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Global attractors for Cahn–Hilliard equations with nonconstant mobility

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Abstract

We address, in a three-dimensional spatial setting, both the viscous and the standard Cahn–Hilliard equation with a nonconstant mobility coefficient. As shown in Barrett and Blowey (1999 *Math. Comput.* **68** 487–517), one cannot expect the uniqueness of the solution to the related initial and boundary value problems. Nevertheless, referring to Ball's theory of generalized semiflows, we are able to prove the existence of compact quasi-invariant global attractors for the associated dynamical processes settled in the natural 'finite energy' space. A key point in the proof is a careful use of the energy equality, combined with the derivation of a 'local compactness' estimate for systems with supercritical nonlinearities, which may have an independent interest. Under growth restrictions on the configuration potential, we also show the existence of a compact global attractor for the semiflow generated by the (weaker) solutions to the nonviscous equation characterized by a 'finite entropy' condition.

Mathematics Subject Classification: 35K55, 35B41, 35B45

1. Introduction

In this paper we address the initial and (homogeneous Neumann) boundary value problem for the equation

$$u_t - \operatorname{div}(b(u)\nabla(\varepsilon u_t - \Delta u + W'(u) + f)) = 0, \tag{1.1}$$

which is settled in a smooth and bounded domain $\Omega \subset \mathbb{R}^3$ and corresponds for $\varepsilon = 0$ to the standard, and for $\varepsilon > 0$ to the *viscous*, Cahn–Hilliard equation with *nonconstant* mobility function $b(\cdot)$. In particular, b is assumed to depend on u in a globally Lipschitz way and is not allowed to degenerate. In the above relation, W is possibly a nonconvex configuration potential and f a source which is included in view of possible applications to conserved phase field models (where u is an order parameter and f represents a coupling term depending on the temperature, see, e.g. [8]). Relation (1.1) is complemented by homogeneous Neumann boundary conditions for both u and the *chemical potential* $w := (\varepsilon u_t - \Delta u + W'(u) + f)$.

The dependence of the mobility on the variable u is very relevant for physical applications. Actually, as u represents the density of one component in a binary alloy, one expects that the diffusion of mass is influenced by the actual configuration, i.e. by the value of u. In fact, it is just due to difficulties arising in the analysis of (1.1) that, in the mathematical literature, b has been generally replaced by a constant function.

The most relevant work devoted to the mathematical study of (1.1) for nonconstant (but nondegenerate) b is [5], where, for zero source f and no viscosity (i.e. $\varepsilon = 0$), the existence and uniqueness of the solution, together with additional regularity properties, are proved in space dimensions 1 and 2. In contrast, in the three-dimensional case, only the existence of a weak solution is shown (the uniqueness would hold in 3D for a class of more regular solutions, but the authors cannot prove this further regularity). The results of [5], which are also complemented by numerical investigations, are very sharp and it seems rather difficult to fill the regularity gap which prevents from having well posedness in 3D. Actually, some more recent work [13] has been devoted to improve the regularity of solutions, but still only in the 2D case.

Here, we aim to analyse, referring just to the 3D setting, the long-time behaviour of (1.1) from the point of view of global attractors and considering both the viscous and the nonviscous cases. Due to the quoted difficulties, this analysis is far from being trivial. Actually, the use of more or less standard tools seems possible only for the viscous equation and if the potential W has a controlled growth at ∞ (cf (2.12)). Indeed, in this case, uniqueness holds at least for $t > 0$ (for $t \geqslant 0$ if the initial datum is more regular) and we have uniform regularization properties. Instead, if $\varepsilon = 0$ and/or we are in the (physically relevant) situation of fast growing or even *singular* (i.e. uniformly taking the value $+\infty$ outside a bounded interval, cf (2.13)) potentials, we then have to proceed much more carefully.

In such a framework, the existence of solutions and dissipativity of the process are still easy to show, but then we have to face the following three main difficulties.

(i) We have no uniqueness result. Thus, we have to refer to some machinery which is suitable for dealing with problems with lack of uniqueness. Among the various possible choices (we quote in particular the alternative possibility of working in the space of trajectories, cf, e.g. [9, 22]), we decided to refer to Ball's theory of *generalized semiflows* [2, 3] which has the advantage of being very close to the standard physical interpretation. Namely, the system still gives rise to a dynamical process settled in a phase space $\mathcal{V}$ of states (rather than, for instance, of trajectories). To be more precise, due to point (ii), a further generalization of Ball's approach, recently devised in [18] (see also [14]), will be used.

(ii) Not all the estimates we perform can be rigorously carried out in the regularity framework which appears to be the natural one for (1.1). Namely, one has to proceed through approximation and passage to the limit. However, due to the lack of uniqueness, it is not obvious whether all the solutions with the natural regularity can be reached by the approximation procedure. Thus, we have to restrict ourselves to consider solutions which are the limit of more regular sequences for which the estimates can be rigorously shown. This has a consequence on the structure of the global attractor, which turns out to be only *quasi-invariant* rather than *fully invariant* as in the standard cases (cf [18, definition 2.8] and remark 2.9 for more details on this point).

(iii) Finally, despite the strictly parabolic character of the system, we cannot prove any uniform in time regularization property of solutions (which, by the way, would also lead to uniqueness). For this reason, we have to get the asymptotic compactness of the process through a different procedure, which in our opinion can have an independent interest and might be applied to other systems with supercritical or fast growing nonlinearities. Actually, we combine the use of the energy equality, which is a consequence of the variational structure of (1.1) and is satisfied by all solutions in our regularity class, with a 'locally uniform' regularization property. Namely, we can show that there exist a set K_0, compact in the phase space $\mathcal{V}$, and a number $\delta > 0$, both independent of the initial data, such that all admissible solutions $u = u(t)$ starting from a given set B bounded in $\mathcal{V}$, after some $T_0 > 0$ depending only on the radius of B in $\mathcal{V}$, satisfy that

$$\forall\, t \geqslant T_0, \qquad \exists\, \tau = \tau(t) \in [0, 3/2] : \forall\, s \in [t+\tau, t+\tau+\delta], \qquad u(s) \in K_0. \quad (1.2)$$

Such a property (combined with the energy equality) turns out to imply the asymptotic compactness of the semiflow. We observe that a similar argument has been recently used in a different context in the paper [16] (cf proposition 1.1). Unfortunately, we are not able to show (1.2) for all 'singular potentials' W (i.e. those equalling $+\infty$ outside a bounded interval, here normalized to $(-1, 1)$ for convenience), but only for those of a subclass (introduced in [21, (H5)] and called here 'separating' potentials, see definition 2.10), which turn out to explode sufficiently fast in the proximity of ± 1. In particular, this class does *not* seem to include the *logarithmic* potential

$$W(r) = (1+r)\log(1+r) + (1-r)\log(1-r) - \frac{\lambda}{2}r^2, \qquad r \in (-1, 1), \quad (1.3)$$

where λ is a positive parameter, relevant in concrete physical situations. We also remark that, due to the nonuniform character of (1.2), the resulting global attractor will be compact in $\mathcal{V}$, but not necessarily bounded in a 'better' space.

The procedure sketched above can be applied both for $\varepsilon > 0$ and for $\varepsilon = 0$. Of course, if $\varepsilon > 0$ and W has a controlled growth at infinity, we have uniform regularization and, at least, unique continuation of solutions. Thus, the global attractor can be intended in the framework of the standard theory for single-valued semigroups (cf, e.g. [24]). Moreover, in this case one could prove with only technical difficulties further regularity properties of the attractor.

Finally, in the nonviscous case $\varepsilon = 0$ we can also prove the existence of a compact set in $\mathcal{V}$ which uniformly attracts *less regular* solutions, namely, those taking values in a larger phase space $\mathcal{H}$, which we call 'finite entropy'. Actually, by approximation, the existence in this class (which was not considered in [5]) can be proved by means of an estimate of the entropy type (cf, e.g. [10]), the validity of which, however, seems to require a growth restriction on W (excluding from this result any singular potential). Moreover, the entropy estimate turns out to have a dissipative character, yielding the existence of an absorbing set in $\mathcal{H}$. Then we prove that from any initial datum $u_0 \in \mathcal{H}$ there starts *at least* one solution u, which, for $t > 0$, lies in the energy space $\mathcal{V}$, which is compactly embedded in $\mathcal{H}$. Clearly, this implies the existence of a global attractor bounded in the 'better' space $\mathcal{V}$. Of course, also in this case, we have no uniqueness and are still forced to use the 'generalized semiflows' machinery described above.

The rest of this paper is organized as follows. In section 2, after recalling some preliminary material, we present our hypotheses and state our main results, with the exception of those related to entropy solutions. The related proofs are given in section 3. Finally, section 4 is devoted to the analysis of solutions in the finite entropy class.

2. Notation and main results

Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Let us set $H := L^2(\Omega)$ and denote by $(\cdot, \cdot)$ the scalar product in H and by $\|\cdot\|$ the related norm. The same symbols are also used to represent

H^3 and its scalar product and norm. The symbol $\|\cdot\|_X$ will indicate the norm in the generic Banach space X. Let us also assume that

$$b \in W^{1,\infty}(\mathbb{R};\mathbb{R}), \qquad \exists\, \alpha, \mu > 0 : \alpha \leqslant b(r) \leqslant \mu \quad \forall\, r \in \mathbb{R}. \tag{2.1}$$

Then we set $V := H^1(\Omega)$, endowed with its standard scalar product and norm. Letting $u : \Omega \to \mathbb{R}$ be a *measurable* function, we introduce the pair of elliptic operators $B, B_u : V \to V'$ (where V' is the topological dual of V), respectively, given by

$$\langle Bv, z\rangle = \int_\Omega \nabla v \cdot \nabla z, \qquad \langle B_u v, z\rangle = \int_\Omega b(u) \nabla v \cdot \nabla z, \tag{2.2}$$

the notation $\langle\cdot,\cdot\rangle$ standing for the duality between V' and V. Then, we clearly have

$$\langle B_u v, z\rangle \leqslant \mu \|v\|_V \|z\|_V, \qquad \langle B_u v, v\rangle \geqslant \alpha \|\nabla v\|^2 \tag{2.3}$$

for all u, v, z as before. If u additionally depends on time (i.e. it is a measurable function defined on $\Omega \times (0, T)$ for some $T > 0$), then B_u is naturally extended to time-dependent functions. Namely, $B_u : L^2(0, T; V) \to L^2(0, T; V')$ is still a continuous and coercive operator given by

$$\int_0^T \langle B_u v, z\rangle := \int_0^T \int_\Omega b(u(x,t)) \nabla v(x,t) \cdot \nabla z(x,t) \,\mathrm{d}x\,\mathrm{d}t. \tag{2.4}$$

In the following we will adopt the convention

$$\zeta_\Omega := |\Omega|^{-1} \langle \zeta, 1\rangle \tag{2.5}$$

for $\zeta \in V'$, where $|\Omega|$ stands for the Lebesgue measure of Ω. Let us also set

$$V_0' := \{\zeta \in V' : \zeta_\Omega = 0\}, \qquad H_0 := H \cap V_0', \quad V_0 := V \cap V_0 \tag{2.6}$$

and observe that, in general, if $v \in V$ and $\zeta \in V'$, then

$$\langle \zeta - \zeta_\Omega, v\rangle = \langle \zeta - \zeta_\Omega, v - v_\Omega\rangle = \langle \zeta, v - v_\Omega\rangle. \tag{2.7}$$

Moreover, if u is as in (2.3), then B_u is bijective from V_0 to V_0', so that we can define its inverse $\mathcal{N}_u$, which fulfils, for all $v \in V$, $\zeta \in V'$,

$$\langle B_u v, \mathcal{N}_u(\zeta - \zeta_\Omega)\rangle = \langle \zeta - \zeta_\Omega, v\rangle. \tag{2.8}$$

Let us now come to the assumptions on the potential W. We let I be an open interval of $\mathbb{R}$ containing 0 (possibly unbounded or even coinciding with the whole real line), $\lambda > 0$, $c_W \geqslant 0$ and assume that

$$W \in C^2(I;\mathbb{R}), \qquad W'(0) = 0, \tag{2.9}$$

$$W(r) \geqslant 3\lambda r^2 - c_W \qquad \forall\, r \in I, \tag{2.10}$$

$$W''(r) \geqslant -\lambda \qquad \forall\, r \in I. \tag{2.11}$$

Assumption (2.10) states that the growth rate of W for large values of r is sufficiently fast to compensate its possible nonconvexity (2.11) near 0. Of course, (2.10) holds automatically whenever, for large r, it is $W(r) \sim \eta |r|^q$ for some $\eta > 0$ and $q > 2$. Additionally, we will assume either (2.12) or (2.13). The first is a *controlled growth* condition (the choice of $p \in [2, \infty]$ will be made precise later):

$$I = \mathbb{R}, \qquad \exists\, K_W > 0 : W''(r) \leqslant K_W(1 + |r|^{p-2}) \quad \forall\, r \in \mathbb{R}. \tag{2.12}$$

Of course, the larger the p, the weaker (2.12) and, conventionally, we assume that for $p = \infty$, (2.12) just means $I = \mathbb{R}$. The second condition identifies the so-called *singular* potentials:

$$I = (-1, 1), \qquad \lim_{|r| \to 1^-} W'(r) r = +\infty. \tag{2.13}$$

In particular, in case (2.13) holds, then (2.10) follows immediately. Note that the domain I of W has been normalized to $(-1, 1)$ just for the sake of simplicity. We also let

$$f \in H. \tag{2.14}$$

Let us now introduce the *energy* of the system (possibly taking the value $+\infty$ for some v) as

$$\mathcal{E}(v) := \int_\Omega \left(\frac{|\nabla v|^2}{2} + W(v) + fv \right), \qquad \text{for } v \in V. \tag{2.15}$$

Then we define the space of data of *finite energy* as

$$\mathcal{V} := \{v \in V : W(v) \in L^1(\Omega)\}. \tag{2.16}$$

By continuity of the embedding $V \subset L^6(\Omega)$ it is clear that if (2.12) holds with $p \leqslant 6$, then it is in fact $\mathcal{V} = V$. Otherwise, $\mathcal{V}$ can be a proper subset of V.

We also set $\beta(r) := W'(r) + \lambda r$ for $r \in I$. On account of (2.11), β is a monotone function which will be sometimes identified with a *maximal monotone* operator from H to itself (note that the maximality of β follows from the second condition in (2.13) if W is singular). Then we define

$$\mathcal{W} := \{v \in H^2(\Omega) : \partial_{\boldsymbol{n}} v = 0 \text{ on } \partial\Omega, \beta(v) \in L^2(\Omega)\}. \tag{2.17}$$

The set $\mathcal{W}$ is nothing else than the domain (in H) of the *subdifferential* $\partial\mathcal{E}(v)$, where $\mathcal{E}$ is now seen as a (bounded from below) functional on H. We can also introduce a metric structure on $\mathcal{V}$ by setting

$$\mathrm{d}_{\mathcal{V}}(v, z) := \|v - z\| + \|W(v) - W(z)\|_{L^1(\Omega)} \qquad \forall\, v, z \in \mathcal{V}. \tag{2.18}$$

Proceeding as in [17, lemma 3.8], one can easily show that $\mathcal{V}$ is a complete metric space with the distance $\mathrm{d}_{\mathcal{V}}$. Of course, the contribution of the second term on the right-hand side given above is redundant and could be omitted so that $\mathcal{V} = V$, in case (2.12) holds with $p < 6$. Analogously, $\mathcal{W}$ is endowed with the distance

$$\mathrm{d}_{\mathcal{W}}(v, z) := \|v - z\|_{H^2(\Omega)} + \|\beta(v) - \beta(z)\| \qquad \forall\, v, z \in \mathcal{W}, \tag{2.19}$$

where the second term on the right-hand side is included only in case (2.13) holds; otherwise, it can be omitted. It is clear that $\mathcal{W}$ is also a complete metric space. Assuming (2.11) (respectively, (2.13)), we will take $m > 0$ (respectively, $m \in (0, 1)$) and consider the metric-closed subset of $\mathcal{V}$ given by

$$\mathcal{V}_m := \{v \in \mathcal{V} : \ |v_\Omega| \leqslant m\}. \tag{2.20}$$

We also define, analogously, a closed subset $\mathcal{W}_m$ of $\mathcal{W}$. The proof of the following result, which collects further properties of $\mathcal{V}$, $\mathcal{W}$ and of the energy $\mathcal{E}$, can be performed by standard semicontinuity arguments (cf also [17, lemma 4.2]), and it is thus omitted.

Lemma 2.1. *The functional $\mathcal{E}$ is sequentially weakly lower semicontinuous on V. Moreover, $\mathcal{W}$ is compactly embedded into $\mathcal{V}$, namely, any bounded sequence in $\mathcal{W}$ admits a subsequence converging in $\mathcal{V}$. Finally, the convergence $v_n \to v$ in $\mathcal{V}$ is equivalent to the coupling of*

$$v_n \to v \quad \text{weakly in } V \qquad \text{and} \qquad \limsup_{n \nearrow \infty} \mathcal{E}(v_n) \leqslant \mathcal{E}(v). \tag{2.21}$$

■

For m as above, the initial datum u_0 is then chosen such that

$$u_0 \in \mathcal{V}_m. \tag{2.22}$$

We are now ready to introduce our first notions of solutions to (1.1). We will treat the viscous ($\varepsilon > 0$) and the 'standard' ($\varepsilon = 0$) equations altogether.

Definition 2.2. *We call a (global)* energy solution *to problem* (P_ε) *if* $\varepsilon > 0$ *(respectively, to problem* (P_0) *if* $\varepsilon = 0$*) one function* $u : \Omega \times (0,\infty) \to \mathbb{R}$ *such that, for all* $T > 0$*, the regularity properties*

$$u \in H^1(0,T;V') \cap L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega)), \qquad \varepsilon^{1/2}u \in H^1(0,T;H), \tag{2.23}$$

$$W'(u) \in L^2(0,T;H), \qquad w \in L^2(0,T;V) \tag{2.24}$$

are fulfilled, and u satisfies, in the space V' *and for almost all times in* $(0,\infty)$*, the equations*

$$u_t + B_u w = 0, \tag{2.25}$$

$$w = \varepsilon u_t + Bu + W'(u) + f \tag{2.26}$$

and, a.e. in Ω*, the initial condition*

$$u|_{t=0} = u_0. \tag{2.27}$$

Moreover, we say that an energy solution is regularizing *if the properties*

$$u \in L^\infty(\tau,T;H^2(\Omega)), \qquad u_t \in L^2(\tau,T;V), \qquad \beta(u) \in L^\infty(\tau,T;H), \tag{2.28}$$

hold for all $\tau > 0$, $T \geqslant \tau$.

In the above statement, (1.1) has been split, for convenience, as a system of the two equations (2.25)–(2.26). Testing (2.25) by 1, one immediately sees that, for any energy solution to (P_ε) or to (P_0) corresponding to the initial datum u_0, there holds

$$(u(t))_\Omega = (u_0)_\Omega =: u_\Omega \qquad \forall t \geqslant 0. \tag{2.29}$$

Thus, $\mathcal{V}_m$ can be used as a *phase space* for the dynamical processes associated with problems (P_ε), (P_0).

Let us now come to mathematical results, and we start by establishing existence and, conditionally, uniqueness.

Theorem 2.3. *Let (2.1), (2.9)–(2.11), either (2.12) with* $p = \infty$ *or (2.13), (2.14) and (2.22) hold, and let* $\varepsilon \geqslant 0$*. Then, there exists at least one energy solution u to problem* (P_ε) *(if* $\varepsilon = 0$*, to problem* (P_0)*). Moreover, if* u_1, u_2 *are a pair of energy solutions either to* (P_ε) *or to* (P_0)*, satisfying, for some* $\tau \geqslant 0$*, property (2.28) and such that* $u_1(\tau) = u_2(\tau)$*, then* $u_1 \equiv u_2$ *on* $[\tau,\infty)$*. Finally, only in the case* $\varepsilon > 0$*, and if (2.12) holds with* $p \leqslant 6$*, then problem* (P_ε) *admits at least one* regularizing *solution u which, if* $u_0 \in \mathcal{W}$*, satisfies (2.28) also for* $\tau = 0$.

The proofs of the above theorem, and of the ones which follow, are all postponed to the next section. Note that the existence part of the above statement, at least for $\varepsilon = 0$, follows more or less the lines of [5, theorem 2.2], so that we do not claim originality here. Note also that, if $\varepsilon > 0$ and (2.12) holds with $p \leqslant 6$, uniqueness is satisfied starting from $\tau = 0$ if $u_0 \in \mathcal{W}$ and from any $\tau > 0$ if $u_0 \in \mathcal{V}$ (namely, we have *unique continuation* of trajectories). Instead, the uniqueness part might be vacuous (because we cannot prove the existence of regularizing solutions) in all other cases (in particular, if $\varepsilon = 0$).

Theorem 2.3 will be proved by working on an approximate statement that we now introduce. First of all, we replace W by a regularized potential W_n, with n intended to go to ∞ in the limit, constructed this way. Recalling that $\beta = (W' + \lambda\,\mathrm{Id})$ is monotone by (2.11), we denote as β_n its Yosida approximation of index n^{-1}, i.e.

$$\beta_n := n\left[\mathrm{Id} - \left(\mathrm{Id} + \frac{\beta}{n}\right)^{-1}\right], \tag{2.30}$$

where Id stands for the identity function. Next, we define $W_n' := \beta_n - \lambda\,\mathrm{Id}$. Then, W_n' is (globally in $\mathbb{R}$) Lipschitz continuous (the Lipschitz constant of course depending on n) and it tends to W' in the sense of *G-convergence* (see, e.g. [1, chapter 3]). This in particular entails that, if $r_n \to r$ and $\beta_n(r_n) \to \tilde{\beta}$ as $n\nearrow\infty$, then it turns out that $\tilde{\beta} = \beta(r)$.

Moreover, defining W_n by integration and choosing appropriately the integration constant, one has that (2.9), (2.11) (and possibly (2.12)) still hold for W_n, uniformly in n. Moreover, $W_n(r) \leqslant W(r)$ for all $n \in \mathbb{N}$, $r \in I$, and, in place of (2.10), there holds, at least for n sufficiently large,

$$W_n(r) \geqslant 2\lambda r^2 - c_W \qquad \forall n \in \mathbb{N}, \quad r \in \mathbb{R}. \tag{2.31}$$

Next, we replace $u_0 \in \mathcal{V}_m$ by a regularizing sequence $\{u_{0,n}\} \subset H^2(\Omega) \cap \mathcal{V}_m$ tending to u_0 in V for $n\nearrow\infty$. Namely, we define $u_{0,n}$ as the unique solution to the elliptic problem

$$u_{0,n} \in H^2(\Omega), \qquad \frac{1}{n} B u_{0,n} + u_{0,n} = u_0. \tag{2.32}$$

Then, it is well known [12] that $(u_{0,n}) \subset H^3(\Omega)$, $u_{0,n} \to u_0$ strongly in V, and

$$\|u_{0,n}\|_V \leqslant \|u_0\|_V, \qquad \|u_{0,n} - u_0\| \leqslant n^{-1/2}\|u_0\|_V \quad \forall n \in \mathbb{N}, \tag{2.33}$$

so that, by standard properties of subdifferentials and using (2.32), we also have

$$\begin{aligned}\int_\Omega W_n(u_{0,n}) &\leqslant \int_\Omega W_n(u_0) + (\beta_n(u_{0,n}), u_{0,n} - u_0) - \frac{\lambda}{2}\|u_{0,n}\|^2 + \frac{\lambda}{2}\|u_0\|^2 \\ &\leqslant \int_\Omega W_n(u_0) - \frac{\lambda}{2}\|u_{0,n}\|^2 + \frac{\lambda}{2}\|u_0\|^2 \leqslant \int_\Omega W(u_0) + \sigma_n,\end{aligned} \tag{2.34}$$

where σ_n goes to 0 as $n\nearrow\infty$. Then, if $\varepsilon > 0$, the replacements of W with W_n and of u_0 with $u_{0,n}$ give rise to a new problem $(\mathrm{P}_{n,\varepsilon})$. If $\varepsilon = 0$, we additionally take $\varepsilon = \varepsilon_n > 0$ in (2.26), where $(\varepsilon_n) \subset (0, 1)$ is some sequence going to 0 as $n\nearrow\infty$ (e.g. $\varepsilon_n = n^{-1}$), and we get a problem we call $(\mathrm{P}_{n,\varepsilon_n})$. It is clear that, at least formally, problems $(\mathrm{P}_{n,\varepsilon})$ and $(\mathrm{P}_{n,\varepsilon_n})$ tend, as $n\nearrow\infty$, to (P_ε) and (P_0), respectively. By a standard application of the Faedo–Galerkin method (cf [5] for the details), one can also show that, for every $n \in \mathbb{N}$, each problem $(\mathrm{P}_{n,\varepsilon})$, $(\mathrm{P}_{n,\varepsilon_n})$ admits one and only one (global in time) solution (in both cases we denote it as u_n). Actually, both the uniqueness property and the global character of the solutions are not directly guaranteed by the Faedo–Galerkin method, but they can be shown proceeding along the lines of the next section (we thus omit the details). Moreover, setting for $T > 0$

$$\mathcal{Y}_T := H^2(0, T; H) \cap W^{1,\infty}(0, T; V) \cap H^1(0, T; H^2(\Omega)), \tag{2.35}$$

it can be proved with only technical difficulties that, for all $n \in \mathbb{N}$,

$$u_n \in \mathcal{Y}_T \qquad \forall T > 0. \tag{2.36}$$

Let us then come to the long-time issue. To begin with, we need some preliminary work, starting with a simple property satisfied by all solutions.

Lemma 2.4. *Let u be an energy solution either to problem* (P_ε) *or to problem* (P_0) *and let $T > 0$. Then, $u(t) \in \mathcal{V}_m$ for* all *(not just a.e.) $t \in [0, T]$. Moreover, for* all $t, t_1, t_2 \geqslant 0$, *u satisfies the* energy equality

$$\mathcal{E}(u(t_2)) - \mathcal{E}(u(t_1)) + \int_{t_1}^{t_2} \int_\Omega (b(u)|\nabla w|^2) + \varepsilon \int_{t_1}^{t_2} \|u_t\|^2 = 0 \tag{2.37}$$

and the dissipativity estimate

$$\mathcal{E}(u(t)) \leqslant \mathcal{E}(u_0)\mathrm{e}^{-\kappa t} + C_0, \tag{2.38}$$

where $\kappa, C_0 > 0$ are computable constants, independent of both the initial data and ε (they can depend on m, cf (2.20), instead).

However, to define the dynamical processes associated with (P_ε) and (P_0), it seems necessary to restrict the classes of admissible solutions. Actually, while the above lemma shows uniform dissipativity for *all* energy solutions, as we look for some form of parabolic regularization in time, we readily realize that energy solutions need not be smooth enough to prove rigorously sharper estimates. For this reason, we 'essentially' (cf remark 2.6) have to work on problem ($P_{n,\varepsilon}$) (or on problem (P_{n,ε_n})) and then take the limit $n\nearrow\infty$. Let us then introduce a useful notion of convergence: given $T>0$, we say that a sequence (u_j) of functions from $\Omega\times(0,T)$ to $\mathbb{R}$ tends to u *weakly in* $\mathcal{V}_T$ if the following two properties hold:

$$u_j \to u \qquad \text{weakly in } H^1(0,T;V')\cap L^2(0,T;H^2(\Omega)) \quad (\text{also in } H^1(0,T;H), \text{ if } \varepsilon>0), \tag{2.39}$$

$$u_j \to u \qquad \text{weakly star in } L^\infty(0,T;V). \tag{2.40}$$

We can thus introduce the class of solutions for which we will prove the existence of the global attractor.

Definition 2.5. *Let u be an* energy solution *to problem* (P_ε) *(respectively, to problem* (P_0)*). We say that u is* limiting *if there exists a sequence* (u_j)*, with* $(u_j)\subset\mathcal{Y}_T$ *for all* $T>0$*, such that, as* $j\nearrow\infty$*,* u_j *tends to u weakly in* $\mathcal{V}_T$ *for all* $T>0$*; moreover, for each* $j\in\mathbb{N}$*, there exists* $n_j\in\mathbb{N}$*, with* $n_j\nearrow\infty$ *for* $j\nearrow\infty$ *(if* $\varepsilon=0$ *we also ask existence of* $(0,1)\ni\varepsilon_j\to 0$*), such that* u_j *solves, in the usual sense,*

$$u_{j,t}+B_{u_j}w_j=0, \tag{2.41}$$

$$w_j=\varepsilon_j u_{j,t}+Bu_j+W'_{n_j}(u_j)+f \tag{2.42}$$

(with $\varepsilon_j\equiv\varepsilon$ *if* $\varepsilon>0$*) and, finally, there hold*

$$W'_{n_j}(u_j)\to W'(u) \qquad \text{weakly in } L^2(0,T;H), \qquad \forall T>0, \tag{2.43}$$

$$\exists\,\sigma_j\searrow 0:\quad \mathcal{E}_{n_j}(u_j(0))\leqslant\mathcal{E}(u_0)+\sigma_j. \tag{2.44}$$

Here, for $n\in\mathbb{N}$*, the approximate energy* $\mathcal{E}_n$ *is defined as in (2.15), but with* W_n *replacing* W*.*

Remark 2.6. In the proof of theorem 2.3 we will show that the class of limiting solutions is not empty by passing to the limit $n\nearrow\infty$ in ($P_{n,\varepsilon}$) (or (P_{n,ε_n})). In particular, we then have that (2.44) is satisfied thanks to (2.33) and (2.34). However, this natural procedure seems not 'robust' enough (especially from the viewpoint of taking subsequences) to guarantee that the resulting set of solutions satisfies the desired semiflow properties (in particular, (H4) in definition 2.8). This is the reason why in definition 2.5 we are forced to generalize the method a bit (which, as a further consequence, possibly enlarges the class of limiting solutions).

Remark 2.7. Clearly, due to nonuniqueness, there might be energy solutions which are not limiting. Even though it is not excluded that (some of) these solutions may have the same good regularity properties as the limiting ones, they have to be forcedly excluded from the long-time analysis.

To study the long-time dynamics of 'limiting solutions', we also need to introduce a suitable extension (cf [18, section 2.2]) of the concept of 'generalized semiflow' introduced by Ball in the celebrated papers [2, 3].

Definition 2.8. *We say that a family $\mathcal{S}$ of maps from* $[0, \infty)$ *to a metric space X is a* limiting semiflow *on the* phase space X *if the following properties hold.*

(H1) *For all $u_0 \in X$ there exists at least one $u \in \mathcal{S}$ such that $u(0) = u_0$* (existence property).
(H2) *For all $u \in \mathcal{S}$ and every $\tau \geqslant 0$, the function u^τ defined for $t \geqslant 0$ by $u^\tau(t) := u(t + \tau)$ still belongs to $\mathcal{S}$* (translation invariance).
(H4) *For all sequence $(u_k) \subset \mathcal{S}$ such that $u_{0,k} := u_k(0)$ tends in X to some u_0, there exist $u \in \mathcal{S}$ such that $u(0) = u_0$ and a nonrelabelled subsequence of k such that, for all $t \geqslant 0$, $u_k(t) \to u(t)$ in X* (upper semicontinuity w.r.t. initial data).

Remark 2.9. In the above definition, we kept Ball's notation. The property missing here (obviously named (H3) in Ball's papers) essentially states that if we *concatenate* a pair of solutions u_1, u_2, respectively, defined on $[0, t_1]$ and on $[t_1, t_2]$ and such that $u_1(t_1) = u_2(t_1)$, we still obtain a solution. Unfortunately, it is very difficult to prove such a condition for semiflows constructed by means of an approximation-limit argument, and the main reason is that the values in t_1 of the trajectories approximating the two solutions may be incompatible. From the point of view of asymptotics, it is shown in [18, theorem 2.9] (reported here as theorem 2.15) that the global attractor $\mathcal{A}$ still exists and is unique for limiting semiflows, under the natural conditions of dissipativity and eventual compactness. However, $\mathcal{A}$ turns out to be just *quasi-invariant* (i.e. any $u_0 \in \mathcal{A}$ can be seen as the initial value of a *complete orbit* of the semiflow taking its values in $\mathcal{A}$ for all $t \in \mathbb{R}$) rather than *fully invariant* (as it happens in Ball's theory). We refer to [18, subsection 2.4] for further remarks on this point and to [14] for an alternative approach based on multivalued semiflows.

The limiting solutions to (P_ε) and (P_0) turn out to fit the above definition, at least for a restricted class of potentials, including also some *singular* cases.

Definition 2.10. *We say that the potential W is* separating *if the following conditions are fulfilled. First, (2.13) holds. Second, for all $v \in \mathcal{W}$, there holds that* $\max\{|v(x)|, x \in \bar{\Omega}\} < 1$. *Third, there exists an increasingly monotone function $\phi : [0, \infty) \to [0, 1)$ such that*

$$\|v\|^2_{C^0(\bar{\Omega})} \leqslant \phi(\|v\|^2_{W^{1,6}(\Omega)} + \|\beta(v)\|^2) \qquad \forall\, v \in \mathcal{W}. \tag{2.45}$$

Remark 2.11. An easy refinement of the argument in [21, proposition 2.10] shows that a sufficient condition for W to be separating is that W' explodes sufficiently fast in the proximity of ± 1. Namely, in 3D, W is separating provided that, for some $c > 0$ (recall that $\beta = W' + \lambda\,\mathrm{Id}$),

$$\beta(r) \geqslant \frac{c}{(1-r)^3}, \qquad -\beta(r) \geqslant \frac{c}{(1+r)^3}, \tag{2.46}$$

respectively, in the left neighbourhood of 1 and in the right neighbourhood of -1. Actually, it is shown in [21, proposition 2.10] that if $v \in \mathcal{W}$ (which entails that the argument of ϕ in (2.45) is finite thanks to the continuous embedding $H^2(\Omega) \subset W^{1,6}(\Omega)$) and (2.46) holds, then the maximum on $\bar{\Omega}$ of v is strictly smaller than 1, in a way that (monotonically) depends on the distance $\mathrm{d}_{\mathcal{W}}(v, 0)$. Of course, we cannot exclude that a refinement of the argument in [21, proposition 2.10] might show that the class of separating potentials is in fact wider.

Remark 2.12. We point out that there exist other approaches which may lead to a separation property such as (2.45). For instance, in the case of a constant mobility, a comparison method has been successfully used in [15] for a wide class of potentials. However, it seems hard to adapt that argument to the nonconstant mobility setting.

The key point to show the existence of the global attractor is the following 'local regularization' property of limiting solutions.

Lemma 2.13. *Let (2.1), (2.9)–(2.11) and (2.14) hold. Let also W either satisfy (2.12) for $p = \infty$ or be* separating. *Let u be a limiting solution either to* (P_ε) *or to* (P_0)*. Then, there exist T_0 depending on $\mathcal{E}(u_0)$ and $C_0, \delta > 0$ independent of T_0, u_0 and ε, such that, for all $T \geqslant T_0$, there holds the property*

$$\exists\, \tau = \tau(T) \in [0, 3/2] : \mathrm{d}_{\mathcal{W}}(u(t), 0) \leqslant C_0 \qquad \forall\, t \in [T + \tau, T + \tau + \delta]. \tag{2.47}$$

Condition (2.47) states that we are not able to prove *uniform in time* regularization for limiting solutions. Nevertheless, at least for a sequence of intervals whose length δ is uniformly controlled from below, the limiting solutions take values in a bounded ball of $\mathcal{W}$, which is a relatively compact set of $\mathcal{V}$ thanks to lemma 2.1. In this sense, (2.47) can be thought as a 'locally uniform' regularization property. We can now state the following.

Theorem 2.14. *Let (2.1), (2.9)–(2.11) and (2.14) hold. Let also W either satisfy (2.12) for $p = \infty$ or be* separating. *Then, the* limiting solutions *to* (P_ε) *(respectively, to* (P_0)*) constitute a* limiting semiflow $\mathcal{S}_\varepsilon$ *(respectively, $\mathcal{S}_0$) on $\mathcal{V}_m$.*

In the following statement [18, theorem 2.9] (see also [2, theorem 3.3]) we collect the definition of global attractor for a limiting semiflow and the basic tool to prove its existence.

Theorem 2.15. *Let $\mathcal{S}$ be a limiting semiflow on the metric space X. Then, a compact set $\mathcal{A} \subset X$ is the global attractor for $\mathcal{S}$ if it is compact, quasi-invariant and it attracts all bounded sets of X w.r.t. its metric. The attractor $\mathcal{A}$ exists if and only if $\mathcal{S}$ satisfies the following properties.*

(A1) *There exists a metric bounded set $B_0 \subset X$ such that any $u \in \mathcal{S}$ eventually takes values in B_0* (point dissipativity).

(A2) *For all $(u_k) \subset \mathcal{S}$ such that $(u_{0,k})$ is bounded in X (where $u_{0,k} := u_k(0)$) and all (t_k) such that $t_k \nearrow \infty$, there exist $u_\infty \in X$ and a (nonrelabelled) subsequence of k such that $u_k(t_k) \to u_\infty$ in X* (asymptotic compactness).

Finally, if $\mathcal{A}$ exists, it is then unique.

Property (2.47) and a careful use of the energy equality are also the key tools to prove the following theorem.

Theorem 2.16. *Let (2.1), (2.9)–(2.11) and (2.14) hold. Let also W either satisfy (2.12) for $p = \infty$ or be* separating. *Then, the semiflows $\mathcal{S}_\varepsilon$, $\mathcal{S}_0$ admit compact global attractors $\mathcal{A}_\varepsilon$, $\mathcal{A}_0$ in the sense of theorem 2.15. Moreover, $\mathcal{A}_\varepsilon$ is bounded w.r.t. the $\mathcal{W}_m$-metric defined in (2.19), if (2.12) holds with $p \leqslant 6$.*

3. Proofs

In what follows, the symbols c, c_i and C_i, with $i \geqslant 0$, will denote positive constants, depending on the data b, W, f of the problem, but independent of ε, u_0, of time and of approximation parameters (e.g. of n in problems ($\mathrm{P}_{n,\varepsilon}$), ($\mathrm{P}_{n,\varepsilon_n}$)). In particular, small letters c and c_i will be used in the computations and capital letters C_i in the resulting estimates. Dependence on m (cf (2.22)) is allowed. Moreover, the value of c may vary even inside a single line. The symbol c_Ω will denote some embedding constants depending only on Ω. Finally, $\mathfrak{c}$, $\mathfrak{c}_i$, $i \geqslant 0$, will stand for positive constants with additional dependences (e.g. on time or on u_0), specified on occurrence.

Proof of theorem 2.3. Let, for $n \in \mathbb{N}$, u be a solution either to problem ($\mathrm{P}_{n,\varepsilon_n}$) or to problem ($\mathrm{P}_{n,\varepsilon}$) introduced in the previous section. The subscript n is omitted in the notation of u just for brevity. We now perform some *a priori* estimates, with the purpose of removing the n-approximation. Note that we can take advantage of regularity (2.36) so that all of the procedure below makes sense rigorously. Moreover, there holds

$$\mathcal{E}(v), \mathcal{E}_n(v) \geqslant \eta \|v\|_V^2 - c \qquad \forall\, v \in V,\ n \in \mathbb{N}, \tag{3.1}$$

where $\eta > 0$ depends on λ, c_W, f and is independent of n. Testing (2.25) by w, (2.26) by u_t, taking the sum and using (2.1), we obtain the approximate energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_n(u) + \int_\Omega (b(u)|\nabla w|^2) + \varepsilon \|u_t\|^2 = 0, \tag{3.2}$$

which holds at least a.e. in time. Hence, by (2.1), $\mathcal{E}_n$ is a Liapounov functional. To get dissipativity, we also have to test (2.26) by $u - u_\Omega$. By (2.7), (2.29) and the Young and Poincaré–Wirtinger inequality (in the form

$$\|v - v_\Omega\|^2 \leqslant c_\Omega \|\nabla v\|^2 \qquad \forall\, v \in V \tag{3.3}$$

and for some $c_\Omega > 0$), we get

$$\begin{aligned}\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\|u - u_\Omega\|^2 + 2\|\nabla u\|^2 + 2\int_\Omega (W_n'(u) + f)(u - u_\Omega) &= 2\int_\Omega w(u - u_\Omega) \\ &= 2\int_\Omega (w - w_\Omega)(u - u_\Omega) \\ &\leqslant 2\|u - u_\Omega\|\|w - w_\Omega\| \leqslant \|\nabla u\|^2 + c_\Omega^2 \|\nabla w\|^2.\end{aligned} \tag{3.4}$$

Let us observe that, by monotonicity of $r \mapsto \beta_n(r) = W_n'(r) + \lambda r$,

$$W_n'(r)(r - r_0) \geqslant W_n(r) - W_n(r_0) - \lambda(r^2 + r_0^2) \qquad \forall\, r, r_0 \in \mathbb{R}. \tag{3.5}$$

Thus, noting that, thanks also to (2.22),

$$W_n(u_\Omega) \leqslant W(u_\Omega) \leqslant c = c(W, m) \tag{3.6}$$

and using (2.31) and (3.3), the last term on the left-hand side of (3.4) is estimated by

$$\begin{aligned}2\int_\Omega (W_n'(u) + f)(u - u_\Omega) &\geqslant 2\int_\Omega W_n(u) - 2\lambda\|u\|^2 + \int_\Omega f(u - u_\Omega) - c \\ &\geqslant \int_\Omega W_n(u) - \frac{1}{2}\|\nabla u\|^2 - c.\end{aligned} \tag{3.7}$$

Thus, summing together (3.2) and $2^{-1}\alpha c_\Omega^{-2} \times$(3.4) and taking (2.1) and (3.7) into account, we readily get, for some $\kappa, C_0 > 0$ with the same dependences as the generic c,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}_n(u) + \frac{\alpha\varepsilon}{2c_\Omega^2}\|u - u_\Omega\|^2\right) + \kappa(\mathcal{E}_n(u) + \|\nabla w\|^2 + \varepsilon\|u_t\|^2) \leqslant C_0. \tag{3.8}$$

By Gronwall's lemma, this gives (2.38) (cf lemma 2.4), with $\mathcal{E}_n$ in place of $\mathcal{E}$. Next, we test (2.26) by Bu. Using (2.11) together with the Hölder and Young inequalities, we infer

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t}\|\nabla u\|^2 + \|Bu\|^2 \leqslant \|f\|^2 + (2\lambda + 1)\|\nabla u\|^2 + \|\nabla w\|^2. \tag{3.9}$$

The combination of (3.2) and (3.9) then immediately gives (2.23) (at the n-approximated level).

To get (2.24), it remains to estimate the space averages of $W_n'(u)$ and w_Ω. To do this, we can proceed by using an argument devised in [11] (see also [6, section 5]) which is just sketched here. Namely, we first have to compute (2.26) times $\beta_n(u) - (\beta_n(u))_\Omega$. By standard calculations, this gives

$$\|\beta_n(u) - (\beta_n(u))_\Omega\|^2 \leqslant c(1 + \|\nabla u\|^2 + \|\nabla w\|^2 + \varepsilon^2\|u_t\|^2). \tag{3.10}$$

Proceeding, e.g. as in [6, (5.32)–(5.33)], we also get

$$|(\beta_n(u))_\Omega|^2 \leqslant c\|\beta_n(u) - (\beta_n(u))_\Omega\|^2\|u - u_\Omega\|^2, \tag{3.11}$$

where the constant c depends in particular on m. Then, the coupling of (3.10) and (3.11) together with (2.23) and a further comparison in (2.26) (made in order to estimate w_Ω) readily give (2.24) (note that all constants in the procedure are independent of n). More precisely, integrating (3.2), (3.9) and (3.10) from t to $t+1$, for $t \geqslant \tau \geqslant 0$, recalling (2.38) and also taking (3.11) into account, it is not difficult to infer that, for some monotone function $M : [0,\infty) \to [0,\infty)$ independent of n and possibly new values of C_0, κ, there holds

$$\sup_{t\in(\tau,\infty)} \int_t^{t+1} (\|u(s)\|^2_{H^2(\Omega)} + \|w(s)\|^2_V + \|\beta_n(u(s))\|^2)\,\mathrm{d}s \leqslant M(\mathcal{E}(u_0))\mathrm{e}^{-\kappa\tau} + C_0. \tag{3.12}$$

To proceed, we now prove that, as at least a subsequence of n goes to infinity, the solutions to $(\mathrm{P}_{n,\varepsilon})$ (or $(\mathrm{P}_{n,\varepsilon_n})$) pass to the limit yielding (at least) one solution to (P_ε) (or to (P_0)) which satisfies the same bounds (and w.r.t. the same constants). This is standard and can be performed essentially as in [5] or [6], so we will give very few details.

Actually, the uniform bounds corresponding to (2.23) and (2.24) entail that for a (not relabelled) subsequence of n there holds convergence to a proper limit function. Then to show that this limit function solves (2.25)–(2.26) in the original form and satisfies (2.27) one has to pass to the limit the nonlinear terms. In particular, to identify the term depending on W', the G-convergence $W'_n \to W'$ and a standard monotonicity argument are used (note that W' is monotone up to a linear perturbation and refer, e.g. to [1, proposition 3.59, p 361] or [4, proposition 1.1, p 42]). Moreover, to treat the mobility term, one can observe that by (2.1) $b(u)$ converges strongly in $L^a(\Omega\times(0,T))$ for all $a \in [1,\infty)$ and weakly star in $L^\infty(\Omega\times(0,T))$ for all $T > 0$. Thus, by the bound on w in (3.12), also the product $b(u)\nabla w$ (or, analogously, the operator B_u) passes to the proper limit. Let us also notice that the dissipativity bounds (2.38) and (3.12) are still valid in the limit with the same constant C_0 and κ thanks to the semicontinuity property of norms w.r.t. weak or weak star convergences.

Next, let us show (conditional) uniqueness. To start with, note that if u is an energy solution additionally satisfying (2.28) for some $\tau \geqslant 0$, $T \geqslant \tau$, then, by the first of (2.28), the multiplication operator

$$V \to V, \qquad v \mapsto b(u(t))v \tag{3.13}$$

is continuous for a.a. $t \in (0,T)$. Then (2.25) can be rewritten, a.e. in (τ,T), as the relation in V'

$$Bw = -\frac{1}{b(u)}u_t + \frac{b'(u)}{b(u)}\nabla u\cdot\nabla w. \tag{3.14}$$

Moreover, evaluating the H-norm of the latter term on the right-hand side we have

$$\begin{aligned}\left\|\frac{b'(u)}{b(u)}\nabla u\cdot\nabla w\right\| &\leqslant c\|\nabla u\|_{L^6(\Omega)}\|\nabla w\|_{L^3(\Omega)} \leqslant c\|u\|_{H^2(\Omega)}\|\nabla w\|^{1/2}_{L^6(\Omega)}\|\nabla w\|^{1/2}\\ &\leqslant \mathfrak{c}\|w\|^{1/2}_{H^2(\Omega)}\|\nabla w\|^{1/2} \leqslant \frac{1}{2}\|Bw\| + \mathfrak{c}\|w\|_V,\end{aligned} \tag{3.15}$$

where the constants $\mathfrak{c}$ depend also on the $L^\infty(\tau,T;H^2(\Omega))$-norm of u. Thus,

$$w \in L^2(\tau,T;H^2(\Omega)). \tag{3.16}$$

Then, following closely the procedure in [5, proof of theorem 2.2], we consider a pair of solutions u_1, u_2 both satisfying (2.28) (and consequently (3.16)) for some $\tau \geqslant 0$, $T \geqslant \tau$. We also assume that $u_1(\tau) = u_2(\tau)$. Then we set $u := u_1 - u_2$, $w := w_1 - w_2$, compute $((2.25)_1–(2.25)_2) \times \mathcal{N}_{u_1} u$ and subtract $((2.26)_1–(2.26)_2) \times u$ from the result. Note that, actually, $u \in V_0$ a.e. in time. Setting $\zeta := \mathcal{N}_{u_1} u$, so that $B_{u_1}\zeta = u$, we have

$$\frac{\varepsilon}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2 + \|\nabla u\|^2 + (\zeta, (B_{u_1}\zeta)_t) \leqslant \lambda\|u\|^2 - \int_\Omega (b(u_1) - b(u_2))\nabla w_2 \cdot \nabla\zeta. \tag{3.17}$$

Then it is not difficult to see that

$$(\zeta, (B_{u_1}\zeta)_t) \geqslant \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{b(u_1)}{2}|\nabla\zeta|^2 - c\|u_{1,t}\|_{L^6(\Omega)}\|\nabla\zeta\|^2_{L^{12/5}(\Omega)}. \tag{3.18}$$

Moreover, noting that there holds (cf [5, (2.26)] or [20, lemma 1.2])

$$\|\zeta\|_{H^2(\Omega)} \leqslant c(\|u\| + \|u_1\|^2_{H^2(\Omega)}\|\nabla\zeta\|), \tag{3.19}$$

we get

$$\begin{aligned}\|\nabla\zeta\|^2_{L^{12/5}(\Omega)} &\leqslant c\|\nabla\zeta\|^{3/2}\|\zeta\|^{1/2}_{H^2(\Omega)} \leqslant c\|\nabla\zeta\|^{3/2}(\|u\|^{1/2} + \|u_1\|_{H^2(\Omega)}\|\nabla\zeta\|^{1/2})\\ &\leqslant \mathfrak{c}\|\nabla\zeta\|^2 + \mathfrak{c}_0\|u\|^2,\end{aligned} \tag{3.20}$$

the constants $\mathfrak{c}, \mathfrak{c}_0$ also depending on u_1. Finally, using (2.3), the analogue of (3.19) and standard embedding and interpolation inequalities, we get

$$\begin{aligned}-\int_\Omega (b(u_1) - b(u_2))\nabla w_2 \cdot \nabla\zeta &\leqslant c\|u\|_{L^3(\Omega)}\|\nabla w_2\|_{L^6(\Omega)}\|\nabla\zeta\|\\ &\leqslant c\|\nabla u\|^{3/4}\|u\|^{1/4}_{V'}\|w_2\|_{H^2(\Omega)}\|\nabla\zeta\|\\ &\leqslant c\|\nabla u\|^{3/4}\|w_2\|_{H^2(\Omega)}\|\nabla\zeta\|^{5/4} \leqslant \tfrac{1}{4}\|\nabla u\|^2 + c\|w_2\|^{8/5}_{H^2(\Omega)}\|\nabla\zeta\|^2.\end{aligned} \tag{3.21}$$

Thus, noting that, by interpolation,

$$(\mathfrak{c}_0 + \lambda)\|u\|^2 \leqslant \tfrac{1}{4}\|\nabla u\|^2 + \mathfrak{c}\|\nabla\zeta\|^2, \tag{3.22}$$

and since w_2 complies with (3.16), an application of the Gronwall lemma in (3.17) allows us to conclude on the uniqueness of regularizing solutions.

Finally, let us now assume $\varepsilon > 0$ and that (2.12) holds with $p = 6$ and let us prove (2.28). Of course, to be fully rigorous, we should work on the solution to $(\mathrm{P}_{n,\varepsilon})$ and then pass to the limit, but, for brevity and since everything is standard, we assume here directly that u solves (P_ε). Then, testing (2.26) by Bu_t, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\|Bu\|^2 + 2(f, Bu)) + 2\varepsilon\|\nabla u_t\|^2 &\leqslant 2(w - W_n'(u), Bu_t)\\ &\leqslant \varepsilon\|\nabla u_t\|^2 + \frac{1}{\varepsilon}\|\nabla w + W_n''(u)\nabla u\|^2.\end{aligned} \tag{3.23}$$

To estimate the latter term, we use (2.12) (which holds uniformly in n), (2.23), standard interpolation inequalities and the continuous embeddings $V \subset L^6(\Omega)$ and $H^{5/4}(\Omega) \subset L^{12}(\Omega)$. Namely, we have

$$\begin{aligned}\|W_n''(u)\nabla u\|^2 &\leqslant c\|\nabla u\|^2_{L^6(\Omega)}(1 + \|u\|^8_{L^{12}(\Omega)})\\ &\leqslant c\|u\|^2_{H^2(\Omega)}(1 + \|u\|^6_V\|u\|^2_{H^2(\Omega)}).\end{aligned} \tag{3.24}$$

Thus, (3.23) gives, for all $t \geqslant \tau \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|Bu\|^2 + 2(f, Bu)) + \varepsilon\|\nabla u_t\|^2 \leqslant c_\varepsilon(M(\mathcal{E}(u_0))\mathrm{e}^{-\kappa\tau} + C_0)(1 + \|Bu\|^4), \tag{3.25}$$

where $c_\varepsilon > 0$ depends on ε and explodes as $\varepsilon \searrow 0$. Thus, recalling (3.12) and using the standard Gronwall lemma if $u_0 \in \mathcal{W}_m$ and the *uniform* Gronwall lemma [24, lemma I.1.1] if it is just $u_0 \in \mathcal{V}_m$, we get the regularity of u in (2.28), respectively, for $\tau \geqslant 0$ and $\tau > 0$. To conclude, we notice that, since $p = 6$ in (2.12), $\|\beta(u)\| \leqslant c(1 + \|u\|^4_{L^8(\Omega)})$, so that also the third of (2.28) holds. The proof of theorem 2.3 is thus complete. ■

Proof of lemma 2.4. The energy equality can be proved as for the approximated problem (i.e. testing (2.25) by w, (2.26) by u_t and taking the difference). The key point is that, for $\varepsilon > 0$, the regularities (2.23)–(2.24) are sufficient to apply the chain rule formula [7, lemma 3.3, p 73]; actually, all the terms in (2.26) lie in $L^2(0, T; H)$ as well as the test function u_t. As we consider, instead, energy solutions to (P$_0$), u_t is only in $L^2(0, T; V')$ and the terms in (2.26) do not lie, each one separately, in $L^2(0, T; V)$. Nevertheless, since both w and the sum $Bu + W'(u) + f$ lie in $L^2(0, T; V)$, one can use, e.g. [17, lemma 4.1] (note that W satisfies the growth assumption [17, (4.23)] due to (2.10)) and still conclude for (2.37). Observe also that, as a by-product, the function $t \mapsto \mathcal{E}(u(t))$ is absolutely continuous on $[0, T]$. Finally, by the same type of considerations, also the procedure used to get (2.38) (cf the computation leading to (3.8)) can be justified. ■

Proof of lemma 2.13. Let u be a limiting solution either to problem (P$_\varepsilon$) or to (P$_0$), and let u_j, n_j and ε_j be as in definition 2.5. Since $u_j \in \mathcal{Y}_T$ for all $j \in \mathbb{N}$ and $T > 0$, we can test (2.42) by $Bu_{j,t}$. We then get (cf (3.23))

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|Bu_j\|^2 + 2(f, Bu_j)) + 2\varepsilon_j \|\nabla u_{j,t}\|^2 \leqslant 2(w_j - W'_{n_j}(u_j), Bu_{j,t}). \tag{3.26}$$

Next, we test (2.41) by $w_{j,t}$ and subtract from the result the expression obtained by differentiating in time (2.42) and testing it by $u_{j,t}$. Using (2.1) and (2.11) (which still holds for W_{n_j}), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varepsilon_j \|u_{j,t}\|^2 + \int_\Omega (b(u_j)|\nabla w_j|^2)\right) + 2\|\nabla u_{j,t}\|^2 \leqslant 2\lambda \|u_{j,t}\|^2 + c\int_\Omega (|u_{j,t}||\nabla w_j|^2). \tag{3.27}$$

Next, we compute

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_j\|^2 + \|\beta_{n_j}(u_j)\|^2) = 2(u_j, u_{j,t}) + \int_\Omega (\beta'_{n_j}(u_j)\beta_{n_j}(u_j)u_{j,t})$$
$$\leqslant \|u_j\|^2 + \|u_{j,t}\|^2 + \int_\Omega (\beta'_{n_j}(u_j)\beta_{n_j}(u_j)u_{j,t}). \tag{3.28}$$

Then, we estimate the terms on the right-hand sides of (3.26) and (3.27):

$$2(w_j, Bu_{j,t}) \leqslant 4\|\nabla w_j\|^2 + \frac{1}{4}\|\nabla u_{j,t}\|^2, \tag{3.29}$$

$$-2(W'_{n_j}(u_j), Bu_{j,t}) = 2\lambda(u_j, Bu_{j,t}) - 2(\beta_{n_j}(u_j), Bu_{j,t})$$
$$\leqslant \frac{1}{4}\|\nabla u_{j,t}\|^2 + c\|\nabla u_j\|^2 - 2\int_\Omega (\beta'_{n_j}(u_j)\nabla u_j \cdot \nabla u_{j,t}), \tag{3.30}$$

$$\int_\Omega (|u_{j,t}||\nabla w_j|^2) \leqslant c\|u_{j,t}\|_{L^6(\Omega)}\|\nabla w_j\|^{3/2}(\|\nabla w_j\|^{1/2} + \|Bw_j\|^{1/2}). \tag{3.31}$$

Using the analogue of (3.19), one gets from (3.31) that

$$\int_\Omega (|u_{j,t}||\nabla w_j|^2) \leqslant \frac{1}{4}\|u_{j,t}\|_V^2 + c\|\nabla w_j\|^4 + c\|\nabla w_j\|^6 + c\|u_j\|^6_{H^2(\Omega)}. \tag{3.32}$$

Next, we estimate the last terms on the right-hand sides of (3.28) and (3.30) this way:

$$2\int_\Omega \beta'_{n_j}(u_j)(\beta_{n_j}(u_j)u_{j,t} - \nabla u_j \cdot \nabla u_{j,t})$$
$$\leqslant \frac{1}{4}\|u_{j,t}\|_V^2 + c\|u_j\|_{H^2(\Omega)}^6 + c\|\beta_{n_j}(u_j)\|^6 + c\|\beta'_{n_j}(u_j)\|_{L^3(\Omega)}^3. \tag{3.33}$$

Summing now (3.26), (3.27) and (3.28), using on $H^2(\Omega)$ the equivalent norm $(\|\cdot\|^2+\|B\cdot\|^2)^{1/2}$ and owing to (3.29)–(3.33), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}(\|u_j\|_{H^2(\Omega)}^2 + 2(f, Bu_j) + \varepsilon_j\|u_{j,t}\|^2 + \int_\Omega(b(u_j)|\nabla w_j|^2) + \|\beta_{n_j}(u_j)\|^2) + (1+2\varepsilon_j)\|\nabla u_{j,t}\|^2$$
$$\leqslant c_1(1 + \|u_{j,t}\|^2 + \|\nabla w_j\|^6 + \|u_j\|_{H^2(\Omega)}^6 + \|\beta_{n_j}(u_j)\|^6 + \|\beta'_{n_j}(u_j)\|_{L^3(\Omega)}^3). \tag{3.34}$$

Again, by interpolation and recalling (2.1), we have

$$c_1\|u_{j,t}\|^2 \leqslant \tfrac{1}{2}\|\nabla u_{j,t}\|^2 + c\|u_{j,t}\|_{V'}^2 \leqslant \tfrac{1}{2}\|\nabla u_{j,t}\|^2 + c\|\nabla w_j\|^2. \tag{3.35}$$

To proceed, we start considering the simpler case when $I = \mathbb{R}$ (i.e. (2.12) holds with $p = \infty$). Then, noting as $\upsilon_n := (\mathrm{Id} + n^{-1}\beta)^{-1}$ the *resolvent* of β of index n^{-1} and recalling that, for all n, υ_n is 1-Lipschitz and satisfies $\beta_n = \beta \circ \upsilon_n$ and $\upsilon_{n_j}(0) = 0$ (due to (2.9)), it is not difficult to realize that

$$\|\beta'_{n_j}(u_j)\|_{L^3(\Omega)}^3 \leqslant \|\beta'(\upsilon_{n_j}(u_j))\|_{L^3(\Omega)}^3 \leqslant \gamma(\|u_j\|_{L^\infty(\Omega)}^2), \tag{3.36}$$

where we have set, for $s \geqslant 0$,

$$\gamma_0(s) := |\Omega| \max\{\beta'(r)^3 + \beta'(-r)^3, r \in [0, s]\}, \qquad \gamma(r) := \gamma_0(s^{1/2}), \tag{3.37}$$

so that $\gamma : [0, \infty) \to [0, \infty)$ is monotone. By continuity of the embedding $H^2(\Omega) \subset L^\infty(\Omega)$, it follows that, for all $c_2 \geqslant 0$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_2 + \|u_j\|_{H^2(\Omega)}^2 + 2(f, Bu_j) + \varepsilon_j\|u_{j,t}\|^2 + \int_\Omega (b(u_j)|\nabla w_j|^2) + \|\beta_{n_j}(u_j)\|^2\right)$$
$$+\left(\frac{1}{2} + 2\varepsilon_j\right)\|\nabla u_{j,t}\|^2$$
$$\leqslant c_1(1 + \|\nabla w_j\|^6 + \|u_j\|_{H^2(\Omega)}^6 + \|\beta_{n_j}(u_j)\|^6 + \gamma(c_\Omega\|u_j\|_{H^2(\Omega)}^2)). \tag{3.38}$$

Then, noting as y_j the function whose time derivative appears on the left-hand side and then choosing $c_2 > 0$ so that $y_j \geqslant 0$ (note that this can be done independently of the initial datum), the above relation can be interpreted in the form

$$y'_j(t) \leqslant \psi(y_j(t)), \tag{3.39}$$

where $\psi : [0, \infty) \to [0, \infty)$ is a suitable monotone function depending on γ, but independent of j. Now, let us observe that u_j satisfies the analogue of (3.12), namely,

$$\sup_{t\in(T_0,\infty)} \int_t^{t+1} (\|u_j(s)\|_{H^2(\Omega)}^2 + \|w_j(s)\|_V^2 + \|\beta_{n_j}(u_j(s))\|^2)\,\mathrm{d}s \leqslant M(\mathcal{E}(u_0) + \sigma_j)\mathrm{e}^{-\kappa T_0} + C_0, \tag{3.40}$$

where σ_j is as in (2.44) and $T_0 > 0$. Actually, the above surely holds if u_j is a solution to some $(\mathrm{P}_{n,\varepsilon})$ (or $(\mathrm{P}_{n,\varepsilon_n})$). Here, although the choice of u_j is slightly more general (cf remark 2.6), it is easy to see that (3.40) is still satisfied. Then, taking j_0 large enough so that $\sigma_j \leqslant \mathcal{E}(u_0)$ for all $j \geqslant j_0$, it is clear that also T_0 can be chosen so that the right-hand side given above is less than $2C_0$. Thus, we can directly assume $T_0 = 1$ for simplicity of notation and without loss of generality. In what follows, we denote as $(j)_i$ a subsequence of $(j)_{j\geqslant j_0} =: (j)_0$ obtained by successive extractions. Namely, for all i, $(j)_i$ is a subsequence of $(j)_{i-1}$. The indices belonging to $(j)_i$ will be still indicated just by j, for notational simplicity.

Relation (3.40) readily implies that there exists $C_1 > 0$ such that, for all $m \in \mathbb{N}$ (recall we assumed $T_0 = 1$) and all $j \in (j)_0$, there exists $t_{m,j} \in [m, m+1/2]$ such that $y_j(t_{m,j}) \leqslant C_1$. Then, defining

$$\Psi(s) := \int_{C_1}^{s} \frac{\mathrm{d}r}{\psi(r)} \tag{3.41}$$

and solving (3.39), it is clear that

$$\Psi(y_j(t)) = \Psi(y_j(t_{m,j})) + (t - t_{m,j}) \leqslant t - t_{m,j}, \tag{3.42}$$

at least for all $t \geqslant t_{m,j}$ such that the above relation makes sense. This implies in particular that there exist $\delta \in (0, 1/4]$ and $C_2 > 0$, both independent of ε, m, j and u_0 and such that

$$\|u_j\|^2_{L^\infty(t_{m,j},t_{m,j}+2\delta;H^2(\Omega))} + \|\beta_{n_j}(u_j)\|^2_{L^\infty(t_{m,j},t_{m,j}+2\delta;H)} \leqslant \Psi^{-1}(\delta) = C_2 < \infty, \tag{3.43}$$

which holds for all $m \in \mathbb{N}$ and $j \in (j)_0$. Since the sequence $j \mapsto t_{1,j}$, $j \in (j)_0$ ranges in the compact interval $[1, 3/2]$, it is clear that we can extract a subsequence $(j)_1$ and find a point $t_1 \in [1, 3/2]$, such that

$$\|u_j\|^2_{L^\infty(t_i,t_i+\delta;H^2(\Omega))} + \|\beta_{n_j}(u_j)\|^2_{L^\infty(t_i,t_i+\delta;H)} \leqslant C_2 \tag{3.44}$$

holds for $i = 1$ and for all $j \in (j)_1$. Proceeding by induction, for all $N \in \mathbb{N}$ we then find $(j)_N$ such that (3.44), where t_i is some point in $[i, i+1/2]$, holds for all $i \leqslant N$ and $j \in (j)_N$. At the end, we can thus extract a diagonal subsequence $(j)_\infty$, which gives (3.44) for all $j \in (j)_\infty$ and all $i \in \mathbb{N}$. Since $(j)_\infty$ is a subsequence of $(j)_0$, taking the lim inf for $j \nearrow \infty$, $j \in (j)_\infty$, of (3.44) and recalling that, by (2.43), $\beta_{n_j}(u_j)$ tends to $\beta(u)$ weakly in $L^2(0, T; H)$ for all $T > 0$, we finally get that u satisfies the *locally uniform* regularization estimate

$$\|u\|^2_{L^\infty(t_i,t_i+\delta;H^2(\Omega))} + \|\beta(u)\|^2_{L^\infty(t_i,t_i+\delta;H)} \leqslant C_2, \tag{3.45}$$

with t_i as above. This can be also rewritten as

$$\mathrm{d}^2_{\mathcal{W}}(u(t), 0) \leqslant C_2 \qquad \text{for a.e. } t \in \bigcup_{i=1}^{\infty} [t_i, t_i + \delta] \tag{3.46}$$

and clearly entails the validity of (2.47) in case W satisfies (2.12) with $p = \infty$.

To conclude the proof, we have to face the case when W is a *separating potential* (cf definition 2.10). Then the procedure does not change till (3.35). After that point, the last inequality in (3.36) is replaced by (notice that now υ_{n_j} takes values into $(-1, 1)$)

$$\begin{aligned} \|\beta'_{n_j}(\upsilon_{n_j}(u_j))\|^3_{L^3(\Omega)} &\leqslant \gamma(\|\upsilon_{n_j}(u_j)\|^2_{L^\infty(\Omega)}) \leqslant \gamma(\phi(\|\upsilon_{n_j}(u_j)\|^2_{W^{1,6}(\Omega)} + \|\beta(\upsilon_{n_j}(u_j))\|^2)) \\ &\leqslant \zeta(\|u_j\|^2_{W^{1,6}(\Omega)} + \|\beta_{n_j}(u_j)\|^2) \leqslant \zeta(c_\Omega \|u_j\|^2_{H^2(\Omega)} + \|\beta_{n_j}(u_j)\|^2), \end{aligned} \tag{3.47}$$

where γ is as in (3.37), ϕ as in definition 2.10 and we have set $\zeta : [0, \infty) \to [0, \infty)$ as $\zeta := \gamma \circ \phi$. Actually, ζ is still a monotone function. Note that in (3.47) we also used the 1-Lipschitz continuity of υ_{n_j}, Sobolev's embeddings and that $\upsilon_{n_j}(0) = 0$ for all j. At this point, one gets an expression similar to (3.38), where the last term $\gamma(c_\Omega \|u_n\|^2_{H^2(\Omega)})$ is suitably replaced by the right-hand side of (3.47) (possibly up to a modification of the expression of ζ). From this point on, the proof goes through with no further change. ■

Proof of theorem 2.14. Property (H1) is an easy consequence of the proof of theorem 2.3. In particular, (2.39)–(2.40), (2.43) and (2.44) follow from (3.8), (3.9), (3.10)–(3.11) and (2.32)–(2.34). Property (H2) is immediate. To show (H4), let us first extract a nonrelabelled subsequence of k such that, for some function u,

$$u_k \to u \qquad \text{weakly in } \mathcal{V}_T \quad \forall\, T > 0. \tag{3.48}$$

To do this, let us take $R > 0$ such that $\mathrm{d}_{\mathcal{V}}(u_{0,k}, 0) \leqslant R$ for all k. Then, since any of the (u_k) satisfies the analogue of (2.38), setting $(k)_0 := (k)_{k\in\mathbb{N}}$, for all $N \geqslant 1$ we can extract a subsequence $(k)_N$ of $(k)_{N-1}$ such that, for some function u, u_k tends to u at least weakly in $\mathcal{V}_N$ as $k \in (k)_N$ goes to infinity. Thus, taking a diagonal subsequence $(k)_\infty$, it is clear that (3.48) holds. From this point on, we will work on this subsequence. Proceeding similarly with the proof of theorem 2.3 (i.e. passing to the limit in (2.25)–(2.26)), one sees immediately that u is an energy solution either to (P_ε) or to (P_0) and in particular it satisfies $u(0) = u_0$. Actually, for all $T > 0$ and $k \in (k)_\infty$, by the analogue of (3.12) we have

$$\|W'(u_k)\|_{L^2(0,T;H)} \leqslant \mathfrak{c} \tag{3.49}$$

(where $\mathfrak{c}$ might depend on T and R), so that, by the *strong* convergence $u_k \to u$ in $L^2(0, T; H)$, following from (3.48), and the usual monotonicity argument, we infer (without extracting any other subsequence and for all $T > 0$)

$$W'(u_k) \to W'(u) \qquad \text{weakly in } L^2(0, T; H). \tag{3.50}$$

Let us now show that u is limiting, which is a bit more difficult. Since u_k is limiting, for all $k \in (k)_\infty$ there exist an increasing sequence $j \mapsto n_k^j$ (if $\varepsilon = 0$, also a decreasing sequence $j \mapsto \varepsilon_k^j$, otherwise we intend that $\varepsilon_k^j \equiv \varepsilon$) and a sequence of functions $j \mapsto u_k^j$, where $u_k^j \in \mathcal{Y}_T$ for all $T > 0$, solving, in the usual sense,

$$u_{k,t}^j + B_{u_k^j} w_k^j = 0, \tag{3.51}$$

$$w_k^j = \varepsilon_k^j u_{k,t}^j + B u_k^j + W'_{n_k^j}(u_k^j) + f. \tag{3.52}$$

Moreover, as $j \nearrow \infty$, u_k^j tends to u_k in the sense specified in definition 2.5. Then it is clear that there exists $\mathfrak{c} > 0$ depending on R and T but independent of j and k, such that

$$\|u_k^j\|_{H^1(0,T;V')} + \varepsilon^{1/2}\|u_k^j\|_{H^1(0,T;H)} + \|u_k^j\|_{L^\infty(0,T;V)} + \|u_k^j\|_{L^2(0,T;H^2(\Omega))} \leqslant \mathfrak{c} \qquad \forall\, j,\ k. \tag{3.53}$$

Next, for each $k \in (k)_\infty$ we can choose an index j_k such that the sequences $k \mapsto j_k$, $k \mapsto n_k^{j_k}$ are strictly increasing (and, if $\varepsilon = 0$, $k \mapsto \varepsilon_k^{j_k}$ is strictly decreasing) and

$$\|u_k^{j_k} - u_k\|_{C^0([0,k];H)} \leqslant 1/k, \tag{3.54}$$

so that, with no further extraction of subsequences (the limit is already identified), $k \mapsto u_k^{j_k}$ tends to the above-constructed u strongly in $C^0([0, T]; H)$ and weakly in $\mathcal{V}_T$ for all $T > 0$. This shows that (2.39) and (2.40), intended as $j_k \nearrow \infty$, hold for the limit function u. Moreover, being for all k (cf (2.44))

$$\mathcal{E}_{n_k^{j_k}}(u_k^{j_k}(0)) \leqslant \mathcal{E}(u_{0,k}) + \sigma_k^{j_k}, \tag{3.55}$$

it is clear that one can also take (j_k) in such a way that $j \mapsto \sigma_k^{j_k}$ is decreasing and tends to 0, which shows that (2.44) holds for u since $\mathcal{E}(u_{0,k})$ tends to $\mathcal{E}(u_0)$ with k by the hypothesis of convergence $u_{0,k} \to u_0$ in $\mathcal{V}$ and thanks to lemma 2.1. Next, noticing that, again,

$$u_k^{j_k} \to u \quad \text{strongly in } L^2(0, T; H), \qquad \|W'_{n_k^{j_k}}(u_k^{j_k})\|_{L^2(0,T;H)} \leqslant \mathfrak{c}, \tag{3.56}$$

for all $T > 0$, and using that $n_k^{j_k} \nearrow \infty$ with k so that $W'_{n_k^{j_k}}$ G-converges to W', one readily gets (2.43) for the limit u by the usual monotonicity argument and still for the whole sequence $(k)_\infty$. Thus, u is limiting.

The proof of (H4), however, is not yet concluded since we still have to check that, choosing an arbitrary $t \geqslant 0$, $u_k(t)$ tends to $u(t)$ *strongly* in $\mathcal{V}$, which is *not* a consequence of the 'weak' convergence in $\mathcal{V}_T$ holding for all $T \geqslant 0$. Actually, by the uniform bound on u_k corresponding to (3.53), it is clear that $u_k \to u$ in $C_w(0, T; V)$ so that, for all $t \geqslant 0$, one can only deduce that $u_k(t)$ tends to $u(t)$ *weakly* in V.

Thus, let us pick T larger than the chosen t and so large (in a way that only depends on R) that, by (2.47), for all k in our subsequence there exists $\tau_k \in [0, 3/2]$ such that $\mathrm{d}_{\mathcal{W}}(u_k(s), 0) \leqslant C_0$ for all $s \in [T + \tau_k, T + \tau_k + \delta]$, where we remark once more that δ and C_0 are independent of k and R.

Now, the weak convergence in $\mathcal{V}_T$ and (2.3) guarantee that

$$w_k \to w \qquad \text{weakly in } L^2(0, T; V). \tag{3.57}$$

Moreover, since $(\tau_k) \subset [0, 3/2]$, which is a compact set, there exist $S \in [T, T + 3/2 + \delta]$ and a subsequence $(k)_*$ of $(k)_\infty$ such that, at least for sufficiently large $k \in (k)_*$, it is $\mathrm{d}_{\mathcal{W}}(u_k(S), 0) \leqslant C_0$. This, by lemma 2.1, entails that $u_k(S)$ tends to u in $\mathcal{V}$ so that, in particular, $\mathcal{E}(u_k(S))$ tends to $\mathcal{E}(u(S))$ at least as $k \in (k)_*$ goes to ∞. Next, writing the energy equality (2.37) for u_k on the interval $(0, S)$ gives (possibly for $\varepsilon = 0$)

$$\mathcal{E}(u_k(S)) - \mathcal{E}(u_{0,k}) = -\int_0^S \int_\Omega b(u_k)|\nabla w_k|^2 - \int_0^S \int_\Omega \varepsilon |u_{k,t}|^2. \tag{3.58}$$

Thus, taking the limit $k \nearrow \infty$ in $(k)_*$, noting that the left-hand side converges to $\mathcal{E}(u(S)) - \mathcal{E}(u_0)$, and using the energy equality for u we get (still possibly for $\varepsilon = 0$)

$$\lim_{k \nearrow \infty} \left(\int_0^S \int_\Omega b(u_k)|\nabla w_k|^2 + \int_0^S \int_\Omega \varepsilon |u_{k,t}|^2 \right) = \int_0^S \int_\Omega b(u)|\nabla w|^2 + \int_0^S \int_\Omega \varepsilon |u_t|^2, \tag{3.59}$$

which readily entails (in the case $\varepsilon > 0$, also thanks to the *weak* convergence $u_{k,t} \to u_t$ in $L^2(0, S; H)$ following from (3.48))

$$\limsup_{k \nearrow \infty} \int_0^S \int_\Omega b(u_k)|\nabla w_k|^2 \leqslant \int_0^S \int_\Omega b(u)|\nabla w|^2. \tag{3.60}$$

Let us now notice that, by (2.1),

$$\begin{aligned} \alpha \int_0^S \|\nabla w_k - \nabla w\|^2 &\leqslant \int_0^S \int_\Omega (b(u_k)|\nabla w_k - \nabla w|^2) \\ &= \int_0^S \int_\Omega b(u_k)|\nabla w_k|^2 + \int_0^S \int_\Omega b(u_k)|\nabla w|^2 - 2\int_0^S \int_\Omega b(u_k)\nabla w_k \cdot \nabla w. \end{aligned} \tag{3.61}$$

Here, the latter two terms on the right-hand side converge to the expected limits since $|\nabla w|^2 \in L^1(\Omega \times (0, S))$ and there hold the convergences $b(u_k) \to b(u)$ (weakly star in $L^\infty(\Omega \times (0, S))$ and strongly in $L^a(\Omega \times (0, S))$ for all $a \in (1, \infty)$) and $\nabla w_k \to \nabla w$ (weakly in $L^2(0, S; H)$), which in particular entails

$$b(u_k)\nabla w_k \to b(u)\nabla w \qquad \text{weakly in } L^2(0, S; H). \tag{3.62}$$

Thus, taking the lim sup in (3.61) and using (3.60), we obtain

$$w_k \to w \qquad \text{strongly in } L^2(0, S; V). \tag{3.63}$$

Then, recalling that $S \geqslant T$, we have in particular

$$w_k \to w \qquad \text{strongly in } L^2(0, T; V). \tag{3.64}$$

Notice that, *a priori*, the latter convergence holds only for the subsequence $(k)_*$ but, in fact, being the limit already identified, it is valid for the whole sequence $(k)_\infty$. Thus, we can now come back to (3.58), which we rewrite with t in place of S. Using (3.64) (and also (3.48) if $\varepsilon > 0$), and recalling that $t \leqslant T$, we then get

$$\limsup_{k\nearrow\infty} \mathcal{E}(u_k(t)) \leqslant \mathcal{E}(u_0) - \int_0^t \int_\Omega b(u)|\nabla w|^2 - \int_0^t \int_\Omega \varepsilon |u_t|^2, \tag{3.65}$$

so that, by comparison with the limit energy equality and thanks to lemma 2.1, we finally get

$$\lim_{k\nearrow\infty} \mathcal{E}(u_k(t)) = \mathcal{E}(u(t)), \tag{3.66}$$

which implies that $u_k(t) \to u(t)$ in $\mathcal{V}$ and concludes the proof of (H4) and of the theorem. ■

Remark 3.1. The main reason which forced us to use the complicated 'local compactness' argument is the presence of the nonconstant mobility $b(\cdot)$. Actually, for constant b, once (3.57) is known, one can immediately pass to (3.65) and get directly the strong convergence $\mathcal{E}(u_k(t)) \to \mathcal{E}(u(t))$. Instead, for nonconstant b, without the help of (3.64) it is not clear whether the semicontinuity property

$$\int_0^t \int_\Omega b(u)|\nabla w|^2 \leqslant \liminf_{k\nearrow\infty} \int_0^t \int_\Omega b(u_k)|\nabla w_k|^2 \tag{3.67}$$

(which is necessary to prove (3.65)) holds. Actually, at this stage, the integrand on the right-hand side is only bounded in $L^1(\Omega)$ and not even known to converge pointwise.

Proof of theorem 2.16. Condition (A1) of theorem 2.15 is an immediate consequence of (2.38). Let us then show (A2). With the notation of (A2), let us first point out that, since $(u_{0,k})$ is a bounded set in $\mathcal{V}_m$, by (2.47) there exist $\delta \in [0, 1/4]$ and $C_0 > 0$ such that for all (sufficiently large, depending on the 'radius' of $(u_{0,k})$) $k \in \mathbb{N}$ there exists $\tau_k \in [0, 3/2]$ with

$$\mathrm{d}_{\mathcal{W}}(u_k(t), 0) \leqslant C_0 \qquad \forall\, t \in [t_k - 2 + \tau_k, t_k - 2 + \tau_k + \delta]. \tag{3.68}$$

In particular, we can extract a subsequence of (k), not relabelled, such that $\tau_k \to \tau \in [0, 3/2]$. Setting then $v_k(s) := u_k(t_k - 2 + \tau + \delta/2 + s)$, since there eventually holds that $|\tau_k - \tau| \leqslant \delta/2$, we then have that, still up to the extraction of a further subsequence, $v_k(0)$ tends to some v_0 in $\mathcal{V}$. Moreover, by (H2), $(v_k) \subset \mathcal{S}_\varepsilon$ (possibly for $\varepsilon = 0$), i.e. it is a limiting solution and, clearly,

$$u_k(t_k) = v_k(2 - \tau - \delta/2). \tag{3.69}$$

Thus, the same argument used to show (H4) in the proof of theorem 2.14 permits us to say that a subsequence of $v_k(2 - \tau - \delta/2)$ admits a proper limit (which coincides, by the way, with an element of the semiflow evaluated at the time $2 - \tau - \delta/2$) in the metric topology of $\mathcal{V}$. This gives (A2).

Finally, if (2.12) holds with $p \leqslant 6$, it is clear from the uniform regularization property (2.28) that $\mathcal{A}_\varepsilon$ is bounded in $\mathcal{W}_m$. This concludes the proof of the theorem. ■

4. Entropy solutions

In this section we show that if $\varepsilon = 0$, (2.12) holds with $p \in (2, 6)$ and, in place of (2.10)–(2.11), we have

$$W''(r) \geqslant \eta |r|^{p-2} - \lambda \qquad \forall\, r \in I = \mathbb{R}, \tag{4.1}$$

where $\eta > 0$ and $p \in (2, 6)$ is the same exponent as in (2.12), then there exist weaker solutions to the analogue of problem (P_0), corresponding to the choice of an initial datum u_0 satisfying (cf (2.20))

$$u_0 \in \mathcal{H}_m := \{v \in H : |v_\Omega| \leqslant m\}. \tag{4.2}$$

To show this, we proceed in a somehow reverse order, by first deriving some estimates and then inferring a precise statement. We notice that still a rigorous procedure should rely on approximation and passage to the limit arguments (i.e. working on ($\mathrm{P}_{n,\varepsilon_n}$) or some analogue and then letting $n\nearrow\infty$). Nevertheless, for brevity (and since all works are in the previous section) we prefer to consider here directly, although formally, a limit solution u. Thus, let us set

$$\mu(s) := \int_0^s \frac{\mathrm{d}r}{b(r)}, \qquad \hat{\mu}(s) := \int_0^s \mu(r)\,\mathrm{d}r. \tag{4.3}$$

Of course, by (2.1), $\hat{\mu}$ satisfies

$$\frac{1}{\mu}s^2 \leqslant 2\hat{\mu}(s) \leqslant \frac{1}{\alpha}s^2 \qquad \forall\, s \in \mathbb{R}. \tag{4.4}$$

Let us now perform an estimate of the *entropy* type. Namely, let us test (2.25) by $\mu(u)$, (2.26) by Bu and take the sum. Noting that two opposite terms cancel, we infer

$$2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{\mu}(u) + \|Bu\|^2 + 2\int_\Omega W''(u)|\nabla u|^2 \leqslant \|f\|^2. \tag{4.5}$$

Adding $\|u\|^2 + 2\lambda\|\nabla u\|^2$ to both sides and using (4.1) and the Poincaré–Wirtinger inequality (3.3), we readily obtain

$$2\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \hat{\mu}(u) + \|u\|^2_{H^2(\Omega)} + 2\eta\int_\Omega (|u|^{p-2}|\nabla u|^2) \leqslant c_3(1 + \|\nabla u\|^2), \tag{4.6}$$

where c_3 on the right-hand side depends on λ, m (cf (4.2)) and on the H-norm of f.

Once the initial datum $u_0 \in \mathcal{H}_m$ is given, owing to (4.4) and noting that

$$c_3(1 + \|\nabla u\|^2) \leqslant \tfrac{1}{2}\|u\|^2_{H^2(\Omega)} + c\|u\|^2 + c, \tag{4.7}$$

an application of Gronwall's lemma in (4.6) gives, for $T > 0$, the *a priori* estimate (notice that we control the full V-norm of $|u|^{(p-2)/2}u$ since we know that $|u_\Omega| \leqslant m$)

$$\|u\|_{L^\infty(0,T;H)\cap L^2(0,T;H^2(\Omega))} + \||u|^{(p-2)/2}u\|_{L^2(0,T;V)} \leqslant \mathfrak{c}. \tag{4.8}$$

Here and below the constants $\mathfrak{c} > 0$ may depend on T and u_0. Hence, by Sobolev's embeddings, we also have

$$\|u\|_{L^p(0,T;L^{3p}(\Omega))} \leqslant \mathfrak{c}. \tag{4.9}$$

To show the existence of an *entropy* solution, we have to see that (4.8) and (4.9) are sufficient to take the limit in equations (2.25)–(2.26) (recall that we should work on some approximation here). By interpolation of Lebesgue spaces, we actually have (here we just use that $p > 2$)

$$\|u\|_{L^{(3p-2)(p-1)/3(p-2)}(0,T;L^{2(p-1)}(\Omega))} \leqslant \mathfrak{c}, \tag{4.10}$$

whence, by (2.12) and with the help of a comparison of terms in (2.26), we have

$$\|W'(u)\|_{L^{(3p-2)/3(p-2)}(0,T;H)} + \|w\|_{L^{(3p-2)/3(p-2)}(0,T;H)} \leqslant \mathfrak{c}. \tag{4.11}$$

However, equation (2.25) makes no sense in that form as u has only the above regularity. Nevertheless, taking $v \in H^3_{\boldsymbol{n}}(\Omega)$ (the (closed) subspace of $H^3(\Omega)$ consisting of functions with 0 normal derivative on $\partial\Omega$), a.e. in $(0, T)$ we can formally write

$$\langle B_u w, v\rangle = \int_\Omega b(u)\nabla w \cdot \nabla v = -\int_\Omega w b'(u)\nabla u \cdot \nabla v - \int_\Omega b(u) w \Delta v. \tag{4.12}$$

Thus (2.25) has the weak correspondent

$$\langle u_t, v\rangle - \int_\Omega w b'(u)\nabla u\cdot\nabla v - \int_\Omega b(u) w\Delta v = 0. \tag{4.13}$$

Let us prove that (4.13) does make sense in the present regularity framework. Actually, noticing that, by (2.1),

$$\|b(u)\|_{L^\infty(\Omega\times(0,T))} + \|b'(u)\|_{L^\infty(\Omega\times(0,T))} \leqslant c, \tag{4.14}$$

we readily obtain that

$$\frac{|\langle u_t, v\rangle|}{\|v\|_{H^3(\Omega)}} \leqslant c\|w\|(\|\nabla u\| + 1). \tag{4.15}$$

Thus, observing that by (4.8) and interpolation ∇u is bounded in $L^4(0,T;H)$, noting that for $p < 6$ it is $(3p-2)/3(p-2) > 4/3$, taking the supremum w.r.t. $v \in H^3_{\boldsymbol{n}}(\Omega)$ in (4.15) and integrating the result over $(0,T)$, we get

$$\|u_t\|_{L^1(0,T;(H^3_{\boldsymbol{n}})'(\Omega))} \leqslant \mathfrak{c}, \tag{4.16}$$

which shows that (4.13) makes sense, provided that we interpret the first term as a duality between $H^3_{\boldsymbol{n}}(\Omega)$ and its dual (notice that, in fact, $H^3_0(\Omega)$ is contained in $H^3_{\boldsymbol{n}}(\Omega)$, so that, in particular, (4.13) can be seen as a relation in $H^{-3}(\Omega)$).

Let us now see that, more precisely, (2.25) passes to the limit (i.e. that we have sufficient compactness to remove some kind of approximation). Actually, by (2.1), (4.8), (4.16) and the generalized Aubin lemma [23, corollary 4], we have the convergences (holding at least for suitable subsequences of the approximating solutions, as usual) of

$$b(u), b'(u), \quad \text{strongly in } L^a(\Omega\times(0,\infty)) \qquad \forall\, a\in[1,\infty), \tag{4.17}$$

$$u, \quad \text{strongly in } L^b(0,T;V) \qquad \forall\, b\in[1,4). \tag{4.18}$$

Moreover, using that p is *strictly* less than 6 and modifying a bit the argument leading to (4.11), we can show that there exist exponents $a_* > 4/3$ and $b_* > 2$ such that

$$w \qquad \text{converges weakly in } L^{a_*}(0,T;L^{b_*}(\Omega)). \tag{4.19}$$

Thus, it is easy to see that (4.17)–(4.19) allow us to pass to the limit in (4.13). Also the limit of (2.26) is then easily taken since (4.18) and the weak convergence coming from the first of (4.11) easily allow us to identify the limit of $W'(u)$ by the usual monotonicity argument. We have thus proven the following.

Theorem 4.1. *Let (2.1), (2.9), (2.12) and (4.1) with* $p \in (2,6)$ *hold. Let* f *and* u_0 *satisfy (2.14) and (4.2), respectively. Then, there exists at least one couple* (u,w) *complying with the regularity properties (4.8), (4.9), (4.11) and (4.16) and such that (4.13) holds for all* $v \in H^3_{\boldsymbol{n}}(\Omega)$ *and a.e. in* $(0,T)$*. Moreover, we have*

$$(w - W'(u)) = Bu + f \quad \text{in } H, \qquad \text{a.e. in } (0,T), \tag{4.20}$$

and the initial condition (2.27) holds in H (indeed, by (4.8) and (4.16), $u \in C_w([0,T];H)$*). We call such a function u an* entropy solution *to problem* (P_0).

Let us now study the long-time behaviour of entropy solutions which, in a sense that might be specified following the lines of the previous two sections (cf definition 2.5), have a *limiting* character. Still, we prefer to work formally and do not enter the details of the approximation-limit argument, which should be very close to that sketched in the previous sections. It is anyway worth noting that for entropy solutions (which are less regular than the 'energy' ones) we do not expect, *a fortiori*, any uniqueness property. Our last result in this paper is the following.

Theorem 4.2. *Under the assumptions of theorem 4.1, the set $\mathcal{S}_{\rm entr}$ of* limiting *entropy solutions to* (P$_0$) *constitutes a* limiting semiflow *on $\mathcal{H}_m$. Furthermore, $\mathcal{S}_{\rm entr}$ admits the global attractor $\mathcal{A}_{\rm entr}$ which is compact in $\mathcal{H}_m$ and bounded in $\mathcal{V}_m$.*

Proof of theorem 4.2. The key point is to show that the entropy estimate (4.6) derived above also has a *dissipative* character. Let us then take $M > 0$ (whose value will be chosen later) and set

$$u_M := \max\{-M, \min\{u, M\}\}. \tag{4.21}$$

Then, it is clear that

$$c_3(1 + \|\nabla u\|^2) \leqslant c_3 + \frac{c_3}{M^{p-2}} \int_\Omega (|u|^{p-2}|\nabla u|^2) + c_3 \int_\Omega |\nabla u_M|^2 \tag{4.22}$$

and, by interpolation, for all $\sigma > 0$ the latter term can be controlled as follows:

$$\begin{aligned} c_3 \int_\Omega |\nabla u_M|^2 &\leqslant \sigma \|u_M\|^2_{H^{5/4}(\Omega)} + c(\sigma)\|u_M\|^2 \\ &\leqslant c_4\sigma\|u\|^2_{H^{5/4}(\Omega)} + c(\sigma)\|u_M\|^2 \leqslant c_5\sigma\|u\|^2_{H^2(\Omega)} + c(\sigma, M, \Omega). \end{aligned} \tag{4.23}$$

We used here the fact that the truncation operator $u \mapsto u_M$ is continuous from $H^s(\Omega)$ into itself for all $s < 3/2$ (cf, e.g. [19, remark 0.1]).

Thus, choosing σ such that $c_5\sigma = 1/2$ and M so large that $c_3/M^{p-2} \leqslant \eta$, (4.6) gives

$$2\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \hat{\mu}(u) + \frac{1}{2}\|u\|^2_{H^2(\Omega)} + \eta \int_\Omega (|u|^{p-2}|\nabla u|^2) \leqslant c_6, \tag{4.24}$$

where c_6 depends on m but is independent of the choice of u_0. By (4.4) and Gronwall's lemma, (4.24) readily gives dissipativity in the space $\mathcal{H}_0$ as well as, for all $\tau > 0$, the analogue of (3.12), namely,

$$\sup_{t\in(\tau,\infty)} \int_t^{t+1} \|u(s)\|^2_{H^2(\Omega)}\,\mathrm{d}s \leqslant c_7\|u_0\|^2 \mathrm{e}^{-\kappa'\tau} + C_0', \tag{4.25}$$

for suitable c_7, κ', C_0' independent of u_0. At this point, writing the energy equality in the form (3.2) (with $\mathcal{E}$ in place of $\mathcal{E}_n$) and using the uniform Gronwall lemma, we immediately obtain the existence of a uniformly absorbing set bounded in $\mathcal{V}$ and hence compact in $\mathcal{H}$. This fact implies the existence of the attractor and concludes the proof.

Remark 4.3. Coming back to the local compactness argument in the previous section, one can readily see that $\mathcal{A}_{\rm entr}$ is not only bounded but even *compact* in the space $\mathcal{V}$.

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