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# ANALYSIS OF A PHASE-FIELD MODEL FOR TWO-PHASE COMPRESSIBLE FLUIDS

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A model describing the evolution of a binary mixture of compressible, viscous, and macroscopically immiscible fluids is investigated. The existence of global-in-time weak solutions for the resulting system coupling the compressible Navier–Stokes equations governing the motion of the mixture with the Allen–Cahn equation for the order parameter is proved without any restriction on the size of initial data.

*Keywords*: Compressible Navier–Stokes system; Allen–Cahn dynamics; existence of weak solutions.

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## 1. Introduction

A fluid-mechanical theory for two-phase mixtures of fluids faces a well-known mathematical difficulty: the movement of the interfaces is naturally amenable to a Lagrangian description, while the bulk fluid flow is usually considered in the Eulerian framework. The phase-field methods overcome this problem by postulating the existence of a "diffuse" interface spread over a possibly narrow region covering the "real" sharp interface boundary. A phase variable $\chi$ is introduced to demarcate the two species and to indicate the location of the interface. A mixing energy is defined in terms of $\chi$ and its spatial gradient the time evolution of which is described by means of a convection–diffusion equation.

As the underlying physical problem still conceptually consists of sharp interfaces, the dynamics of the phase variable remains to a considerable extent purely fictitious. Typically, different variants of Cahn–Hilliard, Allen–Cahn or other types of dynamics are used (see Anderson *et al.*,[4] Feng *et al.*[16]). In this paper, we consider a variant of a model for a two-phase flow undergoing phase changes proposed by Blesgen.[6] This model allows phases to grow or shrink due to changes of densities and incorporates their transport with the current. As pointed out in Ref. 6, the model should be viewed as a first step towards incorporating transport mechanism into the description of phase-formation processes. Although the model certainly needs further generalizations to be applicable to real-world problems, its mathematical analysis carried out in this present paper is already rather involved.

The resulting problem consists of the *Navier–Stokes system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) = \mathrm{div}_x \mathbb{T}, \tag{1.2}$$

governing the evolution of the fluid density $\varrho = \varrho(t,x)$ and the velocity field $\mathbf{u} = \mathbf{u}(t,x)$, coupled with a modified Allen–Cahn equation

$$\partial_t(\varrho\chi) + \mathrm{div}_x(\varrho\chi\mathbf{u}) = -\mu(\varrho, \chi, \Delta\chi), \tag{1.3}$$

$$\varrho\mu = -\Delta\chi + \varrho\frac{\partial f(\varrho,\chi)}{\partial \chi}, \tag{1.4}$$

describing the changes of the phase variable $\chi = \chi(t,x)$.

The rheology of the fluid is described by means of the Cauchy stress–tensor $\mathbb{T} = \mathbb{T}(\varrho, \chi, \nabla_x\chi, \nabla_x\mathbf{u})$,

$$\mathbb{T} = \mathbb{S} - \left(\nabla_x\chi \otimes \nabla_x\chi - \frac{|\nabla_x\chi|^2}{2}\mathbb{I}\right) - p(\varrho,\chi)\mathbb{I}, \tag{1.5}$$

where $\mathbb{S}$ is the conventional Newtonian viscous stress,

$$\mathbb{S}(\chi, \nabla_x\mathbf{u}) = \nu(\chi)\left(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right) + \eta(\chi)\mathrm{div}_x\mathbf{u}\mathbb{I}, \tag{1.6}$$

and $p$ denotes the thermodynamic pressure related to the potential energy $f$ through the formula

$$p(\varrho,\chi) = \varrho^2\frac{\partial f(\varrho,\chi)}{\partial\varrho}. \tag{1.7}$$

Furthermore, following Blesgen[6] we consider the potential energy density in the form

$$f(\varrho, \chi) = W(\chi) + \chi G_1(\varrho) + (1 - \chi) G_2(\varrho), \tag{1.8}$$

with

$$W(\chi) = L(\chi) - b(\chi), \tag{1.9}$$

$$G_i(\varrho) = \Gamma(\varrho) + g_i(\varrho), \quad i = 1, 2, \tag{1.10}$$

where

$$L : (0, 1) \to \mathbb{R} \quad \text{is a convex function.} \tag{1.11}$$

The function $L$ may be singular at the endpoints $\chi = 0, 1$ (see hypothesis (2.2) below), in particular, the case of the so-called *logarithmic potential* (cf. e.g. Ref. 8, p. 170)

$$L(\chi) = \chi \log \chi + (1 - \chi) \log(1 - \chi) \tag{1.12}$$

is included.

System (1.1)–(1.3) may be supplemented by the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \nabla_x \chi \cdot \mathbf{n}|_{\partial\Omega} = 0. \tag{1.13}$$

For technical reasons, and in contrast with Ref. 6, we have assumed that the middle-term in (1.5) is independent of the density in the spirit of a similar model proposed by Anderson *et al.*[4] Models based on the compressible Navier–Stokes system were also developed in the seminal work of Lowengrub and Truskinovsky.[21]

The models based on the *incompressible* Navier–Stokes system have been extensively studied (see Abels,[1,2] Desjardins,[10] Gurtin *et al.*,[19] Nouri and Poupaud,[23] Plotnikov,[24] and the references cited therein). Considerably less rigorous results are available for the *compressible* models. In Ref. 3, the authors studied the model proposed by Anderson *et al.*,[4] based on the compressible Navier–Stokes system, where the phase variable satisfies a Cahn–Hilliard type equation. In comparison with Ref. 3, the analysis of the present system, based on the Allen–Cahn dynamics of the phase-field variable, is mathematically much more delicate. The main difficulty is the lower regularity of the phase variable due to much weaker *a priori* estimates, and last but not least, the presence of the singular potential $L$ in (1.9).

Our main goal is to develop a rigorous *existence theory* for problem (1.1)–(1.13) based on the concept of weak solution for the compressible Navier–Stokes system introduced by Lions.[20] In particular, the theory can handle any initial data of finite energy and the solutions exist globally in time. Unfortunately, we are not able to exclude the possibility that solutions might develop a vacuum state in a finite time, which is one of the major technical difficulties to be overcome. In particular, the existence of *suitable* weak solutions, for which the phase variable ranges between the physically relevant values 0 and 1, is strongly conditioned by a proper choice of the approximation scheme. Moreover, in order to gain higher integrability or even boundedness of the density, we consider an equation of state containing a singular component in the spirit of Carnahan and Starling.[9]

The paper is organized as follows. The basic hypotheses concerning the structural properties of the constitutive functions, together with the main existence theorem, are presented in Sec. 2. In Sec. 3, we introduce the basic approximation scheme in order to construct solutions to our problem. The proof of existence of global-in-time solutions is rather technical and carried over by means of several steps described in Secs. 4–6, respectively.

## 2. Hypotheses and Main Result

### 2.1. *Hypotheses*

It follows immediately from (1.8)–(1.11) that

$$\begin{aligned} f(\varrho,\chi) &= W(\chi) + \Gamma(\varrho) + \chi g_1(\varrho) + (1-\chi)g_2(\varrho) \\ &= L(\chi) - b(\chi) + \Gamma(\varrho) + \chi g_1(\varrho) + (1-\chi)g_2(\varrho). \end{aligned} \tag{2.1}$$

In accordance with (1.11), we assume that

$$L : (0,1) \to (0,\infty) \quad \text{is convex,} \quad \operatorname{ess}\lim_{\chi\to 0+} L'(\chi) = -\infty, \quad \operatorname{ess}\lim_{\chi\to 1-} L'(\chi) = \infty, \tag{2.2}$$

$$b \in C^2_c(0,1), \tag{2.3}$$

meaning $W$ is a perturbation of a singular potential $L$.

Since the quantity $\varrho^2\partial_\varrho f(\varrho,\chi)$ represents the pressure, it is natural to take

$$g_i(\varrho) = a_i \log(\varrho), \quad a_i > 0, \ \ i = 1,2. \tag{2.4}$$

Thus, by (1.7), we have that

$$p(\varrho,\chi) = \varrho^2\Gamma'(\varrho) + \varrho(a_1\chi + a_2(1-\chi)), \tag{2.5}$$

where the latter summand on the right-hand side represents the thermodynamic pressure of a mixture of two species. The component $\Gamma$, identical for both species, penalizes the density changes for large values of the pressure in the spirit of the hard-sphere model.

To state our hypotheses on $\Gamma$, we introduce further notation setting

$$\varrho^2\Gamma'(\varrho) = P(\varrho) \quad \text{or, equivalently,} \quad \Gamma(\varrho) = \int_0^\varrho \frac{P(z)}{z^2}\,\mathrm{d}z, \tag{2.6}$$

where we require the latter integral to be finite. Moreover, we assume that

$$P \in C^1[0,r), \quad P(0) = P'(0) = 0, \quad P' \geq 0, \quad \liminf_{\varrho\to r-} P(\varrho)(r-\varrho)^3 = P_r > 0. \tag{2.7}$$

Thus, $P$ turns out to represent a singular pressure in the spirit of Carnahan and Starling[9] and $r$ stands for the upper threshold of the density.

Finally, we assume that the viscosity coefficients are bounded functions of the phase parameter, more precisely

$$\nu,\, \eta \in C^1[0,1], \quad \nu(\chi) \geq \underline{\nu} > 0, \quad \eta(\chi) \geq 0 \quad \text{for all } \chi \in [0,1], \tag{2.8}$$

where $\underline{\nu}$ is a positive constant.

### 2.2. *Weak solutions*

We shall say that a trio $\{\varrho, \mathbf{u}, \chi\}$ is a *weak solution* of problem (1.1)–(1.13) supplemented with the initial data

$$\varrho(0,\cdot) = \varrho_0, \quad (\varrho\mathbf{u})(0,\cdot) = (\varrho\mathbf{u})_0, \quad (\varrho\chi)(0,\cdot) = (\varrho\chi)_0 \tag{2.9}$$

if

- the density $\varrho$ is a bounded measurable function, $0 \leq \varrho(t,x) \leq r$ for a.a. $(t,x) \in (0,T) \times \Omega$, $\mathbf{u} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$, and the integral identity

$$\int_0^T \int_\Omega (\varrho \partial_t \varphi + \varrho\mathbf{u} \cdot \nabla_x \varphi) \mathrm{d}x\, \mathrm{d}t = -\int_\Omega \varrho_0 \varphi(0,\cdot) \mathrm{d}x \tag{2.10}$$

  holds for any test function $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$;
- the phase function $\chi$ satisfies

$$\chi \in L^\infty(0,T; H^1(\Omega)) \cap L^2(0,T; H^2(\Omega)) \tag{2.11}$$

  together with $0 \leq \chi(t,x) \leq 1$ for a.a. $(t,x) \in (0,T) \times \Omega$. Moreover, $p(\varrho,\chi) \in L^1((0,T) \times \Omega)$, and the integral identity

$$\int_0^T \int_\Omega (\varrho\mathbf{u} \cdot \partial_t \varphi + \varrho(\mathbf{u} \otimes \mathbf{u}) : \nabla_x \varphi) \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_\Omega \mathbb{T} : \nabla_x \varphi \, \mathrm{d}x\, \mathrm{d}t - \int_\Omega (\varrho\mathbf{u})_0 \cdot \varphi(0,\cdot) \mathrm{d}x, \tag{2.12}$$

  holds for any $\varphi \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$, where the Cauchy stress $\mathbb{T}$ satisfies (1.5), (1.6) (note that the regularity conditions on $\mathbf{u}$, $\chi$ and $p$ guarantee, in particular, that $\mathbb{T}$ belongs to $L^1((0,T) \times \Omega)$);
- $\mu \in L^2((0,T) \times \Omega)$, and the integral identity

$$\int_0^T \int_\Omega (\varrho\chi \partial_t \varphi + \varrho\chi\mathbf{u} \cdot \nabla_x \varphi) \mathrm{d}x\, \mathrm{d}t = \int_0^T \int_\Omega \mu\varphi \, \mathrm{d}x\, \mathrm{d}t - \int_\Omega (\varrho\chi)_0 \varphi(0,\cdot) \mathrm{d}x \tag{2.13}$$

  holds for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, where $\mu$ satisfies (1.4), with

$$\varrho^{1/2} W'(\chi) \in L^2((0,T) \times \Omega). \tag{2.14}$$

  Finally, we require that the second condition in (1.13) holds.

### 2.3. *Main result*

Having collected all the preliminary material, we are in a position to formulate the main result of this paper.

**Theorem 2.1.** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2+\lambda}$, $\lambda > 0$. Suppose that the function $f$ is given by (2.1), where the functions $L$, $b$, $\Gamma$, $g_1$, $g_2$ satisfy hypotheses (2.2)–(2.7), and that the viscosity coefficients $\nu$, $\eta$ obey (2.8). Furthermore, let the initial data satisfy*

$$\left\{ \begin{array}{c} 0 < \operatorname*{essinf}_{x\in\Omega} \varrho_0(x) \leq \operatorname*{esssup}_{x\in\Omega} \varrho_0(x) < r, \\ (\varrho\chi)_0 = \varrho_0\chi_0, \ \text{with } 0 < \operatorname*{essinf}_{x\in\Omega} \chi_0(x) \leq \operatorname*{esssup}_{x\in\Omega} \chi_0(x) < 1, \\ \nabla_x \chi_0 \in L^2(\Omega;\mathbb{R}^3), \\ (\varrho\mathbf{u})_0 = \varrho_0\mathbf{u}_0, \ \text{with } \mathbf{u}_0 \in L^2(\Omega;\mathbb{R}^3). \end{array} \right\}. \tag{2.15}$$

*Then problem (1.1)–(1.13) possesses a weak solution $\{\varrho, \mathbf{u}, \chi\}$ in $(0,T)\times\Omega$ in the sense specified in Sec. 2.2.*

The rest of the paper is devoted to the proof of Theorem 2.1.

## 3. Approximation Scheme

The solution $\{\varrho, \mathbf{u}, \chi\}$ will be constructed by means of a multi-level approximation scheme similar to that used in Chap. 7 of Ref. 12. To begin with, we regularize the initial data replacing $\varrho_0$ by $\varrho_{0,\delta}$, $\mathbf{u}_0$ by $\mathbf{u}_{0,\delta}$, and $\chi_0$ by $\chi_{0,\delta}$ where $\delta \in (0, 1/4)$ tends to 0, and the quantities $\varrho_{0,\delta}$, $\mathbf{u}_{0,\delta}$, and $\chi_{0,\delta}$ are smooth in $\overline{\Omega}$ and satisfy a stronger version of hypothesis (2.15), namely

$$\left\{ \begin{array}{c} 0 < \delta < \operatorname*{essinf}_{x\in\Omega} \varrho_{0,\delta} \leq \operatorname*{esssup}_{x\in\Omega} \varrho_{0,\delta}(x) < r - \delta, \\ 0 < \delta < \operatorname*{essinf}_{x\in\Omega} \chi_{0,\delta} \leq \operatorname*{esssup}_{x\in\Omega} \chi_{0,\delta}(x) < 1 - \delta, \\ \|\nabla_x \chi_{0,\delta}\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \\ \|\mathbf{u}_{0\delta}\|_{L^2(\Omega;\mathbb{R}^3)} \leq c \text{ uniformly for } \delta \to 0. \end{array} \right\}. \tag{3.1}$$

Similarly, we introduce

$$f_\delta(\varrho,\chi) = L_\delta(\chi) - b(\chi) + \Gamma_\delta(\varrho) + \chi g_{1,\delta}(\varrho) + (1-\chi) g_{2,\delta}(\varrho), \tag{3.2}$$

with

$$L_\delta \in C^\infty \cap W^{1,\infty}(\mathbb{R}) \quad \text{a convex function}, \tag{3.3}$$

$$g_{1,\delta}, g_{2,\delta} \in C^\infty \cap L^\infty[0,\infty), \quad g'_{1,\delta}, g'_{2,\delta} \geq 0. \tag{3.4}$$

Thanks to (2.2), we can assume that, for all $\delta \in (0, 1/4)$,

$$-L'_\delta(\chi) + b'(\chi) + g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho) < 0 \quad \text{for } \chi > 1-\delta, \varrho \geq 0, \tag{3.5}$$

$$-L'_\delta(\chi) + b'(\chi) + g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho) > 0 \quad \text{for } \chi < \delta, \varrho \geq 0. \tag{3.6}$$

Finally, we take

$$\Gamma_\delta(\varrho) = \int_0^\varrho \frac{P_\delta(z)}{z^2}\,\mathrm{d}z, \tag{3.7}$$

where $P_\delta \in C^1[0,\infty)$ satisfies

$$P'_\delta(\varrho) \geq \delta\varrho + c_*\varrho^{\gamma-1}, \tag{3.8}$$

$$P_\delta(\varrho) \leq c^*(\varrho^\gamma + 1), \tag{3.9}$$

with $c_* = c^*(\delta) > 0$. The exponent $\gamma > 0$ (large enough) will be specified below.

Moreover, we may assume that

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{i,\delta}(s) + sg'_{i,\delta}(s)) \geq -\frac{c_*}{2}s^{\gamma-2} \quad \text{for } i = 1, 2, \tag{3.10}$$

where $c_* > 0$ is the same as in (3.8). Indeed the functions $g_{i,\delta}$ can simply be constructed by truncation and mollification; then it is not difficult to check that $g_{i,\delta}(s) + sg'_{i,\delta}(s)$ are monotone for small values of $s$. Thus, in order to have (3.10), it is sufficient to truncate $g_i(s)$ in a suitable way for large values of $s$. We shall assume that $L_\delta$, $\Gamma_\delta$ and $g_{i,\delta}$, tend to $L$, $\Gamma$, and $g_i$, $i = 1, 2$, respectively, uniformly on compact subsets of $(0,1)$, $(0,r)$ and $(0,\infty)$ (further details will be given in Sec. 6 below).

At the first level of the approximation procedure, the continuity equation (1.1) is supplemented with an artificial viscosity term, the momentum equation (1.2) is replaced by its Faedo–Galerkin approximation, while the Allen–Cahn system (1.3), (1.4) is provided with an extra term in order to keep the energy estimates valid. Then, the resulting *approximate system* reads:

- $\varrho$ is a smooth solution ($\varrho \in C^1([0,T]; C^\nu(\overline{\Omega})) \cap C^0([0,T]; C^{\nu+2}(\overline{\Omega}))$, strictly positive in $[0,T] \times \overline{\Omega}$) (cf. Sec. 7.3.1 of Ref. 12) to the initial-boundary value problem

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = \varepsilon\Delta\varrho, \quad \varepsilon > 0, \quad \text{in } (0,T) \times \Omega, \tag{3.11}$$

$$\nabla_x\varrho \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.12}$$

$$\varrho(0,\cdot) = \varrho_{0,\delta}; \tag{3.13}$$

- $\mathbf{u} \in C^1([0,T]; X_n)$ satisfies the integral identity

$$\begin{aligned}
&\int_0^T\int_\Omega (\varrho\mathbf{u} \cdot \partial_t\varphi + \varrho[\mathbf{u} \otimes \mathbf{u}] : \nabla_x\varphi)\mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^T\int_\Omega \varepsilon(\varphi\nabla_x\mathbf{u}) \cdot \nabla_x\varrho\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad + \int_0^T\int_\Omega \mathbb{T}(\nabla_x\chi, \nabla_x\mathbf{u}) : \nabla_x\varphi\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_{0,\delta}\mathbf{u}_{0,\delta} \cdot \varphi(0,\cdot)\mathrm{d}x
\end{aligned} \tag{3.14}$$

for any test function $\varphi \in C^1_c([0,T);X_n)$, where $X_n$ is a (suitably chosen) finite-dimensional subspace of $C^\infty_c(\Omega;\mathbb{R}^3)$;

- $\chi$, $\mu$ solve in the classical sense the system

$$\partial_t(\varrho\chi) + \mathrm{div}_x(\varrho\chi\mathbf{u}) = -\mu + \varepsilon\Delta\varrho\,\chi, \tag{3.15}$$

$$\nabla_x\chi\cdot\mathbf{n}|_{\partial\Omega} = 0, \tag{3.16}$$

$$(\varrho\chi)(0,\cdot) = \varrho_{0,\delta}\chi_{0,\delta}, \tag{3.17}$$

with

$$\varrho\mu = -\Delta\chi + \varrho\frac{\partial f_\delta(\varrho,\chi)}{\partial\chi}, \tag{3.18}$$

where $f_\delta$ was specified through (3.2)–(3.8).

Given $\varepsilon > 0$, $\delta > 0$ and $n$ finite, the approximate system (3.11)−(3.18) can be solved on the time interval $(0,T)$ by means of the Schauder fixed point argument, similarly to Chap. 7 of Ref. 12. Accordingly, the proof of Theorem 2.1 reduces to performing successively the limits $n\to\infty$, $\varepsilon\to 0$, and, finally, $\delta\to 0$.

## 4. Limit in the Faedo–Galerkin Approximations

### 4.1. *Uniform bounds*

Our first goal is to let $n\to\infty$ in the sequence of solutions $\{\varrho_n,\mathbf{u}_n,\chi_n\}_{n=1}^\infty$ to the approximate problem (3.11)−(3.18). It is a routine matter to check, in accordance with the hypotheses introduced in (3.1)−(3.8), that all quantities are regular, in particular, both (3.11) and (3.15) are satisfied in the classical sense, and the density $\varrho_n$ is bounded below away from zero.

Thus, as the first step, we use (3.11) and (3.18) to rewrite (3.15) in the form

$$\varrho_n\partial_t\chi_n + \varrho_n\mathbf{u}_n\cdot\nabla_x\chi_n = \frac{1}{\varrho_n}\Delta\chi_n - L'_\delta(\chi_n) + b'(\chi_n) + g_{2,\delta}(\varrho_n) - g_{1,\delta}(\varrho_n).$$

Then, by hypotheses (3.1), (3.5), (3.6), combined with the classical maximum principle argument, we obtain that

$$\delta \le \chi_n(t,x) \le 1-\delta \quad \text{for all } (t,x)\in[0,T]\times\bar{\Omega}, \tag{4.1}$$

where we point out that the bound is independent of both $n$ and $\varepsilon$.

Next, we aim to derive a *global energy estimate*. To obtain this, we first test (3.15) by $\mu_n$ and (3.18) by $\partial_t\chi_n$. We then notice that, by (3.11),

$$\int_\Omega \partial_t\varrho_n\chi_n\mu_n = \int_\Omega \varrho_n\chi_n\nabla_x\mu_n\cdot\mathbf{u}_n + \int_\Omega \varrho_n\mu_n\nabla_x\chi_n\cdot\mathbf{u}_n + \varepsilon\int_\Omega \Delta\varrho_n\chi_n\mu_n. \tag{4.2}$$

Thus, standard computation leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\frac{1}{2}|\nabla_x\chi_n|^2+\varrho_n f_\delta(\varrho_n,\chi_n)\right)\mathrm{d}x+\|\mu_n\|^2_{L^2(\Omega)}$$
$$=\int_\Omega(f_\delta(\varrho_n,\chi_n)+\varrho_n\partial_\varrho f_\delta(\varrho_n,\chi_n))\partial_t\varrho_n\,\mathrm{d}x-\int_\Omega\varrho_n\mu_n\nabla_x\chi_n\cdot\mathbf{u}_n\,\mathrm{d}x; \quad (4.3)$$

whence we have to handle the terms on the right-hand side.

The former can be treated expressing $\partial_t\varrho_n$ by means of (3.11). As for the latter term, we use (3.18) that gives rise to

$$-\int_\Omega\varrho_n\mu_n\nabla_x\chi_n\cdot\mathbf{u}_n\mathrm{d}x=\int_\Omega(\Delta\chi_n-\varrho_n\partial_\chi f_\delta)\nabla_x\chi_n\cdot\mathbf{u}_n\mathrm{d}x. \quad (4.4)$$

Then, taking $\mathbf{u}_n$ as a test function in (3.14), multiplying (3.11) by $|\mathbf{u}_n|^2/2$, and adding both relations to (4.3), we check that the term depending on $\Delta\chi_n$ in (4.4) cancels out with the corresponding term in the $\chi$-dependent part of the stress tensor. Consequently, integrating by parts the terms depending on $f_\delta$, and performing some additional manipulation, we end up with the *approximate total energy balance*:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[\frac{1}{2}\varrho_n|\mathbf{u}_n|^2+\frac{1}{2}|\nabla_x\chi_n|^2+\varrho_n f_\delta(\varrho_n,\chi_n)\right]\mathrm{d}x$$
$$+\int_\Omega[\mathbb{S}_n(\nabla_x\mathbf{u}_n):\nabla_x\mathbf{u}_n+|\mu_n|^2]\mathrm{d}x$$
$$-\varepsilon\int_\Omega(f_\delta(\varrho_n,\chi_n)+\varrho_n\partial_\varrho f_\delta(\varrho_n,\chi_n))\Delta\varrho_n\,\mathrm{d}x=0, \quad (4.5)$$

where $\mathbb{S}_n$ is the $n$-approximation of $\mathbb{S}$ given by (1.6).

In order to obtain suitable *uniform bounds* independent of $n$ (and, in fact, of $\varepsilon$), we have to control the last integral. To this end, we first observe that, by (3.7)–(3.8),

$$-\varepsilon\int_\Omega(\Gamma_\delta+\varrho_n\Gamma'_\delta)\Delta\varrho_n\,\mathrm{d}x=\varepsilon\int_\Omega\frac{P'_\delta(\varrho_n)}{\varrho_n}|\nabla_x\varrho_n|^2\,\mathrm{d}x$$
$$\geq\varepsilon\int_\Omega|\nabla_x\varrho_n|^2(\delta+c_*\varrho_n^{\gamma-2})\mathrm{d}x. \quad (4.6)$$

Next, we notice that, by (4.1) and (3.10),

$$-\varepsilon\int_\Omega(W_\delta(\chi_n)+\chi_n(g_{1,\delta}(\varrho_n)+\varrho_n g'_{1,\delta}(\varrho_n))+(1-\chi_n)(g_{2,\delta}(\varrho_n)+\varrho_n g'_{2,\delta}(\varrho_n)))\Delta\varrho_n\,\mathrm{d}x$$
$$\geq -c_\delta\varepsilon\|\nabla_x\varrho_n\|_{L^2(\Omega;\mathbb{R}^3)}\|\nabla_x\chi_n\|_{L^2(\Omega;\mathbb{R}^3)}-\frac{\varepsilon c_*}{2}\int_\Omega\varrho_n^{\gamma-2}|\nabla_x\varrho_n|^2\,\mathrm{d}x$$
$$\geq -\frac{\varepsilon\delta}{2}\|\nabla_x\varrho_n\|^2_{L^2(\Omega;\mathbb{R}^3)}+\varepsilon c_\delta\|\nabla_x\chi_n\|^2_{L^2(\Omega;\mathbb{R}^3)}-\frac{\varepsilon c_*}{2}\int_\Omega\varrho_n^{\gamma-2}|\nabla_x\varrho_n|^2\,\mathrm{d}x, \quad (4.7)$$

where $W_\delta=L_\delta-b$ (cf. (3.2)).

Finally, using Gronwall's lemma in (4.5), we infer that for a suitable subsequence (not relabeled) of $n \nearrow \infty$ there hold:

$$\mathbf{u}_n \to \mathbf{u} \quad \text{weakly in } L^2(0,T;H^1_0(\Omega;\mathbb{R}^3)), \tag{4.8}$$

$$\chi_n \to \chi \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega), \tag{4.9}$$

$$\mu_n \to \mu \quad \text{weakly in } L^2(0,T;L^2(\Omega)), \tag{4.10}$$

where (4.8) follows from Poincaré's and Korn's inequalities, and (4.9) also leans on (4.1). Moreover, we have

$$\|\varrho_n|\mathbf{u}_n|^2\|_{L^\infty(0,T;L^1(\Omega))} \le c \tag{4.11}$$

independently of $n$.

Finally, using (3.7)–(3.8), (4.6) and (4.1), we may infer that

$$\varrho_n \to \varrho \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;L^\gamma(\Omega)) \cap L^2(0,T;H^1(\Omega)). \tag{4.12}$$

### 4.2. *Limit in the continuity equation*

In order to derive further estimates on $\varrho_n$, we adopt (3.11) the procedure described in Lemma 7.5 of Ref. 12. In what follows, we assume that $\gamma \ge 6$ (cf. (3.8)), observing that, in fact, it is enough to take $\gamma = 6$ in most steps.

We rewrite (3.11) as

$$\partial_t \varrho_n + A_\varepsilon \varrho_n = \varrho_n - \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \varrho_n - \varrho_n \mathrm{div}_x \mathbf{u}_n - \mathbf{u}_n \cdot \nabla_x \varrho_n, \tag{4.13}$$

where we have set $A_\varepsilon = \mathrm{Id} - \varepsilon\Delta$, and where $\Delta$ denotes the Laplace operator supplemented with the homogeneous Neumann boundary conditions and Id stands for the identity operator. We also introduce, for $s \in \mathbb{R}$, $\mathcal{H}^{2s} := D(A^s_\varepsilon)$, the domain of $A^s_\varepsilon$, and, correspondingly, for $v \in H^{2s}$, $\|v\|_{2s} := \|A^s_\varepsilon v\|_{L^2(\Omega)}$. Then, by (4.8), (4.12) and standard interpolation and embeddings,

$$\|\mathrm{div}_x(\varrho_n \mathbf{u}_n)\|_{L^1(0,T;\mathcal{H}^{-1/2})} \le c\|\mathrm{div}_x(\varrho_n \mathbf{u}_n)\|_{L^1(0,T;L^{3/2}(\Omega))} \le c. \tag{4.14}$$

Next, thanks to (4.11) and (4.12),

$$\|\varrho_n \mathbf{u}_n\|_{L^\infty(0,T;\mathcal{H}^{-1/4})} \le c\|\varrho_n \mathbf{u}_n\|_{L^\infty(0,T;L^{12/7}(\Omega))} \le c, \tag{4.15}$$

whence

$$\|\mathrm{div}_x(\varrho_n \mathbf{u}_n)\|_{L^\infty(0,T;\mathcal{H}^{-5/4})} \le c. \tag{4.16}$$

Interpolating between (4.14) and (4.16) it then follows that

$$\|\mathrm{div}_x(\varrho_n \mathbf{u}_n)\|_{L^p(0,T;\mathcal{H}^{-1})} \le c \quad \text{for some } p > 2; \tag{4.17}$$

whence, applying the $L^p$-regularity theory to (4.13) (notice that, indeed, $\mathcal{H}^{-1} = (H^1)^*$), we get

$$\|\varrho_n\|_{L^p(0,T;H^1)} \le c \quad \text{for some } p > 2. \tag{4.18}$$

Thus, using once more (4.8), we can improve (4.14) to

$$\|\mathrm{div}_x(\varrho_n \mathbf{u}_n)\|_{L^p((0,T)\times\Omega)} \le c \quad \text{for some } p > 1. \tag{4.19}$$

Applying the $L^p$-theory to (4.13), we finally have

$$\partial_t \varrho_n \to \partial_t \varrho, \quad \Delta \varrho_n \to \Delta \varrho \quad \text{weakly in } L^p((0,T)\times\Omega) \quad \text{for some } p > 1. \tag{4.20}$$

Consequently, in accordance with (4.12),

$$\varrho_n \to \varrho \quad \text{strongly in } L^p((0,T)\times\Omega) \quad \text{for all } p \in [1,\gamma) \tag{4.21}$$

and

$$\varrho_n \to \varrho \quad \text{in } C_w([0,T]; L^\gamma(\Omega)). \tag{4.22}$$

Then, by (4.8), we also get

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega), \tag{4.23}$$

so that we can take the limit $n \nearrow \infty$ in (3.11).

### 4.3. *Limit in the Allen–Cahn equation*

Firstly, we notice that, by (4.8), (4.9) and (4.12) (recall that $\gamma \geq 6$),

$$\|\varrho_n \chi_n\|_{L^\infty(0,T;L^6(\Omega))} \leq c, \tag{4.24}$$

$$\|\varrho_n \chi_n \mathbf{u}_n\|_{L^2(0,T;L^3(\Omega))} \leq c. \tag{4.25}$$

Moreover, expanding $\mathrm{div}_x(\varrho_n \chi_n \mathbf{u}_n)$ by Leibnitz formula and using again (4.18), we easily see that

$$\|\mathrm{div}_x(\varrho_n \chi_n \mathbf{u}_n)\|_{L^p((0,T)\times\Omega)} \leq c \quad \text{for some } p > 1. \tag{4.26}$$

Since the same bound holds for the right-hand side of (3.15), relations (4.1) and (4.20) give rise to

$$\|\partial_t(\varrho_n \chi_n)\|_{L^p((0,T)\times\Omega)} \leq c \quad \text{for some } p > 1. \tag{4.27}$$

Now, let us handle the last term in (3.18) (cf. (1.8), (1.10)). We obtain

$$\varrho_n \frac{\partial f_\delta(\varrho_n, \chi_n)}{\partial \chi} = \varrho_n W_\delta'(\chi_n) + \varrho_n (g_{1,\delta}(\varrho_n) - g_{2,\delta}(\varrho_n)). \tag{4.28}$$

By (4.21), (3.4) and Lebesgue's theorem, we then infer that

$$\varrho_n (g_{1,\delta}(\varrho_n) - g_{2,\delta}(\varrho_n)) \to \varrho (g_{1,\delta}(\varrho) - g_{2,\delta}(\varrho)) \quad \text{strongly in } L^2((0,T)\times\Omega). \tag{4.29}$$

Moreover, by virtue of (4.9), (4.22), (4.1) and (3.3),

$$\varrho_n W_\delta'(\chi_n) \to \varrho \overline{W_\delta'(\chi)} \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T; L^6(\Omega)). \tag{4.30}$$

Finally, due to (4.10) and (4.21), we have

$$\varrho_n \mu_n \to \varrho \mu \quad \text{weakly in } L^1((0,T)\times\Omega). \tag{4.31}$$

Thus we can take the limit in (3.18) to get

$$\varrho \mu = -\Delta \chi + \varrho \overline{W_\delta'(\chi)} + \varrho (g_{1,\delta}(\varrho) - g_{2,\delta}(\varrho)). \tag{4.32}$$

Next, we test (3.18) by $\chi_n$ and integrate over $(0,T)\times\Omega$:

$$\int_0^T\int_\Omega |\nabla_x\chi_n|^2\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \varrho_n\mu_n\chi_n\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \varrho_n W_\delta'(\chi_n)\chi_n\,\mathrm{d}x\,\mathrm{d}t - \int_0^T\int_\Omega \varrho_n(g_{1,\delta}(\varrho_n) - g_{2,\delta}(\varrho_n))\chi_n\,\mathrm{d}x\,\mathrm{d}t. \tag{4.33}$$

Observe that

$$\partial_t(\varrho_n W_\delta'(\chi_n)) = \partial_t\varrho_n(W_\delta'(\chi_n) - W_\delta''(\chi_n)\chi_n) + W_\delta''(\chi_n)\partial_t(\varrho_n\chi_n); \tag{4.34}$$

whence, by (4.1), (3.3), (4.20) and (4.27),

$$\|\partial_t(\varrho_n W_\delta'(\chi_n))\|_{L^p((0,T)\times\Omega)} \le c \quad \text{for some } p>1. \tag{4.35}$$

This implies that (4.30) can be improved to

$$\varrho_n W_\delta'(\chi_n) \to \varrho\overline{W_\delta'(\chi)} \quad \text{in } C_w([0,T];L^6(\Omega)). \tag{4.36}$$

Using (4.9), we deduce

$$\varrho_n W_\delta'(\chi_n)\chi_n \to \varrho\overline{W_\delta'(\chi)}\chi \quad \text{weakly in } L^2((0,T)\times\Omega). \tag{4.37}$$

Thus, we can take the limit $n\nearrow\infty$ in (4.33). Indeed, the first term on the right-hand side can be treated in a similar (and in fact simpler) way. Comparing the result with (4.32) integrated in space and time and using also Poincaré's inequality in the form as in Lemma 3.1 of Ref. 15, we finally obtain

$$\chi_n \to \chi \quad \text{strongly in } L^2(0,T;H^1(\Omega)) \tag{4.38}$$

and, consequently, $\overline{W_\delta'(\chi)} = W_\delta'(\chi)$. Thus, we can take the limit of (3.18).

Finally, we aim to take the limit of (3.15). Here, we simply notice that, by (4.38), (4.1) and Lebesgue's theorem,

$$\chi_n \to \chi \quad \text{strongly in } L^p((0,T)\times\Omega) \quad \text{for all } p\in[1,\infty). \tag{4.39}$$

Thus, by virtue of the strong convergence established in (4.38) and (4.21), it is easy to pass to the limit. In particular, the last term on the right-hand side is treated by means of (4.38) and (4.20).

### 4.4. *Limit in the momentum equation*

We choose $\varphi\in C_c^1([0,T);X_m)$ for fixed $m\in\mathbb{N}$ and examine the identity (3.14) for $n\ge m$. Our aim is to let $n\nearrow\infty$. First, we consider the stress−tensor $\mathbb{T}$ given by (1.5). It is clear that the components specified in (1.6) admit limits thanks to hypothesis (2.8) and relations (4.8) and (4.39). Next, the terms depending only on $\chi$ in (1.5) are treated by means of (4.38). Finally, to take the limit of the pressure term contained in $\mathbb{T}$, we observe that, thanks to (4.6),

$$\varrho_n \to \varrho \quad \text{weakly in } L^\gamma(0,T;L^{3\gamma}(\Omega)); \tag{4.40}$$

whence the desired conclusion follows by (4.22), interpolation and Lebesgue's theorem (cf. assumption (3.9)).

Now, let us notice that, by means of (4.15), the integral relation (3.14), and Ascoli's theorem (cf. e.g. Corollary 2.1 of Ref. 12), we have

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \quad \text{in } C_w([0,T]; L^{12/7}(\Omega)); \tag{4.41}$$

whence, by (4.8),

$$\varrho_n(\mathbf{u}_n \otimes \mathbf{u}_n) \to \varrho(\mathbf{u} \otimes \mathbf{u}) \quad \text{weakly in } L^2(0,T; L^{4/3}(\Omega)). \tag{4.42}$$

Finally, to treat the first term on the right-hand side of (3.14) we need to prove that

$$\varrho_n \to \varrho \quad \text{strongly in } L^2(0,T; H^1(\Omega)). \tag{4.43}$$

To obtain this, we proceed similarly to the Allen–Cahn equation. Namely, we test (3.11) by $\varrho_n$ and integrate over $(0,T) \times \Omega$. We then notice that

$$\int_\Omega \operatorname{div}_x(\varrho_n \mathbf{u}_n)\varrho_n \,\mathrm{d}x = \int_\Omega \frac{\varrho_n^2}{2} \operatorname{div}_x \mathbf{u}_n \,\mathrm{d}x, \tag{4.44}$$

and the same holds for the limit functions $\varrho$, $\mathbf{u}$. Thus, using (4.22) to treat the term depending on the initial datum, we may infer that

$$\limsup_{n \nearrow \infty} \int_0^T \int_\Omega |\nabla_x \varrho_n|^2 \,\mathrm{d}x\,\mathrm{d}t \le \int_0^T \int_\Omega |\nabla_x \varrho|^2 \,\mathrm{d}x\,\mathrm{d}t, \tag{4.45}$$

which implies (4.43). Thus, the limit of (3.14) holds for any $\varphi \in C_c^1([0,T]; X_m)$ and, due to arbitrariness of $m$ and density of $\cup X_m$, holds also for $\varphi \in C_c^1([0,T) \times \Omega)$, as desired.

### 4.5. *Limit in the energy inequality*

In order to perform the passage $\varepsilon \searrow 0$, we need to prove that the solution constructed above still satisfies a suitable version of the equality (4.5). Notice that this cannot be achieved by using test functions in the limit equations as at this level the solutions are no longer regular. Instead we take the limit in (4.5) for $n \nearrow \infty$. Integrating (4.5) over $(0,\tau)$, $\tau \in (0,T]$, we obtain

$$\begin{aligned}
&\int_\Omega \left( \frac{1}{2} \varrho_n(\tau)|\mathbf{u}_n(\tau)|^2 + \frac{1}{2}|\nabla_x \chi_n(\tau)|^2 + \varrho_n(\tau) f_\delta(\varrho_n(\tau), \chi_n(\tau)) \right) \mathrm{d}x \\
&\quad + \int_0^\tau \int_\Omega \mathbb{S}_n(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n \,\mathrm{d}x\,\mathrm{d}t + \int_0^\tau \int_\Omega |\mu_n|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon \int_0^\tau \int_\Omega (\varrho_n^{-1} P_\delta'(\varrho_n) + \chi_n h_{1,\delta}'(\varrho_n) + (1-\chi_n) h_{2,\delta}'(\varrho_n)) |\nabla_x \varrho_n|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \varepsilon \int_0^\tau \int_\Omega (W_\delta'(\chi_n) + h_{1,\delta}(\varrho_n) - h_{2,\delta}(\varrho_n)) \nabla_x \chi_n \cdot \nabla_x \varrho_n \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_\Omega \left( \frac{1}{2} \varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{1}{2}|\nabla_x \chi_{0,\delta}|^2 + \varrho_{0,\delta} f_\delta(\varrho_{0,\delta}, \chi_{0,\delta}) \right) \mathrm{d}x,
\end{aligned} \tag{4.46}$$

where $h_{i,\delta}(s) = g_{i,\delta}(s) + sg'_{i,\delta}(s)$, $i = 1, 2$. Our aim is to take the lim inf as $n \nearrow \infty$ in the above equality.

First notice, by (4.38) and (4.43), that

$$|\nabla_x \varrho_n|^2 \to |\nabla_x \varrho|^2, \quad \nabla_x \varrho_n \nabla_x \chi_n \to \nabla_x \varrho \cdot \nabla_x \chi \quad \text{strongly in } L^1((0,T) \times \Omega), \tag{4.47}$$

and, on the other hand,

$$\chi_n h'_{1,\delta}(\varrho_n) + (1 - \chi_n) h'_{2,\delta}(\varrho_n) + W'_\delta(\chi_n) + h_{1,\delta}(\varrho_n) - h_{2,\delta}(\varrho_n) \tag{4.48}$$

converges to the corresponding limit weakly-(*) in $L^\infty((0,T) \times \Omega)$. Moreover,

$$\int_0^T \int_\Omega \varrho_n^{-1} P'_\delta(\varrho_n) |\nabla_x \varrho_n|^2 \,\mathrm{d}x \,\mathrm{d}t = \int_0^T \int_\Omega |\nabla_x Q_\delta(\varrho_n)|^2 \,\mathrm{d}x \,\mathrm{d}t, \tag{4.49}$$

where $(Q'_\delta)^2 = P'_\delta/\varrho$. Moreover, we check easily that $Q_\delta(\varrho_n) \to Q_\delta(\varrho)$ weakly in $L^2(0,T; H^1(\Omega))$.

Concerning the other terms, we observe that

$$\int_0^T \int_\Omega \nu(\chi_n) \nabla_x \mathbf{u}_n : \nabla_x \mathbf{u}_n \,\mathrm{d}x \,\mathrm{d}t = \|\psi_n\|^2_{L^2((0,T)\times\Omega)}, \tag{4.50}$$

where

$$\psi_n = \nu^{1/2}(\chi_n) \nabla_x \mathbf{u}_n \to \psi = \nu^{1/2}(\chi) \nabla_x \mathbf{u} \quad \text{weakly in } L^2((0,T) \times \Omega; \mathbb{R}^{3\times3}). \tag{4.51}$$

Thus, one can compute the lim inf of all terms on the left-hand side of (4.46) except those on the first line, evaluated pointwise in time. To deal with these, one has to perform one more integration in terms of the energy equality. Thus, we obtain that

$$\begin{aligned} &\int_t^{t+\tau} \int_\Omega \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{1}{2} |\nabla_x \chi|^2 + \varrho f_\delta(\varrho, \chi) \right) \mathrm{d}x \,\mathrm{d}s \\ &\quad \le \liminf_{n \nearrow \infty} \int_t^{t+\tau} \int_\Omega \left( \frac{1}{2} \varrho_n |\mathbf{u}_n|^2 + \frac{1}{2} |\nabla_x \chi_n|^2 + \varrho_n f_\delta(\varrho_n, \chi_n) \right) \mathrm{d}x \,\mathrm{d}s \end{aligned} \tag{4.52}$$

for all $\tau > 0$. Then, thanks to semi-continuity of norms with respect to the weak or weak-(*) convergence, we obtain the limit form of the energy estimate. Dividing by $\tau$ and letting $\tau \searrow 0$, the limit energy inequality is then recovered in the original form and for a.a. value $t$ of the time variable.

## 5. Artificial Viscosity Limit

Our aim is to let $\varepsilon \searrow 0$. In accordance with the preceding step, we can assume to have a family of approximate solutions $\{\mathbf{u}_\varepsilon, \varrho_\varepsilon, \mu_\varepsilon, \chi_\varepsilon\}_{\varepsilon>0}$ satisfying (3.11)–(3.18). At this stage, the regularity properties of $\mathbf{u}_\varepsilon, \varrho_\varepsilon, \mu_\varepsilon, \chi_\varepsilon$ are those established in the previous step. In addition, as stated in Sec. 4.5, we also know that it is possible to perform the limit $n \nearrow \infty$ in (4.46).

### 5.1. *Limit in the continuity equation*

We rewrite (the $\varepsilon$-version) of (4.13) as

$$\partial_t \varrho_\varepsilon + \varepsilon A \varrho_\varepsilon = \varepsilon \varrho_\varepsilon - \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon). \tag{5.1}$$

Here, similarly to Sec. 4.2, $A = \mathrm{Id} - \Delta$ (Id being the identity operator), with the homogeneous Neumann boundary conditions, and, for $s \in \mathbb{R}$, $\mathcal{H}^{2s} := D(A^s)$ with the natural norms. By means of the $\varepsilon$-analogues of (4.8) and (4.12), it is not difficult to conclude that

$$\|\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)\|_{L^2(0,T;\mathcal{H}^{-1})} + \|\mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon)\|_{L^\infty(0,T;\mathcal{H}^{-5/4})} \leq c. \tag{5.2}$$

Here and hereafter, the constants $c$ are independent of $\varepsilon$. Standard parabolic estimates (namely, testing (5.1) by $\varepsilon\varrho_\varepsilon$) yield

$$\|\varrho_\varepsilon\|_{H^1(0,T;\mathcal{H}^{-1})} + \varepsilon^{1/2}\|\varrho_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} + \varepsilon\|\varrho_\varepsilon\|_{L^2(0,T;\mathcal{H}^1)} \leq c. \tag{5.3}$$

Using the energy inequality we have

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_w([0,T];L^6(\Omega)), \tag{5.4}$$

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2(0,T;H_0^1(\Omega;\mathbb{R}^3)), \tag{5.5}$$

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \quad \text{weakly in } L^2(0,T;L^3(\Omega;R^3)), \tag{5.6}$$

in particular, we can pass to the limit in (5.1).

Since $\varrho_\varepsilon$, $\mathbf{u}_\varepsilon$ satisfy (5.1), we can use the regularization procedure introduced by DiPerna and Lions[11] to show that $\varrho_\delta, \mathbf{u}_\delta$ represent a *renormalized* solution of Eq. (1.1). Namely, there holds

$$\begin{aligned} &\int_0^T\int_\Omega (b(\varrho_\delta)\partial_t\varphi + b(\varrho_\delta)\mathbf{u}_\delta \cdot \nabla_x\varphi + (b(\varrho_\delta) - b'(\varrho_\delta)\varrho_\delta)\mathrm{div}_x\mathbf{u}_\delta\varphi)\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\int_\Omega b(\varrho_{0,\delta})\varphi(0,\cdot)\mathrm{d}x, \end{aligned} \tag{5.7}$$

for any $\varphi \in C_c^\infty([0,T) \times \bar{\Omega})$ and any $b \in W^{1,\infty}[0,\infty)$. It is easy to see, at least formally, that such a formula can be deduced by testing the limit equation of (5.1) by $b'(\varrho_\delta)\varphi$ for any $\varphi \in C_c^\infty([0,T) \times \bar{\Omega})$ and integrating by parts.

### 5.2. *Limit in the Allen–Cahn system and in the momentum equation*

Our aim is to let $\varepsilon \searrow 0$ in the system

$$\partial_t(\varrho_\varepsilon\chi_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon\chi_\varepsilon\mathbf{u}_\varepsilon) = -\mu_\varepsilon + \varepsilon\Delta\varrho_\varepsilon\chi_\varepsilon, \tag{5.8}$$

$$\varrho_\varepsilon\mu_\varepsilon = -\Delta\chi_\varepsilon + \varrho_\varepsilon\frac{\partial f_\delta(\varrho_\varepsilon,\chi_\varepsilon)}{\partial\chi_\varepsilon}. \tag{5.9}$$

Let us first notice that, by the energy inequality, the $\varepsilon$-analogue of (4.1), (5.4) and Poincaré's inequality (once more in the form of Lemma 3.1 of Ref. 15), we have

$$\mu_\varepsilon \to \mu \quad \text{weakly in } L^2((0,T)\times\Omega), \tag{5.10}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;H^1(\Omega)) \cap L^\infty((0,T)\times\Omega). \tag{5.11}$$

Thus, using once more (5.4), we also obtain

$$\varrho_\varepsilon\chi_\varepsilon \to \varrho\chi \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;L^6(\Omega)). \tag{5.12}$$

Moreover, by (5.5),

$$\|\varrho_\varepsilon\chi_\varepsilon\mathbf{u}_\varepsilon\|_{L^2(0,T;L^3(\Omega))} \le c. \tag{5.13}$$

The energy inequality, hypothesis (3.4), relations (4.1) and (5.4), and a comparison of (5.9) give rise to

$$\|\varrho_\varepsilon\mu_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} + \|\Delta\chi_\varepsilon\|_{L^2(0,T;L^{3/2}(\Omega))} \le c. \tag{5.14}$$

Let us pick $\phi \in C^1_c([0,T);H^2_0(\Omega))$ and notice that

$$\begin{aligned} &\varepsilon\int_0^T\!\!\int_\Omega \Delta\varrho_\varepsilon\chi_\varepsilon\phi\,\mathrm{d}x\,\mathrm{d}t \\ &\quad = -\varepsilon\int_0^T\!\!\int_\Omega \phi\nabla_x\varrho_\varepsilon\cdot\nabla_x\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t - \varepsilon\int_0^T\!\!\int_\Omega \chi_\varepsilon\nabla_x\varrho_\varepsilon\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{5.15}$$

Thus, testing (5.8) by $\phi$, integrating over $(0,T)\times\Omega$, and making use of relations (5.13), (5.10) and (5.3), we easily see that

$$\|\partial_t(\varrho_\varepsilon\chi_\varepsilon)\|_{L^2(0,T;H^{-1}(\Omega))+L^2(0,T;L^1(\Omega))} \le c, \tag{5.16}$$

whence (5.12) is improved to

$$\varrho_\varepsilon\chi_\varepsilon \to \varrho\chi \quad \text{in } C_w([0,T];L^6(\Omega)) \tag{5.17}$$

and, consequently, by (5.5), we also have

$$\varrho_\varepsilon\chi_\varepsilon\mathbf{u}_\varepsilon \to \varrho\chi\mathbf{u} \quad \text{weakly in } L^2(0,T;L^3(\Omega;\mathbb{R}^3)). \tag{5.18}$$

In order to pass to the limit in (5.8), we use $\phi$ as a test function and observe that the right-hand side of (5.15) can be transformed into

$$\varepsilon\int_0^T\!\!\int_\Omega \phi\varrho_\varepsilon\Delta\chi_\varepsilon\,\mathrm{d}x\,\mathrm{d}t + 2\varepsilon\int_0^T\!\!\int_\Omega \varrho_\varepsilon\nabla_x\chi_\varepsilon\cdot\nabla_x\phi\,\mathrm{d}x\,\mathrm{d}t + \varepsilon\int_0^T\!\!\int_\Omega \varrho_\varepsilon\chi_\varepsilon\Delta\phi\,\mathrm{d}x\,\mathrm{d}t, \tag{5.19}$$

where all terms clearly go to 0 for $\phi$ as above.

Next, we take the limit in (5.9), which is more involved. First, we observe that

$$W'_\delta(\chi_\varepsilon) \to \overline{W'_\delta(\chi)} \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;H^1(\Omega)). \tag{5.20}$$

Thus, due to (5.4),

$$\varrho_\varepsilon W'_\delta(\chi_\varepsilon) \to \varrho\overline{W'_\delta(\chi)} \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;L^3(\Omega)), \tag{5.21}$$

and, thanks to (5.17),

$$\varrho_\varepsilon\chi_\varepsilon W'_\delta(\chi_\varepsilon) \to \varrho\chi\overline{W'_\delta(\chi)} \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;L^3(\Omega)). \tag{5.22}$$

At this stage, we follow step by step the procedure developed in Sec. 2.6 of Ref. 3. Accordingly, we focus on the principal steps only. First, we claim that, by (5.11) and (5.17),

$$\int_0^T\int_\Omega \varrho\chi_\varepsilon^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \varrho\chi^2 \,\mathrm{d}x\,\mathrm{d}t. \tag{5.23}$$

As a matter of fact, we can simply check that

$$\chi_\varepsilon^2 \to \overline{\chi^2} \quad \text{weakly in } L^2(0,T;H^1(\Omega)), \tag{5.24}$$

whence

$$\varrho\chi_\varepsilon^2 - \varrho\chi^2 = (\varrho\chi_\varepsilon^2 - \varrho_\varepsilon\chi_\varepsilon^2) + (\varrho_\varepsilon\chi_\varepsilon^2 - \varrho_\varepsilon\chi_\varepsilon\chi) + (\varrho_\varepsilon\chi_\varepsilon\chi - \varrho\chi^2), \tag{5.25}$$

where the last three summands on the right-hand side go to 0 due to (5.11), (5.17) and (5.24). Thus, it follows that

$$\chi_\varepsilon \to \chi \quad \text{strongly in } L^p(Q_T^+) \quad \text{for all } p \in [1,\infty), \tag{5.26}$$

where $Q_T^+(Q_T^0)$ denotes the subset of $(0,T)\times\Omega$ where $\varrho > 0$ $(\varrho = 0)$. Moreover, since $\varrho$ is non-negative, it is clear that

$$\varrho_\varepsilon \to \varrho \quad \text{strongly in } L^p(Q_T^0) \quad \text{for all } p \in [1,6). \tag{5.27}$$

Now, we may rewrite (5.9) as

$$\varrho_\varepsilon\mu_\varepsilon = -\Delta\chi_\varepsilon + \varrho_\varepsilon W'_\delta(\chi_\varepsilon) + h_\delta(\varrho_\varepsilon), \tag{5.28}$$

where we have set $h_\delta(s) = s(g_{1,\delta}(s) - g_{2,\delta}(s))$. At this level, taking the limit (in the sense of distributions) yields

$$\overline{\varrho\mu} = -\Delta\chi + \varrho\overline{W'_\delta(\chi)} + \overline{h_\delta(\varrho)}. \tag{5.29}$$

Now, as in Sec. 4.3, we test (5.28) by $\chi_\varepsilon$ and integrate over $(0,T)\times\Omega$. It is clear, thanks to (5.20)–(5.22), that the desired conclusion

$$\chi_\varepsilon \to \chi \quad \text{strongly in } L^2(0,T;H^1(\Omega)) \tag{5.30}$$

follows as soon as we can prove that

$$\int_0^T\int_\Omega (\overline{\varrho\mu\chi} - \overline{h_\delta(\varrho)\chi})\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega (\overline{\varrho\mu}\chi - \overline{h_\delta(\varrho)}\chi)\mathrm{d}x\,\mathrm{d}t. \tag{5.31}$$

Note that, by (5.27),

$$\iint_{Q_T^0} \overline{h_\delta(\varrho)}\chi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^0} h_\delta(\varrho)\chi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^0} \overline{h_\delta(\varrho)\chi} \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{5.32}$$

whereas, thanks to (5.26),

$$\iint_{Q_T^+} \overline{h_\delta(\varrho)}\chi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^+} \overline{h_\delta(\varrho)\chi} \,\mathrm{d}x\,\mathrm{d}t. \tag{5.33}$$

Analogously,

$$\iint_{Q_T^0} \overline{\varrho\mu}\chi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^0} \varrho\mu\chi \,\mathrm{d}x\,\mathrm{d}t = 0 \tag{5.34}$$

and, on the other hand,

$$\iint_{Q_T^0} \overline{\varrho\mu\chi} \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^0} \varrho\overline{\mu\chi} \,\mathrm{d}x\,\mathrm{d}t = 0. \tag{5.35}$$

Moreover, still by (5.26),

$$\iint_{Q_T^+} \overline{\varrho\mu}\chi \,\mathrm{d}x\,\mathrm{d}t = \iint_{Q_T^+} \overline{\varrho\mu\chi} \,\mathrm{d}x\,\mathrm{d}t. \tag{5.36}$$

Thus, collecting (5.32)–(5.36), we get (5.31), and, consequently, (5.30). Moreover, we arrive at the relation

$$\overline{\varrho\mu} = -\Delta\chi + \varrho W_\delta'(\chi) + \overline{h_\delta(\varrho)}. \tag{5.37}$$

In order to identify the remaining two terms, we need strong convergence of $\varrho_\varepsilon$, whose proof will be discussed later.

Finally, we pass to the limit in the momentum equation, that now reads

$$\begin{aligned} &\int_0^T\int_\Omega (\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t\varphi + \varrho_\varepsilon[\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon] : \nabla_x\varphi)\mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T\int_\Omega \mathbb{T}(\nabla_x\chi_\varepsilon, \nabla_x\mathbf{u}_\varepsilon) : \nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_{0,\delta}\mathbf{u}_{0,\delta} \cdot \varphi(0,\cdot)\mathrm{d}x, \end{aligned} \tag{5.38}$$

for $\varphi$ as in (2.12). Actually, it follows from (5.38) that $\partial_t(\varrho_\varepsilon\mathbf{u}_\varepsilon)$ is uniformly bounded in some negative order Sobolev space. Thus, using (5.4) and (5.5), we conclude as before that

$$\varrho_\varepsilon\mathbf{u}_\varepsilon \to \varrho\mathbf{u} \quad \text{in } C_w([0,T]; L^{12/7}(\Omega)) \tag{5.39}$$

as well as

$$\varrho_\varepsilon(\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) \to \varrho(\mathbf{u} \otimes \mathbf{u}) \quad \text{weakly in } L^2(0,T; L^{4/3}(\Omega)). \tag{5.40}$$

Next, thanks to (5.5) and (5.30), the tensor $\mathbb{T}$ can be treated as in Sec. 4.4, with the exception of the pressure term $p_\delta(\varrho_\varepsilon, \chi_\varepsilon)$, whose treatment also requires the strong convergence of $\varrho_\varepsilon$.

### 5.3. *Conclusion of the proof*

Our main task is to obtain strong $L^1$-convergence of the densities $\varrho_\varepsilon$. For the sake of clarity, we just give the highlights of this procedure, referring to the next section where the same arguments will be repeated (in fact in an even more delicate situation).

- As a first step, we proceed as in Sec. 6.2.1 below, i.e. we use the analogue of the test function in (6.24). By just adapting the notation, we then arrive at the analogue of (6.26). In the present situation, thanks to (3.8)–(3.9), this gives in particular

$$p(\varrho_\varepsilon, \chi_\varepsilon) \to \overline{p(\varrho, \chi)} \quad \text{weakly in } L^{(\gamma+1)/\gamma}((0,T)\times\Omega). \tag{5.41}$$

- Second, we have to perform Lions' argument as in Sec. 6.5. Following step by step that procedure, we then arrive at a pointwise (a.e.) convergence $\varrho_\varepsilon \to \varrho$, which permits in particular to identify the remaining limits in (5.37) and (5.41). The only difference with respect to estimate (6.45) consists in the presence of two other terms

$$-\varepsilon \int_0^T \psi \int_\Omega \xi \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}(\mathrm{div}_x(1_\Omega \nabla_x \varrho_\varepsilon))\mathrm{d}x\,\mathrm{d}t$$

  and

$$\varepsilon \int_0^T \psi \int_\Omega \xi \nabla_x \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon \cdot \nabla_x \Delta^{-1}(1_\Omega \varrho_\varepsilon)\mathrm{d}x\,\mathrm{d}t,$$

  which tend to zero as $\varepsilon \searrow 0$ due to (5.15), (5.39) and (5.45).

Having the strong convergence of $\varrho_\varepsilon$ at our disposal, it is now clear that we can take the limit $\varepsilon \searrow 0$ in all equations. To conclude, it remains to pass to the limit in the energy inequality (cf. (4.46)). To do this, we can use the argument similar to that in Sec. 4.5. The main difference is that now we also have to test (5.1) by $\varrho_\varepsilon$ and deduce that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho_\varepsilon^2\,\mathrm{d}x + \varepsilon\int_\Omega |\nabla_x \varrho_\varepsilon|^2\,\mathrm{d}x \le -\frac{1}{2}\int_\Omega \varrho_\varepsilon^2 |\mathrm{div}_x \mathbf{u}_\varepsilon|\mathrm{d}x. \tag{5.42}$$

This immediately leads to

$$\varepsilon^{1/2}\|\varrho_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le c. \tag{5.43}$$

More precisely, integrating (5.42) in time, we obtain

$$\begin{aligned}\varepsilon\int_0^t\int_\Omega |\nabla_x \varrho_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}s &\le -\frac{1}{2}\int_\Omega \varrho_\varepsilon^2(t)\mathrm{d}x + \frac{1}{2}\int_\Omega \varrho_{0,\delta}^2\,\mathrm{d}x \\ &\quad - \frac{1}{2}\int_0^T\int_\Omega \varrho_\varepsilon^2 \mathrm{div}_x \mathbf{u}_\varepsilon\,\mathrm{d}x\,\mathrm{d}s,\end{aligned} \tag{5.44}$$

so that, taking the lim sup as $\varepsilon \searrow 0$, using semicontinuity of norms w.r.t. weak convergence, and comparing the result with the limit momentum equation (5.7) (in the renormalized form (5.7) with $b(\varrho) = \varrho$), we may infer that

$$\varepsilon^{1/2} \varrho_\varepsilon \to 0 \quad \text{strongly in } L^2(0,T;H^1(\Omega)). \tag{5.45}$$

Thus, the limit energy inequality can be computed as in Sec. 4.5.

## 6. Artificial Pressure Limit

Our ultimate goal is to let $\delta \searrow 0$. To this end, consider a family $\{\varrho_\delta, \mathbf{u}_\delta, \chi_\delta\}_{\delta>0}$ of the approximate solutions constructed in the previous part. Accordingly, we choose the initial data in (3.1) in such a way that

$$\left\{ \begin{array}{c} \operatorname*{essinf}_\Omega \varrho_0 \leq \varrho_{0,\delta}(x) \leq \operatorname*{esssup}_\Omega \varrho_0 < r,\ x \in \Omega,\ \varrho_{0,\delta} \to \varrho_0, \\ \operatorname*{essinf}_\Omega \chi_0 \leq \chi_{0,\delta}(x) \leq \operatorname*{esssup}_\Omega \chi_0,\ x \in \Omega,\ \chi_{0,\delta} \to \chi_0 \text{ a.e. in } \Omega, \\ \mathbf{u}_{0,\delta} \to \mathbf{u}_0 \text{ in } L^2(\Omega;\mathbb{R}^3) \end{array} \right\} \tag{6.1}$$

as $\delta \searrow 0$.

In addition, it is a routine to construct a family of convex functions $L_\delta \in C^\infty(\mathbb{R})$ such that

$$L_\delta(\chi) \nearrow L(\chi) \quad \text{for any } \chi \in (0,1), \tag{6.2}$$

and

$$\begin{aligned} L'_\delta(\chi) &\geq L'(1-\delta) \quad \text{for all } \chi > 1-\delta, \\ L'_\delta(\chi) &\leq L'(\delta) \quad \text{for all } \chi < \delta \end{aligned} \tag{6.3}$$

for a suitable sequence $\delta \searrow 0$. Consequently, in accordance with hypothesis (2.2), we can find the functions $g_{1,\delta}$, $g_{2,\delta}$ such that (3.4)–(3.6) hold, and, in addition,

$$\left. \begin{array}{ll} g_{i,\delta}(\varrho) \nearrow a_i \log(\varrho) & \text{for } \varrho \geq 1, \\ g_{i,\delta}(\varrho) \searrow a_i \log(\varrho) & \text{for } 0 \leq \varrho \leq 1 \end{array} \right\} \quad \text{as } \delta \searrow 0,\ i = 1,2, \tag{6.4}$$

$$g'_{i,\delta}(\varrho) \to \frac{a_i}{\varrho} \quad \text{in } C(0,\infty) \quad \text{as } \delta \searrow 0,\ i = 1,2. \tag{6.5}$$

Finally, we take

$$P_\delta(\varrho) = \delta\varrho^2 + \begin{cases} P(\varrho) & \text{if } 0 \leq \varrho \leq r-\delta, \\ P(r-\delta) + ([\varrho - r - 1]^+)^\gamma & \text{if } \varrho \geq r-\delta. \end{cases} \tag{6.6}$$

### 6.1. *Uniform bounds*

To begin, we recall (cf. (4.1)) that the functions $\chi_\delta$ satisfy the uniform bound

$$\delta \leq \chi_\delta(t,x) \leq 1-\delta \quad \text{for a.a. } t \in (0,T) \times \Omega. \tag{6.7}$$

Moreover, the energy inequality

$$\begin{aligned}\int_0^T\int_\Omega \left[\frac{1}{2}(\varrho_\delta|\mathbf{u}_\delta|^2 + |\nabla_x\chi_\delta|^2) + \varrho_\delta f_\delta(\varrho_\delta,\chi_\delta)\right]\mathrm{d}x\partial_t\psi\,\mathrm{d}t \\ - \int_0^T\int_\Omega [\mathbb{S}_\delta(\nabla_x\mathbf{u}_\delta) : \nabla_x\mathbf{u}_\delta + |\mu_\delta|^2]\mathrm{d}x\psi\,\mathrm{d}t \\ \geq -\int_\Omega \left[\frac{1}{2}(\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + |\nabla_x\chi_{0,\delta}|^2) + \varrho_\delta f_\delta(\varrho_{0,\delta},\chi_{0,\delta})\right]\mathrm{d}x \end{aligned} \tag{6.8}$$

holds for any $\psi \in C_c^\infty[0,T)$, $\psi \geq 0$, $\psi(0) = 1$.

It follows from hypothesis (2.15), (3.1) and (6.1) that

$$\left|\int_\Omega \left[\frac{1}{2}\left(\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + |\nabla_x\chi_{0,\delta}|^2\right) + \varrho_\delta f_\delta(\varrho_{0,\delta},\chi_{0,\delta})\right]\mathrm{d}x\right| \leq c,$$

where the constant is independent of $\delta$. Consequently, we deduce the following uniform estimates:

$$\{\sqrt{\varrho_\delta}\mathbf{u}_\delta\}_{\delta>0} \quad \text{bounded in } L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)), \tag{6.9}$$

$$\{\chi_\delta\}_{\delta>0} \quad \text{bounded in } L^\infty(0,T;H^1(\Omega)), \tag{6.10}$$

$$\{\varrho_\delta\Gamma_\delta(\varrho_\delta)\}_{\delta>0} \quad \text{bounded in } L^\infty(0,T;L^1(\Omega)). \tag{6.11}$$

In particular,

$$\{\varrho_\delta\}_{\delta>0} \quad \text{bounded in } L^\infty(0,T;L^\gamma(\Omega)). \tag{6.12}$$

Moreover,

$$\{\mu_\delta\}_{\delta>0} \quad \text{is bounded in } L^2((0,T)\times\Omega), \tag{6.13}$$

and, as a direct consequence of Korn's inequality and hypothesis (2.8),

$$\{\mathbf{u}_\delta\}_{\delta>0} \quad \text{is bounded in } L^2(0,T;H^1(\Omega;\mathbb{R}^3)). \tag{6.14}$$

Now, it follows from (2.10) and the uniform bounds (6.12), (6.14) that

$$\varrho_\delta \to \varrho \quad \text{in } C_{\text{weak}}([0,T];L^\gamma(\Omega)), \tag{6.15}$$

$$\mathbf{u}_\delta \to \mathbf{u} \quad \text{weakly in } L^2(0,T;H_0^1(\Omega;\mathbb{R}^3)), \tag{6.16}$$

at least for suitable subsequences. Similarly, (6.10) yields

$$\chi_\delta \to \chi \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T;H^1(\Omega)), \tag{6.17}$$

where, as consequence of (6.11), (4.1),

$$0 \leq \varrho(t,x) \leq r \quad \text{for a.a. } t,x, \tag{6.18}$$

$$0 \leq \chi(t,x) \leq 1 \quad \text{for a.a. } t,x. \tag{6.19}$$

As a matter of fact, (6.18) follows as the functional on $L^2((0,T)\times\Omega)$ associated to the function $Z_\delta(r) = r\Gamma_\delta(r)$ is "essentially" convex and converges to $Z(r) = r\Gamma(r)$ in the sense of Mosco (cf. Ref. 22 and Proposition 3.19, p. 297 and Theorem 3.20, p. 298 in Ref. 5). Thus, (6.18) follows from (6.15), (6.11) and the lim inf-inequality

$$\int_0^T \int_\Omega Z(\varrho) \mathrm{d}x \, \mathrm{d}t \le \liminf_{\delta \searrow 0} \int_0^T \int_\Omega Z_\delta(\varrho_\delta) \mathrm{d}x \, \mathrm{d}t < \infty. \tag{6.20}$$

The next step is to multiply (3.18) by $F(L'_\delta(\chi_\delta))$ and integrate by parts to obtain

$$\begin{aligned}
&\int_\Omega (F'(L'_\delta(\chi_\delta))L''_\delta(\chi_\delta)|\nabla_x \chi_\delta|^2 + \varrho_\delta L'_\delta(\chi_\delta) F(L'_\delta(\chi_\delta)))\mathrm{d}x \\
&\quad = \int_\Omega (\varrho_\delta \mu_\delta F(L'_\delta(\chi_\delta)) + \varrho_\delta b'(\chi_\delta) F(L'_\delta(\chi_\delta)) + \varrho_\delta (g_{2,\delta}(\varrho_\delta) - g_{1,\delta}(\varrho_\delta)) F(L'_\delta(\chi_\delta)))\mathrm{d}x,
\end{aligned}$$

where $F$ is a non-decreasing function on $\mathbb{R}$. Taking $F(z)z \approx |z|^{\beta+1}$ for large values of $|z|$, we can use the uniform bounds established in (6.12), (6.13) in order to deduce that

$$\int_0^T \int_\Omega \varrho_\delta |L'_\delta(\chi_\delta)|^{\beta+1} \, \mathrm{d}x \, \mathrm{d}t \le c \quad \text{uniformly with respect to } \delta \searrow 0, \tag{6.21}$$

where

$$0 < \beta = \beta(\gamma) \nearrow 1 \quad \text{provided } \gamma \to \infty \text{ in (6.6)}.$$

Thus going back to (3.18) we may infer that

$$\{\Delta \chi_\delta\}_{\delta > 0} \quad \text{is bounded in } L^\alpha((0,T) \times \Omega) \quad \text{where } 1 < \alpha \nearrow 2 \quad \text{for } \gamma \to \infty. \tag{6.22}$$

Relations (6.10), (6.22), together with the standard elliptic estimates and a simple interpolation argument, yield

$$\{\nabla_x \chi_\delta\}_{\delta > 0} \quad \text{bounded in } L^q((0,T) \times \Omega; \mathbb{R}^3) \quad \text{for a certain } q > 2 \tag{6.23}$$

provided the exponent $\gamma$ in (6.6) was chosen large enough.

### 6.2. *Refined pressure estimates*

One of the principal difficulties in the proof of Theorem 2.1 stems from the fact that the uniform bounds established in the previous part do not imply, in general, any uniform estimates on the pressure $p_\delta(\varrho_\delta, \chi_\delta)$, not even in the space $L^1((0,T) \times \Omega)$.

#### 6.2.1. *Integrability of the pressure term*

Following the strategy of Ref. 14, we use the quantities

$$\varphi = \psi(t) \mathcal{B}\left[ b(\varrho_\delta) - \frac{1}{|\Omega|} \int_\Omega b(\varrho_\delta) \mathrm{d}x \right], \quad \psi \in C_c^\infty(0,T), \tag{6.24}$$

as test functions in the weak formulation of the momentum equation (2.12), where the symbol $\mathcal{B} \approx \mathrm{div}_x^{-1}$ denotes the integral operator introduced by Bogovskii.[7]

The operator $\mathcal{B}$ assigns to each $g \in L^p(\Omega)$ with $\int_\Omega g \,\mathrm{d}x = 0$ a solution $\mathcal{B}[g] \in W_0^{1,p}(\Omega;\mathbb{R}^3)$ of the problem

$$\mathrm{div}_x\, \mathcal{B}[g] = g \quad \text{in } \Omega, \quad \mathcal{B}[g]|_{\partial\Omega} = 0.$$

It can be shown that $\mathcal{B}$ specified in Ref. 7 is a bounded linear operator acting on $L^q(\Omega)$ with values in $W_0^{1,q}(\Omega;\mathbb{R}^3)$ for any $1 < q < \infty$ (see Galdi,[17] Chap. 3), and, moreover, $\mathcal{B}$ can be extended as a bounded linear operator on the dual space $[W^{1,q}(\Omega)]^*$ with values in $L^{q'}(\Omega;\mathbb{R}^3)$ for any $1 < q < \infty$ (see Geissert, Heck and Hieber[18]).

We recall that $\varrho_\delta$ and $\mathbf{u}_\delta$, for any $\varphi \in C_c^\infty([0,T) \times \bar{\Omega})$ and any $b \in W^{1,\infty}[0,\infty)$, satisfy the renormalized equation (5.7).

With (5.7) at hand, we can use the quantities $\varphi$ specified in (6.24) as test functions in (2.12) to obtain

$$\int_0^T \psi \int_\Omega p_\delta(\varrho_\delta, \chi_\delta)\left[b(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\delta)(y)\,\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}t = \sum_{i=1}^6 I_{i,\delta}, \tag{6.25}$$

where

$$I_{1,\delta} = -\int_0^T \psi \int_\Omega \left(\left(\nabla_x\chi_\delta \otimes \nabla_x\chi_\delta - \frac{1}{2}|\nabla_x\chi_\delta|^2\mathbb{I}\right) : \nabla_x\mathcal{B}\left[b(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\delta)\mathrm{d}y\right]\right)\mathrm{d}x\,\mathrm{d}t,$$

$$I_{2,\delta} = \int_0^T \psi \int_\Omega \mathbb{S}_\delta : \nabla_x\mathcal{B}\left[b(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\delta)\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}t,$$

$$I_{3,\delta} = -\int_0^T \psi \int_\Omega \varrho_\delta(\mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \nabla_x\mathcal{B}\left[b(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\delta)\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}t,$$

$$I_{4,\delta} = -\int_0^T \partial_t\psi \int_\Omega \varrho_\delta\mathbf{u}_\delta \cdot \mathcal{B}\left[b(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega b(\varrho_\delta)\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}t,$$

$$I_{5,\delta} = \int_0^T \psi \int_\Omega \varrho_\delta\mathbf{u}_\delta \cdot \mathcal{B}[\mathrm{div}_x(b(\varrho_\delta)\mathbf{u}_\delta)]\mathrm{d}x\,\mathrm{d}t,$$

and

$$I_{6,\delta} = \int_0^T \psi \int_\Omega \varrho_\delta\mathbf{u}_\delta \cdot \mathcal{B}[(b'(\varrho_\delta)\varrho_\delta - b(\varrho_\delta))\mathrm{div}_x\mathbf{u}_\delta - \frac{1}{|\Omega|}\int_\Omega (b'(\varrho_\delta)\varrho_\delta - b(\varrho_\delta))\mathrm{div}_x\mathbf{u}_\delta\,\mathrm{d}y\Bigg]\mathrm{d}x\,\mathrm{d}t.$$

Taking $b(\varrho) = \varrho$, and using the uniform bounds established in (6.9)–(6.23), together with boundedness of the operator $\mathcal{B}$ in $L^q$ and $[W^{1,q}]^*$, we deduce, exactly as in Ref. 14, that all integrals $I_{i,\delta}$, $i = 1, \ldots, 6$ are bounded uniformly for $\delta \searrow 0$ as long as the exponent $\gamma$ in (6.6) is large enough. Consequently, we obtain

$$\left|\int_0^T\int_\Omega p_\delta(\varrho_\delta, \chi_\delta)\left[\varrho_\delta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta(y)\mathrm{d}y\right]\mathrm{d}x\,\mathrm{d}t\right| \le c,$$

with $c$ independent of $\delta \searrow 0$.

Now, it follows from (6.1) that

$$\frac{1}{|\Omega|}\int_\Omega \varrho_\delta(t,\cdot)\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega \varrho_{0,\delta}\,\mathrm{d}x = m_\delta < r,$$

therefore

$$\int_0^T\int_\Omega p_\delta(\varrho_\delta,\chi_\delta)\left(\varrho_\delta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta(y)\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t = J_{1,\delta} + J_{2,\delta},$$

with

$$J_{1,\delta} = \int_{\{\varrho_\delta<(m_\delta+r)/2\}} p_\delta(\varrho_\delta,\chi_\delta)\left(\varrho_\delta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta(y)\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t$$

$$J_{2,\delta} = \int_{\{\varrho_\delta\geq(m_\delta+r)/2\}} p_\delta(\varrho_\delta,\chi_\delta)\left(\varrho_\delta - \frac{1}{|\Omega|}\int_\Omega \varrho_\delta(y)\mathrm{d}y\right)\mathrm{d}x\,\mathrm{d}t$$
$$\geq \frac{r-m_\delta}{2}\int_{\{\varrho_\delta\geq(m_\delta+r)/2\}} p_\delta(\varrho_\delta,\chi_\delta)\mathrm{d}x\,\mathrm{d}t.$$

Since $\Omega$ is a bounded domain, the integrals $J_{1,\delta}$ are evidently bounded, and we may conclude that

$$\{p_\delta(\varrho_\delta,\chi_\delta)\}_{\delta>0},\quad \{p_\delta(\varrho_\delta,\chi_\delta)\varrho_\delta\}_{\delta>0}\quad \text{are bounded in } L^1((0,T)\times\Omega), \tag{6.26}$$

uniformly w.r.t. $\delta \searrow 0$.

### 6.2.2. *Equi-integrability of the pressure term*

Estimate (6.26) is still not sufficient for passing to the limit in the pressure term. In order to establish at least weak convergence of the pressure, we need equi-integrability of the family $\{p_\delta\}_{\delta>0}$. To this end, we make use of hypothesis (2.7).

Analogously as in the previous section, we take the quantities

$$\varphi(t,x) = \psi(t)\mathcal{B}\left[\eta_\delta(\varrho_\delta) - \frac{1}{|\Omega|}\int_\Omega \eta_\delta(\varrho_\delta)\mathrm{d}x\right] \tag{6.27}$$

with $\psi \in C_c^\infty(0,T)$,

$$\eta_\delta = \eta_\delta(\varrho) = \begin{cases} \log(r-\varrho) & \text{if } \varrho \leq r-\delta, \\ \log(\delta) & \text{otherwise,} \end{cases}$$

as test functions in the momentum equation (2.12).

As $P$ satisfies hypothesis (2.7), and $P_\delta$ is given by (6.6), there are constants $c_1 > 0$, $c_2$ such that

$$\Gamma_\delta(\varrho) \geq \frac{c_1}{(r-\varrho)^2} - c_2 \quad \text{for all } 0 \leq \varrho \leq r-\delta,$$

in particular,

$$\Gamma_\delta(\varrho_\delta) \geq c_1(q)|\eta_\delta(\varrho_\delta)|^q - c_2(q) \quad \text{for any } 1 \leq q < \infty, \tag{6.28}$$

and, similarly,

$$\Gamma_\delta(\varrho_\delta) \geq c_1|\eta'_\delta(\varrho_\delta)|^2 - c_2, \tag{6.29}$$

where $\Gamma_\delta$ is given through (3.7).

Thus, exactly as in the previous section, we deduce a uniform bound

$$\int_0^T \int_\Omega |p_\delta(\varrho_\delta)\eta_\delta(\varrho_\delta)| \mathrm{d}x \, \mathrm{d}t \leq c, \quad c \text{ independent of } \delta \searrow 0, \tag{6.30}$$

as soon as we are able to control the integrals on the right-hand side of formula (6.25). We check easily that the most difficult term reads

$$\begin{aligned} I_{6,\delta} = \int_0^T \int_\Omega \psi \varrho_\delta \mathbf{u}_\delta \cdot \mathcal{B}\Big[(\eta'_\delta(\varrho_\delta)\varrho_\delta - \eta_\delta(\varrho_\delta))\mathrm{div}_x \mathbf{u}_\delta \\ - \frac{1}{|\Omega|}\int_\Omega (\eta'_\delta(\varrho_\delta)\varrho_\delta - \eta_\delta(\varrho_\delta))\mathrm{div}_x \mathbf{u}_\delta \, \mathrm{d}y\Big] \mathrm{d}x \, \mathrm{d}t. \end{aligned} \tag{6.31}$$

In accordance with the uniform bounds established in (6.9), (6.11), (6.14), we can use (6.28), (6.29) in order to obtain that

$$\begin{aligned} \Big\| \mathcal{B}\Big[ (\eta'_\delta(\varrho_\delta)\varrho_\delta - \eta_\delta(\varrho_\delta))\mathrm{div}_x \mathbf{u}_\delta \\ - \frac{1}{|\Omega|}\int_\Omega (\eta'_\delta(\varrho_\delta)\varrho_\delta - \eta_\delta(\varrho_\delta))\mathrm{div}_x \mathbf{u}_\delta \, \mathrm{d}x \Big] \Big\|_{L^2(0,T;L^q(\Omega;\mathbb{R}^3))} \leq c(q) \\ \text{for any } q < 3/2. \end{aligned} \tag{6.32}$$

Indeed we have used the embedding $L^1(\Omega) \hookrightarrow [W^{1,q}]^*(\Omega)$ for $q > 3$, together with the result of Geissert *et al.*[18] on boundedness of the operator $\mathcal{B} : [W^{1,q}]^* \to L^{q'}$.

On the other hand, seeing that $H^1(\Omega) \hookrightarrow L^6(\Omega)$, we can take $\gamma > 6$ in (6.6) in order to control the momentum $\varrho_\delta \mathbf{u}_\delta$ by the help of the energy estimates (6.9)–(6.14). Consequently, the integrals $I_{6,\delta}$ are bounded uniformly for $\delta \searrow 0$. Thus we have shown (6.30).

Estimate (6.30) implies equi-integrability of the family $\{p_\delta(\varrho_\delta, \chi_\delta)\}_{\delta>0}$ in the Lebesgue space $L^1((0,T) \times \Omega)$; whence we may conclude that

$$p_\delta(\varrho_\delta, \chi_\delta) \to \overline{p(\varrho, \chi)} \quad \text{weakly in } L^1((0,T) \times \Omega). \tag{6.33}$$

### 6.3. *Convergence of convective terms*

At this stage, we are ready to perform the limit $\delta \searrow 0$ in the approximate equations and to identify the limit system. We start with the convective terms that can be handled in a rather uniform way repeating essentially the arguments of Secs. 4.4 and 5.2.

To begin with, relations (6.9), (6.15), (6.16) give rise to

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \quad \text{weakly-}(^*) \text{ in } L^\infty(0,T; L^p(\Omega;\mathbb{R}^3)) \quad \text{for a certain } p > 6/5$$

provided $\gamma > 0$ is large enough. This can be strengthened to

$$\varrho_\delta \mathbf{u}_\delta \to \varrho \mathbf{u} \quad \text{in } C_w([0,T]; L^p(\Omega;\mathbb{R}^3)) \tag{6.34}$$

by means of (2.12). Thus the same argument leads finally to

$$\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta \to \varrho \mathbf{u} \otimes \mathbf{u} \quad \text{weakly in } L^p((0,T)\times\Omega;\mathbb{R}^{3\times 3}) \quad \text{for a certain } p > 1. \tag{6.35}$$

In the same fashion, relation (6.17) yields

$$\varrho_\delta \chi_\delta \to \varrho\chi \quad \text{in } C_w([0,T]; L^\gamma(\Omega)) \tag{6.36}$$

and

$$\varrho_\delta \chi_\delta \mathbf{u}_\delta \to \varrho\chi\mathbf{u} \quad \text{weakly in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{6.37}$$

Finally, the same argument used to deduce (5.23) yields

$$\int_0^T\int_\Omega \varrho|\chi_\delta|^2 \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T\int_\Omega \varrho|\chi|^2 \mathrm{d}x\,\mathrm{d}t,$$

in other words

$$\chi_\delta \to \chi \quad \text{a.a. in the set } \{(t,x) \in (0,T)\times\Omega \mid \varrho(t,x) > 0\}. \tag{6.38}$$

### 6.4. *Strong convergence of the extra stress*

Following the method developed in Ref. 3 we show strong convergence of $\{\nabla_x \chi_\delta\}_{\delta>0}$. The first part of the argument goes along the lines of Sec. 5.2. First, we let $\delta \searrow 0$ in (3.18) to obtain

$$\int_0^T\int_\Omega \overline{\varrho\mu}\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \nabla_x\chi\cdot\nabla_x\varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (\overline{\varrho L'_\delta(\chi)}\varphi - \overline{\varrho b'(\chi)}\varphi + \overline{\varrho(g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho))}\varphi)\mathrm{d}x\,\mathrm{d}t$$

for any sufficiently regular test function $\varphi$, where the bars denote weak limits in $L^1$. Note that $\{\varrho_\delta L'_\delta(\chi_\delta)\}_{\delta>0}$ is bounded in $L^p((0,T)\times\Omega)$ for a certain $p > 1$ as a consequence of (6.12), (6.21). In particular, taking $\varphi = \chi$ we get

$$\int_0^T\int_\Omega \overline{\varrho\mu}\chi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \nabla_x\chi\cdot\nabla_x\chi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (\overline{\varrho L'_\delta(\chi)}\chi - \overline{\varrho b'(\chi)}\chi + \overline{\varrho(g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho))}\chi)\mathrm{d}x\,\mathrm{d}t. \tag{6.39}$$

On the other hand, we also have

$$\int_0^T\int_\Omega \overline{\varrho\mu\chi} \,\mathrm{d}x\,\mathrm{d}t = \lim_{\delta\to 0}\int_0^T\int_\Omega \nabla_x\chi_\delta\cdot\nabla_x\chi_\delta \,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_\Omega (\overline{\varrho L'_\delta(\chi)\chi} - \overline{\varrho b'(\chi)\chi} + \overline{\varrho(g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho))\chi})\mathrm{d}x\,\mathrm{d}t. \tag{6.40}$$

Now, it follows from (6.38) and the uniform bounds (6.12), (6.13) and (6.17) that

$$\overline{\varrho\mu\chi} = \overline{\varrho\mu}\chi, \quad \overline{\varrho b'(\chi)\chi} = \overline{\varrho b'(\chi)}\chi \quad \text{and}$$
$$\overline{\varrho(g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho))\chi} = \overline{\varrho(g_{2,\delta}(\varrho) - g_{1,\delta}(\varrho))}\chi.$$

At this stage, the above procedure must be modified as the terms $L'_\delta(\chi_\delta)$ are no longer uniformly bounded. We observe, however, that

$$\lim_{\delta\to 0}\int_0^T\int_\Omega \varrho_\delta L'_\delta(\chi_\delta)\chi_\delta \,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega \overline{\varrho L'_\delta(\chi)}\chi\,\mathrm{d}x\,\mathrm{d}t.$$

Indeed,

$$\int_0^T\int_\Omega \varrho_\delta L'_\delta(\chi_\delta)\chi_\delta \,\mathrm{d}x\,\mathrm{d}t = \iint_{\{\varrho>0\}} \varrho_\delta L'_\delta(\chi_\delta)\chi_\delta \,\mathrm{d}x\,\mathrm{d}t + \iint_{\{\varrho=0\}} \varrho_\delta L'_\delta(\chi_\delta)\chi_\delta \,\mathrm{d}x\,\mathrm{d}t,$$

where, as a consequence of (6.38),

$$\iint_{\{\varrho>0\}} \varrho_\delta L'_\delta(\chi_\delta)\chi_\delta \,\mathrm{d}x\,\mathrm{d}t \to \iint_{\{\varrho>0\}} \overline{\varrho L'_\delta(\chi)}\chi\,\mathrm{d}x\,\mathrm{d}t,$$

while, in accordance with (6.12), (6.21),

$$\iint_{\{\varrho=0\}} |\varrho_\delta L'_\delta(\chi_\delta)\chi_\delta|\mathrm{d}x\,\mathrm{d}t \le \|\varrho_\delta^{1/p} L'_\delta(\chi_\delta)\chi_\delta\|_{L^p((0,T)\times\Omega)} \|\varrho_\delta\|_{L^1(\{\varrho=0\})}^{1/p'} \to 0$$

as $\delta \searrow 0$.

In view of the previous arguments, we then arrive at

$$\lim_{\delta\to 0}\int_0^T\int_\Omega |\nabla_x\chi_\delta|^2 \mathrm{d}x\,\mathrm{d}t = \int_0^T\int_\Omega |\nabla_x\chi|^2\mathrm{d}x\,\mathrm{d}t;$$

in other words,

$$\nabla_x\chi_\delta \to \nabla_x\chi \quad \text{(strongly) in } L^2((0,T)\times\Omega;\mathbb{R}^3). \tag{6.41}$$

To conclude this part, we remark that regularity properties (2.11) and (2.14) can be recovered *a posteriori* by testing (the limit of) (3.18) by a suitable truncation of $W'(\chi)$ and then letting the truncation parameter go to 0 in a standard way (the use of a truncation seems necessary since $W'(\chi)$ *a priori* could not have a sufficient regularity to be used as a test function).

### 6.5. *Strong convergence of the density*

In order to complete the proof of Theorem 2.1, we have to show that

$$\overline{p(\varrho,\chi)} = p(\varrho,\chi)$$

in (6.33). To this end, we need strong (a.a. pointwise) convergence of $\{\varrho_\delta\}_{\delta>0}$.

To begin, we use the renormalized continuity Eq. (5.7) to deduce (cf. p. 140 of Ref. 12)

$$\int_\Omega (\overline{\varrho\log(\varrho)} - \varrho\log(\varrho))(\tau,\cdot)\mathrm{d}x + \int_0^\tau\int_\Omega (\overline{\varrho\,\mathrm{div}_x\mathbf{u}} - \varrho\,\mathrm{div}_x\mathbf{u})\mathrm{d}x\,\mathrm{d}t = 0 \tag{6.42}$$

for any $\tau \geq 0$, where, as always, we have used the bar to denote weak limits in $L^1$ of sequences of composed functions.

On the other hand, as the densities $\varrho_\delta$ are bounded in $L^\infty(0,T;L^\gamma(\Omega))$, with $\gamma$ large enough, we can use directly the procedure of Lions,[20] together with the necessary modifications introduced in Ref. 13 to handle the variable viscosity coefficients, to establish the "weak continuity" of the effective viscous pressure, in particular,

$$\int_0^T\int_\Omega \xi\left(\frac{4}{3}\nu(\chi)+\eta(\chi)\right)(\overline{\varrho\,\mathrm{div}_x\mathbf{u}}-\varrho\,\mathrm{div}_x\mathbf{u})\mathrm{d}x\,\mathrm{d}t$$
$$\geq \liminf_{\delta\to 0}\int_0^\tau\int_\Omega \xi(p_\delta(\varrho_\delta,\chi_\delta)\varrho_\delta-\overline{p(\varrho,\chi)}\varrho)\mathrm{d}x\,\mathrm{d}t$$
$$\text{for any } \xi\in C_c^\infty(\Omega), \xi\geq 0. \tag{6.43}$$

The proof of (6.43) is tedious but nowadays well understood (see Ref. 13). The basic idea is to use multipliers of the form

$$\varphi(t,x)=\psi(t)\xi(x)\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta],\quad \varphi(t,x)=\psi(t)\xi(x)\nabla_x\Delta^{-1}[1_\Omega\varrho],$$
$$\psi\in C_c^\infty(0,T),\quad \xi\in C_c^\infty(\Omega), \tag{6.44}$$

as test functions in the momentum equation (2.12) and its asymptotic limit for $\delta\to 0$, respectively. Here, the symbol $\Delta^{-1}$ stands for the inverse of the Laplacian considered on the whole space $\mathbb{R}^3$.

After a straightforward manipulation, we obtain

$$\int_0^T \partial_t\psi\int_\Omega \xi\varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\psi\int_\Omega \xi\varrho_\delta\mathbf{u}_\delta\cdot\nabla_x\Delta^{-1}\,\mathrm{div}_x[\varrho_\delta\mathbf{u}_\delta]\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\psi\int_\Omega \varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta):(\nabla_x\xi\otimes\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta])\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\psi\int_\Omega \xi\varrho_\delta(\mathbf{u}_\delta\otimes\mathbf{u}_\delta):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\psi\int_\Omega p_\delta(\varrho_\delta,\chi_\delta)\nabla_x\xi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\psi\int_\Omega \xi p_\delta(\varrho_\delta,\chi_\delta)\varrho_\delta\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_0^T\psi\int_\Omega\nu(\chi_\delta)\left[\left(\nabla_x\mathbf{u}_\delta+\nabla_x^t\mathbf{u}_\delta-\frac{2}{3}\mathrm{div}_x\mathbf{u}_\delta\mathbb{I}\right)\nabla_x\xi\right]\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t$$
$$+\int_0^T\psi\int_\Omega\nu(\chi_\delta)\xi\left(\nabla_x\mathbf{u}_\delta+\nabla_x^t\mathbf{u}_\delta-\frac{2}{3}\mathrm{div}_x\mathbf{u}_\delta\mathbb{I}\right):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t$$

$$
\begin{aligned}
&+\int_0^T \psi \int_\Omega \eta(\chi_\delta)\mathrm{div}_x \mathbf{u}_\delta(\nabla_x\xi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta] + \xi\,\mathrm{div}_x\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta])\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T \psi \int_\Omega \left[\left(\frac{|\nabla_x\chi_\delta|^2}{2}\mathbb{I} - \nabla_x\chi_\delta\otimes\nabla_x\chi_\delta\right)\nabla_x\xi\right]\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t\\
&+\int_0^T \psi \int_\Omega \xi\left(\frac{|\nabla_x\chi_\delta|^2}{2}\mathbb{I} - \nabla_x\chi_\delta\otimes\nabla_x\chi_\delta\right):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho_\delta]\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{6.45}
$$

and

$$
\begin{aligned}
&\int_0^T \partial_t\psi\int_\Omega \xi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t - \int_0^T\psi\int_\Omega\xi\varrho\mathbf{u}\cdot\nabla_x\Delta^{-1}\,\mathrm{div}_x[\varrho\mathbf{u}]\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\varrho(\mathbf{u}\otimes\mathbf{u}):(\nabla_x\xi\otimes\nabla_x\Delta^{-1}[1_\Omega\varrho])\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\xi\varrho(\mathbf{u}\otimes\mathbf{u}):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\overline{p(\varrho,\chi)}\nabla_x\xi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t + \int_0^T\psi\int_\Omega\xi\overline{p(\varrho,\chi)}\,\varrho\,\mathrm{d}x\,\mathrm{d}t\\
&=\int_0^T\psi\int_\Omega\nu(\chi)\left[\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right)\nabla_x\xi\right]\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\nu(\chi)\xi\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\mathrm{div}_x\mathbf{u}\mathbb{I}\right):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\eta(\chi)\mathrm{div}_x\mathbf{u}(\nabla_x\xi\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho] + \xi\,\mathrm{div}_x\nabla_x\Delta^{-1}[1_\Omega\varrho])\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\left[\left(\frac{|\nabla_x\chi|^2}{2}\mathbb{I}-\nabla_x\chi\otimes\nabla_x\chi\right)\nabla_x\xi\right]\cdot\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t\\
&\quad+\int_0^T\psi\int_\Omega\xi\left(\frac{|\nabla_x\chi|^2}{2}\mathbb{I}-\nabla_x\chi\otimes\nabla_x\chi\right):\nabla_x\nabla_x\Delta^{-1}[1_\Omega\varrho]\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
\tag{6.46}
$$

Now, relation (6.43) can be deduced by letting $\delta \searrow 0$ in (6.45) and comparing the resulting expression with (6.46). This nontrivial step is the heart of the existence theory for the barotropic Navier–Stokes system developed by Lions[20] and extended to variable viscosity coefficients in Ref. 13. The reader may consult Sec. 3.3 of Ref. 3, for an adaptation of this method to the present problem. Let us point out only that the main ingredient is the so-called *commutator lemma*:

**Lemma 6.1.** (See Lemma 4.2 of Ref. 13) *Let* $w \in W^{1,p}(\mathbb{R}^3)$ *and* $\mathbf{V} \in L^2(\mathbb{R}^3;\mathbb{R}^3)$, *where* $p > 6/5$. *Then there exists* $\omega = \omega(p) > 0$ *and* $q(p) > 1$ *such that*

$$
\|\nabla_x\Delta^{-1}\,\mathrm{div}_x[w\mathbf{V}] - w\nabla_x\Delta^{-1}\,\mathrm{div}_x[\mathbf{V}]\|_{W^{\omega,q}(\mathbb{R}^3;\mathbb{R}^3)} \le c(p)\|w\|_{W^{1,p}(\mathbb{R}^3)}\|\mathbf{V}\|_{L^2(\mathbb{R}^3;\mathbb{R}^3)}.
$$

Consequently, in order to show strong convergence of the densities, it is enough to observe that

$$\liminf_{\delta\to 0}\int_0^T\int_\Omega \xi(p_\delta(\varrho_\delta,\chi_\delta)\varrho_\delta - \overline{p(\varrho,\chi)}\,\varrho)\mathrm{d}x\,\mathrm{d}t \geq 0$$
$$\text{for any } \xi\in C_c^\infty(\Omega),\quad \xi\geq 0. \tag{6.47}$$

Indeed, in case (6.47) holds, it follows that $\int_\Omega \overline{\varrho\log(\varrho)}\mathrm{d}x \to \int_\Omega \varrho\log(\varrho)\mathrm{d}x$ a.a. in $(0,T)$ and so $\varrho_\delta\to\varrho$ a.a. in $(0,T)\times\Omega$.

In view of (6.4) and strong convergence of $\{\chi_\delta\}_{\delta>0}$ established in (6.41), relation (6.47) follows as soon as we observe that

$$\liminf_{\delta\to 0}\int_0^T\int_\Omega \xi(P_\delta(\varrho_\delta)\varrho_\delta - \overline{P(\varrho)}\,\varrho)\mathrm{d}x\,\mathrm{d}t \geq 0 \quad \text{for any } \xi\in C_c^\infty(\Omega),\quad \xi\geq 0. \tag{6.48}$$

To see (6.48), it is enough to realize that, due to (6.6), $\varrho\mapsto P_\delta(\varrho)$ are non-decreasing (for $\varrho\geq 0$) for any fixed $\delta>0$, and $P_\alpha\geq P_\beta$ provided $\alpha\leq\beta$; therefore

$$\begin{aligned}
&\liminf_{\delta\to 0}\int_0^T\int_\Omega \xi(P_\delta(\varrho_\delta)\varrho_\delta - \overline{P(\varrho)}\,\varrho)\mathrm{d}x\,\mathrm{d}t\\
&\quad\geq \liminf_{\delta\to 0}\int_0^T\int_\Omega \xi(P_\beta(\varrho_\delta)\varrho_\delta - \overline{P(\varrho)}\,\varrho)\mathrm{d}x\,\mathrm{d}t\\
&\quad\geq \int_0^T\int_\Omega \xi(\overline{P_\beta(\varrho)} - \overline{P(\varrho)})\varrho\,\mathrm{d}x\,\mathrm{d}t \quad \text{for any fixed } \beta,
\end{aligned}$$

where the last inequality follows from monotonicity of $P_\beta$.

However,

$$\overline{P_\beta(\varrho)} - \overline{P(\varrho)} \to 0 \quad \text{in } L^1((0,T)\times\Omega) \quad \text{for } \beta\to 0,$$

as a direct consequence of the equi-integrability property of the pressure established in (6.30).

Summing up relations (6.42)–(6.47) we conclude that

$$\varrho_\delta\to\varrho \quad \text{a.a. in } (0,T)\times\Omega \quad \text{and} \quad \overline{p(\varrho,\chi)} = p(\varrho,\chi),$$

which completes the proof of Theorem 2.1.

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