above functions. More precisely, we observe that (3.82), (3.37), and (3.69) yield

$$\|u_n\|_{L^\infty(0,T;H^{\frac{3}{2}}(\Omega))} + \|v_n\|_{L^\infty(0,T;H^{\frac{3}{2}}(\Gamma))} + \|w_n\|_{L^\infty(0,T;V)} + \|f_0(u_n)\|_{L^2(0,T;H)} \leq c.$$

Therefore, we can use standard weak, weak*, and strong compactness results and have

$$u_n \to u_\infty \quad \text{weakly* in } L^\infty(0,T;H^{\frac{3}{2}}(\Omega)) \text{ and strongly in } C^0([0,T];V), \tag{5.4}$$
$$v_n \to v_\infty \quad \text{weakly* in } L^\infty(0,T;H^{\frac{3}{2}}(\Gamma)) \text{ and strongly in } C^0([0,T];V_\Gamma), \tag{5.5}$$
$$w_n \to w_\infty \quad \text{weakly* in } L^\infty(0,T;V), \tag{5.6}$$
$$f_0(u_n) \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{5.7}$$

for some $u_\infty, v_\infty, w_\infty, \xi$ belonging to the corresponding spaces, at least for a subsequence. On the other hand, (3.84) clearly implies that

$$\partial_t u_n \to 0 \quad \text{and} \quad \partial_t v_n \to 0 \quad \text{strongly in } L^2(0,T;V^*) \text{ and in } L^2(0,T;H_\Gamma), \text{ respectively}, \tag{5.8}$$

so that u_∞ and v_∞ must be time-independent by (5.8). Hence, we can define u_s and v_s as their constant values and we clearly have $(u_s, v_s) \in \Phi_m$. Furthermore, the strong convergence given by (5.4) and (5.5), the weak convergence (5.7), the Lipschitz continuity of f_Γ, and standard monotonicity methods (see, e.g., [2, Proposition 2.5, p. 27] for a similar tool) allow us to conclude that

$$f_\Gamma(v_n) \to f_\Gamma(v_\infty) \quad \text{strongly in } C^0([0,T];H_\Gamma), \quad -1 < u_\infty < 1 \quad \text{a.e. in } \Omega \quad \text{and} \quad \xi = f_0(u_\infty).$$

In particular, ξ is time-independent as well and its constant value is $f_0(u_s)$. At this point, we can let n tend to infinity in (5.2)–(5.3) and obtain similar variational equations for the triplet $(u_\infty, v_\infty, w_\infty)$, the only difference being that time derivatives no longer appear. Moreover, it is straightforward to get rid of time integrations and obtain

$$\int_\Omega \nabla w_\infty(t) \cdot \nabla y \mathrm{d}x = 0 \quad \text{for every } y \in V, \tag{5.9}$$
$$\int_\Omega \nabla u_s \cdot \nabla y \mathrm{d}x + \int_\Omega (f(u_s) - h - w_\infty(t)) y \mathrm{d}x$$
$$+ \int_\Gamma \nabla_\Gamma v_s \cdot \nabla_\Gamma z \mathrm{d}\sigma + \int_\Gamma (f_\Gamma(v_s) + \lambda_\Gamma v_s - h_\Gamma) z \mathrm{d}\sigma = 0 \quad \text{for every } (y,z) \in \mathcal{V}, \tag{5.10}$$

at almost every time $t \in (0,T)$. In particular, by taking $y = w_\infty(t)$ in (5.9), we see that $w_\infty(t)$ must be space-independent. Once this is established, the choice $y = 1$ and $z = 1$ in (5.10) shows that w_∞ is even time-independent. Thus, w_∞ takes a constant value w_s and we conclude that (u_s, v_s) is a steady state. It remains to show that $(u_\omega, v_\omega) = (u_s, v_s)$. To this aim, we note that (5.4) implies that $u_n(0)$ converges to $u_\infty(0) = u_s$ strongly in V. As $u_n(0) = u(t_n)$ converges to u_ω strongly in V by assumption, we conclude that $u_s = u_\omega$. As a similar argument holds for v_s and v_ω, the proof is complete.

Remark 5.1 The above proof clearly shows that different topologies could have been considered in the definition of $\omega(u_0, v_0)$. For instance, any weaker topology, e.g., the weak topology of $\mathcal{V}$, would have worked as well in the conclusion of the proof. A stronger topology can be used, provided that the compactness of $\mathcal{K}_m$ still holds.

6 Proof of Theorem 2.4

We start by analyzing a bit more carefully the stationary problem (2.29). Given $m \in (-1,1)$, we denote by $\mathcal{S}_m$ the family of its solutions, namely,

$$\mathcal{S}_m := \{(u_s, v_s) \in \mathcal{V} : \langle u_s \rangle_\Omega = m;\ \exists w_s \in \mathbb{R} : \text{(2.28)–(2.29) hold}\}. \tag{6.1}$$

Lemma 6.1 *Assume* (2.1)–(2.7) *and fix* $m \in (-1,1)$*. Then, there exists* $M_m > 0$ *depending on* m *such that, for any element* $(u_s, v_s) \in \mathcal{S}_m$*,*

$$|w_s| \le M_m. \tag{6.2}$$

Moreover, there exists $\delta_m > 0$*, also depending on* m*, such that*

$$-1 + \delta_m \le u_s(x) \le 1 - \delta_m, \quad \forall x \in \overline{\Omega}. \tag{6.3}$$

Proof Taking $y \equiv 1$ and $z \equiv 1$ in (2.29), we infer that

$$|\Omega| w_s = \int_\Omega (f(u_s) - h)\mathrm{d}x + \int_\Gamma (f_\Gamma(v_s) + \lambda_\Gamma v_s - h_\Gamma)\mathrm{d}\sigma. \tag{6.4}$$

Choosing instead $y = u_s - m$ and $z = v_s - m$, we find

$$\begin{aligned}&\int_\Omega (|\nabla(u_s - m)|^2 + f(u_s)(u_s - m))\mathrm{d}x + \int_\Gamma (|\nabla_\Gamma(v_s - m)|^2 + f_\Gamma(v_s)(v_s - m))\mathrm{d}\sigma \\ &= \int_\Omega h(u_s - m)\mathrm{d}x - \int_\Gamma \lambda_\Gamma v_s(v_s - m)\mathrm{d}\sigma + \int_\Gamma h_\Gamma(v_s - m)\mathrm{d}x \le c,\end{aligned} \tag{6.5}$$

where the last inequality follows from (2.5) and the fact that $|u_s| \le 1$. On the other hand, estimating the left-hand side from below by means of (3.13), we end up with

$$\|(u_s, v_s)\|_{\mathcal{V}}^2 + \|f(u_s)\|_{L^1(\Omega)} \le c. \tag{6.6}$$

Combining (6.4) and (6.6) and using once more (2.5) to estimate the remaining terms on the right-hand side of (6.4), we readily obtain (6.2) for a suitable choice of M_m.

Next, we notice that, thanks to (2.2) and (2.5), we can find $\delta_m > 0$ such that

$$f(r) - \|h\|_{L^\infty(\Omega)} - M_m \ge 1, \quad \forall\, r \in [1 - \delta_m, 1] \tag{6.7}$$

and a similar relation holds near -1. Moreover, we can assume $1 - \delta_m \ge r_0$ (cf. (2.6)–(2.7)). Thus, take first $y = (u_s - 1 + \delta_m)^+$ and z accordingly, and then $y = -(u_s + 1 - \delta_m)^+$ in (2.29) and using (2.6)–(2.7), it is immediate to obtain (6.3), which concludes the proof.

Corollary 6.1 *Assume* (2.1)–(2.7)*,* (2.30)*, and* $m \in (-1,1)$*. Then,* $\mathcal{S}_m$ *is bounded in* $H^3(\Omega) \times H^3(\Gamma)$*. In particular, this holds for the* ω*-limit set of any solution trajectory.*

Proof Thanks to the separation property (6.3) and the C^1-regularity of f, it is not difficult to obtain (see, e.g., [19, Lemma 2.2]) that $\mathcal{S}_m$ is bounded in $H^2(\Omega) \times H^2(\Gamma)$. Moreover, thanks to this regularity, the stationary problem can be interpreted in the stronger form

$$-\Delta u_s + f(u_s) - h = w_s, \quad \text{a.e. in } \Omega, \tag{6.8}$$

$$-\Delta_\Gamma v_s + f_\Gamma(v_s) + \lambda_\Gamma v_s - h_\Gamma = -\partial_n u_s, \quad \text{a.e. on } \Gamma. \tag{6.9}$$

Using the continuous embedding $H^2(\Omega) \subset L^\infty(\Omega)$ and once more the C^1-regularity of f and f_Γ, we then have, for any $(u_s, v_s) \in \mathcal{S}_m$, $f(u_s) \in V$ and $f_\Gamma(v_s) \in V_\Gamma$. Thus, applying standard regularity results to the coupled elliptic system (6.8)–(6.9) and a simple bootstrap argument (similar to the one given in (3.80)), it is not difficult to obtain $u_s \in H^3(\Omega)$ and $v_s \in H^3(\Gamma)$, thanks also to (2.30). Refining a bit the procedure, we can have uniform H^3-estimates as well.

In the next two lemmas, we give the key step of our procedure which consists in proving the precompactness of any trajectory w.r.t. a better topology.

Lemma 6.2 *Assume* (2.1)–(2.7), (2.30), (2.31), $m \in (-1,1)$, *and* $(u_0, v_0) \in \Phi_m$. *Correspondingly, take R as in* (3.27). *Then, there exist $\beta > 0$, $t_2 = t_2(R)$, and c independent of R such that the corresponding solution (u, v) satisfies*

$$\|u(t)\|_{C^{0,\beta}(\overline{\Omega})} \le c, \quad \forall\, t \ge t_2(R). \tag{6.10}$$

Proof Let us assume $\varphi : (-1,1) \to \mathbb{R}$ to be a smooth and monotone function such that $\varphi(0) = 0$. Then, taking $y = \varphi(u)$ and $z = \varphi(v)$ as test functions in (2.19) and using (2.6)–(2.7), it is not difficult to find

$$\begin{aligned} &\int_\Omega \varphi'(u)|\nabla u|^2 \mathrm{d}x + \int_\Omega f(u)\varphi(u)\mathrm{d}x + \int_\Gamma v_t \varphi(v)\mathrm{d}\sigma + \int_\Gamma \varphi'(v)|\nabla_\Gamma v|^2 \mathrm{d}\sigma \\ &\le \int_\Omega (w+h)\varphi(u)\mathrm{d}x + c(\varphi) \end{aligned} \tag{6.11}$$

(as above, in case $\varphi(r)$ explodes as $|r| \sim 1$, the above estimate should be proved by truncating φ and then passing to the limit). We now first consider the situation when (3.52) holds, i.e., f explodes as a power for $|r| \sim 1$. Indeed, when (2.25) holds, the procedure is simpler, as it will sketched below.

Then, also recall the second of (2.2) and take

$$\varphi_+(r) := (f'(r) - f'(0) + 2\zeta r)\chi_{[0,1]}, \quad \varphi_-(r) := (-f'(r) + f'(0) + 2\zeta r)\chi_{[-1,0]} \tag{6.12}$$

and also set

$$\varphi(r) := \varphi_+(r) + \varphi_-(r), \quad \widehat{\varphi}(r) := \int_0^r \varphi(s)\mathrm{d}s \tag{6.13}$$

where χ denotes the characteristic function and $\zeta \ge 0$ is as in (2.31). Now, as a consequence of (3.69), for any $t \ge t_1(r)$, there exists $\widehat{t} \in (t, t+1)$ such that

$$\|f(v(\widehat{t}\,))\|_{L^1(\Gamma)} \le c. \tag{6.14}$$

Then, we take the function φ in (6.11) as defined by (6.13) and note that, by (3.52),

$$\varphi(r) f(r) \ge \alpha |f(r)|^p - c, \quad \text{where } p = 2 + \gamma^{-1} \in (2,3] \tag{6.15}$$

and

$$\widehat{\varphi}(r) \le c(|f(r)| + 1). \tag{6.16}$$

Thus, we can integrate (6.11) over $(\widehat{t},\widehat{t}+2)$ and estimate the right-hand side as follows:

$$\int_\Omega (w+h)\varphi(u)\mathrm{d}x \le \|w+h\|_{L^6(\Omega)}\|\varphi(u)\|_{L^{\frac{6}{5}}(\Omega)} \le c\|\varphi(u)\|_{L^{\frac{6}{5}}(\Omega)} \le \frac{\alpha}{2}\|f(u)\|^p_{L^p(\Omega)} + c, \quad (6.17)$$

where α is the same as in (6.15). Here, (3.52) has been used again. This finally yields

$$\int_t^{t+1} \|f(u)\|^p_{L^p(\Omega)}\mathrm{d}\tau \le c, \quad \forall\, t \ge t_1(R)+1. \quad (6.18)$$

To conclude, we notice that the same relation can be easily obtained when (2.25) holds, which corresponds, e.g., to f behaving as a logarithm near ± 1 or to the analogue of (3.52), but with $\gamma \in (0,1)$. Indeed, in such a situation, one could simply take $\varphi(r) \sim |f(r)|^{p-1}\operatorname{sign} r$, where the exponent p is chosen such that the antiderivative $\widehat{\varphi}$ is bounded (we omit the details).

Next, we come back to the coupled elliptic problem (3.78)–(3.79) and aim to find additional regularity. Actually, by (3.81), (3.82), and interpolation,

$$\|u\|_{L^3(t,t+1;H^{\frac{11}{6}}(\Omega))} \le c(\|u\|_{C^0([t,t+1];H^{\frac{3}{2}}(\Omega))} + \|u\|_{L^2(t,t+1;H^2(\Omega))}) \le c. \quad (6.19)$$

Hence, by continuity of the trace operator ∂_n from $H^{\frac{11}{6}}(\Omega)$ to $H^{\frac{1}{3}}(\Gamma)$,

$$\|\partial_n u\|_{L^3(t,t+1;L^3(\Gamma))} \le c\|\partial_n u\|_{L^3(t,t+1;H^{\frac{1}{3}}(\Gamma))} \le c. \quad (6.20)$$

Also, on account of (2.5), (3.76) and (6.18), it thus follows that the functions h_1 and h_2 in the right-hand sides of (3.78)–(3.79) satisfy

$$\int_t^{t+1} (\|h_1\|^p_{L^p(\Omega)} + \|h_2\|^p_{L^p(\Gamma)})\mathrm{d}\tau \le c, \quad \forall\, t \ge t_1(R)+1 \quad (6.21)$$

again for a suitable $p \in (2,3]$. Applying Agmon-Douglis-Nirenberg regularity results, we then have

$$\|u\|_{L^p(t,t+1;W^{2,p}(\Omega))} \le c, \quad \forall\, t \ge t_1(R)+1. \quad (6.22)$$

Let us now observe that, by (3.76) and Sobolev embeddings, there also holds

$$\|u\|_{W^{\frac{p+2}{2p},p}(t,t+1;W^{\frac{6-p}{2p},p}(\Omega))} \le c\|u\|_{H^1(t,t+1;V)} \le c, \quad \forall\, t \ge t_1(R). \quad (6.23)$$

Then, by L^p-interpolation, we obtain

$$L^p(t,t+1;W^{2,p}(\Omega)) \cap W^{\frac{p+2}{2p},p}(t,t+1;W^{\frac{6-p}{2p},p}(\Omega)) \subset W^{\rho,p}(t,t+1;W^{r,p}(\Omega)) \quad (6.24)$$

with continuous embedding, where

$$\rho = \vartheta\frac{p+2}{2p}, \quad r = 2(1-\vartheta) + \vartheta\frac{6-p}{2p} \quad (6.25)$$

and ϑ can be taken arbitrarily in $(0,1)$. Then, it is easily seen that $p > 2$ guarantees that ϑ can be chosen so that $\rho > \frac{1}{p}$ and $r > \frac{3}{p}$. Applying once more the Sobolev embedding theorems, we then deduce from (6.24) that

$$\|u\|_{C^{0,\beta}([t,t+1]\times\overline{\Omega})} \le c, \quad \forall\, t \ge t_1(R)+1 \quad (6.26)$$

for some $\beta > 0$, which implies, in particular, (6.10).

Lemma 6.3 *Assume* (2.1)–(2.7), (2.30), (2.31), $m \in (-1,1)$, *and* $(u_0, v_0) \in \Phi_m$. *Correspondingly, take R as in* (3.27). *Then, there exists a time $t_3 > 0$ depending on the trajectory and a constant c which is independent of R such that the corresponding solution* $(u(t), v(t))$ *satisfies*

$$\|u(t)\|_{H^3(\Omega)} + \|v(t)\|_{H^3(\Gamma)} + \|f(u(t))\|_{W^{1,\infty}(\Omega)} \le c, \quad \forall\, t \ge t_3. \tag{6.27}$$

Proof We first claim that there exists $t_3 > 0$ such that

$$-1 + \frac{\delta_m}{2} \le u(x,t) \le 1 - \frac{\delta_m}{2}, \quad \forall\, t \ge t_3, \tag{6.28}$$

where δ_m is as in (6.3). Indeed, we can proceed by contradiction. If we could find a diverging sequence $\{t_n\}$ such that the above inequality does not hold for $t = t_n$, then, by the precompactness deriving from (6.10), $(u(t_n), v(t_n))$ would admit a subsequence converging uniformly in $\overline{\Omega}$ to an element $(u_s, v_s) \in \mathcal{S}_m$. Now, since u_s satisfies (6.3), we have a contradiction. It is then clear that (6.28) and the C^1-regularity of f entail

$$\|f(u(t))\|_{L^\infty(\Omega)} + \|f'(u(t))\|_{L^\infty(\Omega)} \le c, \quad \forall\, t \ge t_3. \tag{6.29}$$

Similarly with (3.78)–(3.79), we now rewrite (the strong formulation of) (2.18)–(2.19) in the more convenient form

$$-\Delta u = k_1 := -f(u) + w + h, \quad \text{in } \Omega, \tag{6.30}$$

$$\partial_t v - \Delta_\Gamma v + \partial_n u = k_2 := -f_\Gamma(v) - \lambda_\Gamma v + h_\Gamma, \quad \text{on } \Gamma \tag{6.31}$$

and notice that, for any $t \ge t_3$, thanks to (3.77), (3.76), and (6.29),

$$\|k_1\|_{L^\infty(t,t+1;V)} \le c \quad \text{and} \quad \|k_2\|_{L^\infty(t,t+1;V_\Gamma)} + \|\partial_t v\|_{L^\infty(t,t+1;L^2(\Gamma))} \le c, \quad \forall\, t \ge t_3. \tag{6.32}$$

Thus, viewing (6.31) as an elliptic equation (i.e., moving $\partial_t v$ to the right-hand side), a further application of [19, Lemma 2.2] gives

$$\|u\|_{L^\infty(t,t+1;H^2(\Omega))} + \|v\|_{L^\infty(t,t+1;H^2(\Gamma))} \le c, \quad \forall\, t \ge t_3. \tag{6.33}$$

To improve the regularity, we have to come back to the full system (2.18)–(2.19), differentiate both equations in time, and take $y = -(t-\tau)w_t$ as a test function in $\partial_t(2.18)$ and $y = (t-\tau)u_{tt}$ and $z = (t-\tau)v_{tt}$ in $\partial_t(2.19)$, where τ is a generic "initial time" taken larger than t_3. Then, observing that

$$\int_\Omega f'(u) u_t u_{tt} \mathrm{d}x = \frac{1}{2}\partial_t \int_\Omega f'(u)|u_t|^2 \mathrm{d}x - \frac{1}{2}\int_\Omega f''(u)|u_t|^3 \mathrm{d}x \tag{6.34}$$

and integrating by parts with respect to time, we end up with

$$\frac{1}{2}\partial_t \Big[(t-\tau)\Big(\int_\Omega |\nabla u_t|^2 \mathrm{d}x + \int_\Gamma |\nabla_\Gamma v_t|^2 \mathrm{d}\sigma + \int_\Omega f'(u)|u_t|^2 \mathrm{d}x\Big)\Big]$$
$$+ (t-\tau)(\|\nabla w_t\|^2_{L^2(\Omega)} + \|v_{tt}\|^2_{L^2(\Omega)})$$

$$\leq (t-\tau)\int_\Gamma |(f_\Gamma'(v)v_t + \lambda_\Gamma v_t)v_{tt}|\mathrm{d}\sigma \\ + \frac{1}{2}\Big(\int_\Omega |\nabla u_t|^2\mathrm{d}x + \int_\Gamma |\nabla_\Gamma v_t|^2\mathrm{d}\sigma + \int_\Omega f'(u)|u_t|^2\mathrm{d}x + (t-\tau)\int_\Omega f''(u)|u_t|^3\mathrm{d}x\Big).$$

At this point, we can split the last term on the second row by means of Young's inequality and observe that, by (6.33), the C^2-regularity of f, standard embedding theorems, and the Poincaré-Wirtinger inequality, we have

$$\int_\Omega f'(u)|u_t|^2\mathrm{d}x + (t-\tau)\int_\Omega f''(u)|u_t|^3\mathrm{d}x \leq c\|u_t\|_V^2 + c\|u_t\|_V(t-\tau)\int_\Omega |\nabla u_t|^2\mathrm{d}x. \tag{6.35}$$

Thus, integrating over $(\tau, \tau+2)$, recalling (3.76), and using assumption (2.1) and Gronwall's lemma, we finally obtain

$$\|u_t\|_{L^\infty(t,t+1;V)} + \|v_t\|_{L^\infty(t,t+1;V_\Gamma)} \leq c, \quad \forall\, t \geq t_3 \tag{6.36}$$

again up to increasing t_3 a bit. With this enhanced regularity at our disposal, we come back to system (6.30)–(6.31) (again seen as a coupled elliptic system). Noting that now

$$\|k_1\|_{L^\infty(t,t+1;V)} + \|-v_t + k_2\|_{L^\infty(t,t+1;V_\Gamma)} \leq c, \quad \forall\, t \geq t_3 \tag{6.37}$$

we then obtain (6.27) by means of standard elliptic regularity results as in the proof of Corollary 6.1. As a concluding remark, it is worth noting that, however, we are no longer able to evaluate the time t_3 in terms of the "initial energy" R.

In order to go on with the proof, we introduce several simplifications which are in fact not restrictive. More precisely, we assume $h = h_\Gamma = 0$ and $m = 0$. Accordingly, the constants δ_m and M_m and the set $\mathcal{S}_m$ introduced above will be simply renamed as δ, M, and $\mathcal{S}$. It is worth noting that, in case $m \neq 0$, one could simply set $u_m := u - m$ and $v_m := v - m$ and notice that the pair (u_m, v_m) solves the system

$$\partial_t u_m - \Delta(-\Delta u_m + f_m(u_m)) = 0, \quad \text{in } \Omega, \tag{6.38}$$

$$\partial_t v_m + (\partial_n u_m)|_\Gamma - \Delta_\Gamma v_m + f_{\Gamma,m}(v_m) + \lambda_\Gamma v_m = 0, \quad \text{on } \Gamma, \tag{6.39}$$

where $f_m(r) := f(r+m)$ and $f_{\Gamma,m}(r) := f_\Gamma(r+m)$. One, then, could simply work on system (6.38)–(6.39).

That said, we can now introduce the function spaces $\mathcal{V}_0$, $\mathcal{H}_0$, $\mathcal{V}_0'$, $\mathcal{W}_0$ as the (closed) subspaces of $\mathcal{V}$, $\mathcal{H}$, $\mathcal{V}'$, $(H^2(\Omega)\times H^2(\Gamma))\cap\mathcal{V}$, respectively, consisting of the functions, or functionals, having zero mean over Ω. The latter space $\mathcal{W}_0$ is endowed with the norm of $H^2(\Omega)\times H^2(\Gamma)$. We can then define the energy functional $\mathcal{E} : \mathcal{V}_0 \to [0, +\infty]$ by

$$\mathcal{E}(u,v) := \frac{1}{2}\|\nabla u\|^2_{L^2(\Omega)} + \int_\Omega F(u)\mathrm{d}x + \frac{1}{2}\|\nabla_\Gamma v\|^2_{L^2(\Gamma)} + \frac{1}{2}\int_\Gamma (2F_\Gamma(v) + \lambda_\Gamma v^2)\mathrm{d}\sigma. \tag{6.40}$$

Due to the singular character of F, $\mathcal{E}$ can very well take the value $+\infty$. We then have to restrict it to a suitable subset of $\mathcal{V}_0$. We let $\mathfrak{c} > 0$ be such that $\|z\|_{C^0(\overline{\Omega})} \leq \mathfrak{c}\|z\|_{H^2(\Omega)}$ for all $z \in H^2(\Omega)$. Next, we define

$$\mathcal{U} := \Big\{(u,v) \in \mathcal{W}_0 : v = u|_\Gamma,\ \exists (u_s, v_s) \in \mathcal{S} : \|u - u_s\|_{H^2(\Omega)} + \|v - v_s\|_{H^2(\Gamma)} < \mathfrak{c}^{-1}\frac{\delta}{2}\Big\}. \tag{6.41}$$

It is clear that $\mathcal{U}$ is an open set in $\mathcal{W}_0$ and $\mathcal{S} \subset \mathcal{U}$. We also have, for any $(u,v) \in \overline{\mathcal{U}}$ (the closure is intended in $H^2(\Omega) \times H^2(\Gamma)$, of course),

$$-1 + \frac{\delta}{2} \leq u(x) \leq 1 - \frac{\delta}{2}, \quad \forall x \in \overline{\Omega}. \tag{6.42}$$

Thus, as a consequence of Lemma 6.3, any global solution to our system eventually lies in $\mathcal{U}$.

Lemma 6.4 *The restriction of $\mathcal{E}$ to $\mathcal{U}$ is twice Fréchet differentiable with respect to the topology of $\mathcal{W}_0$. Moreover, $(\overline{u},\overline{v})$ is a stationary point of $\mathcal{E}$ belonging to $\mathcal{U}$ if and only if $(\overline{u},\overline{v}) \in \mathcal{S}$.*

Proof The key observation is that, thanks to (6.42), $\mathcal{E}$ is uniformly bounded on $\overline{\mathcal{U}}$. Then, we note that the stationary points of $\mathcal{E}$ are precisely those elements $(\overline{u},\overline{v}) \in \mathcal{V}_0$ such that

$$\langle \mathcal{E}'(\overline{u},\overline{v}), (k,\kappa) \rangle = \int_\Omega (\nabla \overline{u} \cdot \nabla k + f(\overline{u})k) \mathrm{d}x + \int_\Gamma (\nabla_\Gamma \overline{v} \cdot \nabla_\Gamma \kappa + (f_\Gamma(\overline{v}) + \lambda_\Gamma \overline{v})\kappa) \mathrm{d}\sigma = 0 \tag{6.43}$$

holds for all $(k,\kappa) \in \mathcal{V}_0$. At this point, the thesis follows by proceeding along the lines of the proof of [6, Proposition 6.4]. Actually, once we restrict $\mathcal{E}$ to $\overline{\mathcal{U}}$, everything is well separated from the singular values ± 1 of f. Thus, we are in the very same regularity setting as the one of [6].

We can now state the Simon-Łojasiewicz inequality which will be needed in what follows [6, Proposition 6.6].

Proposition 6.1 *Let $(\overline{u},\overline{v}) \in \mathcal{U}$ be a critical point of $\mathcal{E}$ and let F, F_Γ be analytic. Then, there exist constants $\vartheta \in (0, \frac{1}{2}]$, $\Lambda > 0$, and $\sigma \in (0, \mathfrak{c}^{-1}\frac{\delta}{2})$ such that*

$$|\mathcal{E}(u,v) - \mathcal{E}(\overline{u},\overline{v})|^{1-\vartheta} \leq \Lambda \|\mathcal{E}'(u,v)\|_{\mathcal{V}'} \tag{6.44}$$

for all $(u,v) \in \mathcal{W}_0$ such that

$$\|(u,v) - (\overline{u},\overline{v})\|_{\mathcal{W}_0} \leq \sigma. \tag{6.45}$$

Proof The proof can be carried out in the same way as in [6]. We point out the only remarkable difference, namely, the occurrence of the $\mathcal{W}_0$-norm in (6.45) (in place of the $\mathcal{V}_0$-norm appearing in [6, Proposition 6.6]). Actually, one can find elements $(u,v) \in \mathcal{V}_0$ which are arbitrarily close to some $(\overline{u},\overline{v}) \in \mathcal{S}$ in the $\mathcal{V}$-norm and such that $\mathcal{E}(u,v) = +\infty$ (see also [6, Remark 6.7]). Instead, if (6.45) holds, then we are sure that (u,v) is uniformly separated from ± 1. In particular, (6.42) holds.

Now, by Corollary 6.1, $\mathcal{S}$ is bounded in $H^3(\Omega) \times H^3(\Gamma)$, hence precompact in $\mathcal{W}_0$. In fact, it is compact, since it is obviously closed in $\mathcal{W}_0$. Given any $(\overline{u},\overline{v}) \in \mathcal{S}$, we can then associate to $(\overline{u},\overline{v})$ the numbers $\vartheta, \Lambda, \sigma$ given by Proposition 6.1. We obtain a covering

$$\mathcal{S} \subset \bigcup_{(\overline{u},\overline{v}) \in \mathcal{S}} B((\overline{u},\overline{v}), \sigma), \tag{6.46}$$

where the ball B is intended in the topology of $\mathcal{W}_0$. By compactness, we can extract a finite subcovering of $\mathcal{S}$. More precisely, we have $\mathcal{S} \subset \mathcal{B}$, where

$$\mathcal{B} := \bigcup_{i=1}^{\bar{i}} B((\overline{u}_i, \overline{v}_i), \sigma_i). \tag{6.47}$$

Actually, since $\sigma_i \leq \mathfrak{c}^{-1}\frac{\delta}{2}$ for all i, we have

$$\mathcal{S} \subset \mathcal{B} \subset \mathcal{U}. \tag{6.48}$$

In particular, any element of $\mathcal{B}$ is separated from ± 1 in the sense of (6.42) and the functional $\mathcal{E}$ is analytic on $\mathcal{B}$. Moreover, due to the finiteness of the covering, the Simon-Łojasiewicz inequality holds in $\mathcal{B}$ with uniform constants ϑ and Λ.

Let us now consider a trajectory (u, v). By Lemma 6.3, (u, v) eventually lies in $H^3(\Omega) \times H^3(\Gamma)$ and, consequently, is precompact in $\mathcal{W}_0$. Then, a simple contradiction argument similar to the one given in the proof of Lemma 6.3 allows to prove that $(u(t), v(t)) \in \mathcal{B}$ for all t larger than some t_4 depending on the trajectory itself. At this point, to complete the proof of the theorem, we notice that $t \mapsto \mathcal{E}(u(t), v(t))$ is a decreasing functional. Moreover, setting $\mathcal{E}_\infty := \lim_{t\to+\infty} \mathcal{E}(u(t), v(t))$, it is easy to see that, for any (u_∞, v_∞) in the ω-limit set of $(u(t), v(t))$, we have $\mathcal{E}(u_\infty, v_\infty) = \mathcal{E}_\infty$.

Then, we can compute, as in [6, Proof of Theorem 2.3] and for $t \geq t_4$, the derivative

$$\begin{aligned} -\partial_t(\mathcal{E}(u(t), v(t)) - \mathcal{E}_\infty)^\vartheta &= -\vartheta\partial_t(\mathcal{E}(u(t), v(t)) - \mathcal{E}(\overline{u}, \overline{v}))(\mathcal{E}(u(t), v(t)) - \mathcal{E}_\infty)^{\vartheta-1} \\ &\geq -\vartheta \frac{\partial_t \mathcal{E}(u(t), v(t))}{\Lambda \|\mathcal{E}'(u(t), v(t))\|_{\mathcal{V}'}} \end{aligned} \tag{6.49}$$

where $(\overline{u}, \overline{v})$ is an element of the ω-limit set such that $(u(t), v(t)) \in B((\overline{u}, \overline{v}), \sigma)$ and σ is such that the Simon-Łojasiewicz inequality holds. At this point, the rest of the proof follows by estimating $\mathcal{E}'(u, v)$ as in [6]. More precisely, taking $y = -w$ in (2.18) and $y = u_t$ and $z = v_t$ in (2.19), we obtain

$$\partial_t \mathcal{E} = -\|\nabla w\|^2_{L^2(\Omega)} - \|v_t\|^2_{L^2(\Gamma)}. \tag{6.50}$$

On the other hand, computing $\mathcal{E}'(u, v)$ as in (6.43) and integrating by parts, we have

$$\langle \mathcal{E}'(u, v), (k, \kappa)\rangle = \int_\Omega (-\Delta u + f(u))k\mathrm{d}x + \int_\Gamma (-\Delta_\Gamma v + f_\Gamma(v) + \lambda_\Gamma v + \partial_n u)\kappa \mathrm{d}\sigma \tag{6.51}$$

for all $(k, \kappa) \in \mathcal{V}_0$. Notice that the integrations by parts are rigorous, since $(u, v) \in H^3(\Omega) \times H^3(\Gamma)$. For the same reason, the system holds in the strong form (1.1)–(1.3). Thus, comparing terms, we obtain

$$\langle \mathcal{E}'(u, v), (k, \kappa)\rangle = \int_\Omega (w - \langle w\rangle_\Omega)k\mathrm{d}x - \int_\Gamma v_t \kappa \mathrm{d}\sigma. \tag{6.52}$$

Indeed, we could subtract the mean $\langle w\rangle_\Omega$, since $(k, \kappa) \in \mathcal{V}_0$. Thus, using the Poincaré-Wirtinger inequality and passing to the supremum w.r.t. $(k, \kappa) \in \mathcal{V}_0$ of unit norm, we have

$$\|\mathcal{E}'(u, v)\|_{\mathcal{V}'} \leq c\|\mathcal{E}'(u, v)\|_{\mathcal{H}} \leq c(\|\nabla w\|_{L^2(\Omega)} + \|v_t\|_{L^2(\Gamma)}). \tag{6.53}$$

Next, estimating the right-hand side of (6.49) with the help of (6.50) and (6.53), we infer

$$\begin{aligned} -\partial_t(\mathcal{E}(u,v)-\mathcal{E}_\infty)^\vartheta &\geq \vartheta \frac{\|\nabla w\|^2_{L^2(\Omega)} + \|v_t\|^2_{L^2(\Gamma)}}{\Lambda c(\|\nabla w\|_{L^2(\Omega)} + \|v_t\|_{L^2(\Gamma)})} \\ &\geq c(\vartheta,\Lambda)(\|\nabla w\|_{L^2(\Omega)} + \|v_t\|_{L^2(\Gamma)}) \end{aligned} \tag{6.54}$$

which is intended to hold for any $t \geq t_4$. Integrating over $(t_4,+\infty)$ and making a further comparison of terms in (1.1), we then obtain

$$u_t \in L^1(t_4,+\infty;V'), \quad \nabla w \in L^1(t_4,+\infty;H), \quad v_t \in L^1(t_4,+\infty;L^2(\Gamma)) \tag{6.55}$$

which readily entails that the *whole* trajectory $(u(t),v(t))$ converges to a single $(\overline{u},\overline{v})$ in $\mathcal{V}_0' \times H$. By precompactness, we have more precisely (2.33), which completes the proof.

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