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Universal attractor for some singular phase transition systems

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Abstract

A singular parabolic system describing the thermal diffusion in a substance possibly subject to a phase transition is introduced. The physical process is described by the variables ϑ (absolute temperature) and χ (order parameter). The latter may have, or not, conserved total mass with respect to time. In both cases, after recalling and sometimes improving some known well-posedness results, the long-time behavior of the system is studied. It is shown that the process is dissipative and the compact universal attractor is constructed. It turns out to attract the trajectories of the system in a rather strong metric which is strictly linked to the constraints imposed to both variables. The techniques used in the proofs seem likely to be applied to other types of evolution systems containing maximal monotone nonlinearities.

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1. Introduction

In this paper we consider some singular parabolic systems coming from the so-called Penrose–Fife model for phase transition phenomena introduced in [20,21]. More in detail, we address the problem of existence of the universal attractor for a rather general class of these models. Moreover, it is worth remarking at once that the techniques used in the present analysis seem suitable to be applied to other types of singular evolution systems.

In order to introduce the precise mathematical problem, let us consider a smooth bounded container $\Omega \subset \mathbb{R}^d$, $1 \le d \le 3$, occupied by the substance undergoing the phase transition. Name ϑ and χ the basic state variables of the process, corresponding to the *absolute* temperature (hence, $\vartheta > 0$) and to the order parameter, respectively. Then, the energy balance equation, describing the evolution of ϑ, can be written in the form [7,9]

$$\partial_t(\vartheta + \lambda(\chi)) - \Delta\alpha(\vartheta) = g, \tag{1.1}$$

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where g is the volumic heat source, $\lambda(\cdot)$ a smooth function accounting for the latent heat, and $\alpha : (0, +\infty) \to \mathbb{R}$ is an increasing and concave function such that

$$\alpha(r) \sim -\frac{1}{r} \quad \text{for } r \sim 0, \qquad \alpha(r) \sim r \quad \text{for } r \sim +\infty. \tag{1.2}$$

Then, (1.1) is coupled with the kinetic equation for the phase variable. We will consider both the *nonconserved*

$$\partial_t \chi - \Delta \chi + \beta(\chi) + \gamma(\chi) \ni -\frac{\lambda'(\chi)}{\vartheta} \tag{1.3}$$

and the *conserved* case

$$\partial_t \chi - \Delta w = 0, \tag{1.4a}$$

$$w \in -\Delta \chi + \beta(\chi) + \gamma(\chi) + \frac{\lambda'(\chi)}{\vartheta}. \tag{1.4b}$$

In both (1.3) and (1.4b), β is taken as a general *maximal monotone graph*, possibly multivalued, coming from the convex part of a *double-well* free energy potential [20] and γ is a smooth function accounting for its nonconvex part. Moreover, relation (1.3) is assumed to be complemented by the homogeneous Neumann boundary condition for χ, while in the case of (1.4) we take such conditions both for χ and for the auxiliary unknown w, generally called *chemical potential*. Furthermore, third type conditions are assumed for ϑ and the whole systems (1.1) and (1.3) and (1.1) and (1.4) are complemented with the Cauchy conditions for ϑ and χ.

Of course, the singular character of the systems above is given by the presence of the constraints $1/\vartheta$, forcing $\vartheta > 0$, and $\beta(\chi)$, that can be chosen to force χ to attain solely values belonging to a bounded interval of $\mathbb{R}$. The presence of this kind of nonlinearities is the main difficulty of (1.1) and (1.3), (1.1) and (1.4), and the related results on well-posedness are relatively recent. Actually, the nonconserved system (1.1) and (1.3) has been first addressed in the paper [7], where existence of a global solution has been proved for a much wider class of functions α than those given by (1.2), provided that λ is Lipschitz continuous along with its first derivative. In case α is chosen as in (1.2), further regularity and uniqueness of the solution have been shown in [9]. In [22] these results have been extended to the fourth-order (conserved) case (1.1) and (1.4) in a more general setting possibly including *thermal memory* effects.

Taking the results of [7,9,22] into account, in this note we address some further questions related to the systems above, with the main task of proving the existence of the universal attractor in both the nonconserved and the conserved case. We have to notice that, while several papers (see, e.g., [2–4,10–12,17]) have been devoted to the analysis of long-time behavior of phase-field models of *Caginalp* [8] type, it seems that very few results have been obtained, up to now, for the parabolic Penrose–Fife models. This seems to be due to the standard (cf. [16]) choice of a *singular* heat flux law of the form $\alpha(r) \sim -1/r$, which gives rise to a strong lack of coercivity of the system with respect to ϑ. Actually, as far as we know, this form of the heat flux has been dealt with only in [13,14,23,24]. More in detail, in [14] the existence of a *weak form* (see below) of the universal attractor is shown in the nonconserved case, provided that a zero-order dissipative term $\varepsilon\vartheta$ for small $\varepsilon > 0$ is added on the left-hand side of (1.1). In the same setting, but only referring to one space dimension, the structure of the attractor is further investigated in [13], where the existence of an *inertial set* [26, Section VIII] is shown. We have to notice that the term $\varepsilon\vartheta$ in [13,14] plays an analogous role as our heat flux law (1.2) in providing further dissipativity for the unknown ϑ.

To our knowledge, the system (1.1) and (1.4) with $\alpha(r) \sim -1/r$, and without the addition of the dissipative term $\varepsilon\vartheta$, has only been studied in the recent papers [23,24], dealing with the conserved and the nonconserved case, respectively. In [23,24] the existence of a uniform attractor is proved in one space dimension; however, a strong constraint is imposed a priori on the initial data, which have to be chosen in a very small phase space. We observe that a function λ of *quadratic* growth at infinity is allowed in [23,24]; however, the analysis is limited to $\beta(r) \sim r^3$, i.e., to the standard double-well case [8].

Due to the difficulties related to the choice $\alpha(r) \sim -1/r$, in this note we address the diffusion law (1.2), which can be motivated by thermodynamical considerations (cf. [7]) and guarantees to the system a good parabolic structure with respect to ϑ. On the other hand, before addressing the question of existence of the attractor, we have to discuss in some detail the properties of the semigroups associated to the systems (1.1) and (1.3) and (1.1) and (1.4). In this concern, we show some continuous dependence theorems which turn out to improve some results in [9,15,22] referring to systems very similar to our ones. Namely, we are able to prove Lipschitz continuity of the semigroup associated to the nonconserved system with respect to a weak metric (cf. (3.24)). In the conserved case, as in [4], we just have 1/2-Hölder continuity, unless some growth conditions are assumed on β. These theorems, beyond constituting a basis for the subsequent asymptotic analysis, appear to deserve an independent interest, indeed.

In the second part of the paper, we prove existence of the universal attractor both for the system (1.1) and (1.3) and for (1.1) and (1.4) in the three-dimensional setting. In order to unify these situations, we limit ourselves to consider *affine* latent heat functions given by $\lambda(r) = br$ for $b \in \mathbb{R}$. In a forthcoming paper we will deal with a nonlinear (and possibly quadratic, cf. [20]) λ. However, this seems to work only in the nonconserved case.

We also point out that our analysis is performed in a phase space $\mathcal{X}$ which is smaller than that considered, e.g., in [13,14]. Such a set $\mathcal{X}$ is chosen precisely as the space of the initial data satisfying the conditions required for having existence of a solution. It is clear that these conditions strongly depend on the constraints imposed on the variables. Actually, we are able to prove that $\mathcal{X}$ is a complete metric space with respect to a suitable metric $d_{\mathcal{X}}$, which is stronger than the metric appearing in the Lipschitz (or Hölder) continuity results since it has to take the constraints into account. Our choice, however, seems to be an appropriate one, since the semigroups are still continuous (but no longer Lipschitz or Hölder continuous) in $d_{\mathcal{X}}$; moreover, we are able to prove the existence of a $d_{\mathcal{X}}$-compact set which *absorbs* any bounded set of $\mathcal{X}$. Of course, as a consequence, we obtain the existence of the compact universal attractor $\mathcal{A}$ both in the nonconserved and in the conserved case. The absorbing set constructed in [14] as a subset of their (bigger) phase space $\mathcal{Z}$, instead, is not able to absorb all the bounded set of $\mathcal{Z}$, but only those subsets which are bounded with respect to a stronger metric, which is similar to the one of our $\mathcal{X}$ and is related to the energy functional on which relies the variational structure of the system. This is the reason why the set constructed in [14] can be defined as a *weak* attractor, while we are able to construct a *strong* attractor $\mathcal{A}$, which attracts all $\mathcal{X}$-metric bounded sets and does this in the proper metric $d_{\mathcal{X}}$.

This approach, which is technically much more complicated since it requires a control of the nonlinear constraints in the dissipativity estimates, has the advantage of giving more information on the solution. Namely, the metric $d_{\mathcal{X}}$ yields some control on the asymptotic behavior of both ϑ and its inverse, and also of the term $\beta(\chi)$, which was not provided by the metric used, e.g., in [14]. Of course, the latter information can be more, or less, relevant depending on the growth conditions that are assumed on β. In general, anyway, this reinforcement of the metric structure of the phase space appears to provide a rather natural framework for studying the dissipativity of parabolic evolution systems with maximal monotone nonlinearities, and applications to several different physical situations should certainly be possible. On the other hand, we have to remark that in this setting it does not seem possible to address the question of existence of an inertial set (cf. [13]), at least using the metric $d_{\mathcal{X}}$. Indeed, the continuity properties of the semigroups with respect to $d_{\mathcal{X}}$ as well as the topological structure of $\mathcal{X}$ appear too weak to apply the related general theory. It might be possible, anyway, to show the existence of some set which attracts exponentially the $d_{\mathcal{X}}$-bounded set, but with the attraction property holding in some metric weaker than $d_{\mathcal{X}}$.

Here is the plan of the paper. In the next section, we shall present some notations and mathematical preliminaries. In Section 3 we will provide our basic hypotheses on data and give a detailed construction of the phase space for our analysis. Moreover, we will state our precise mathematical results. The proofs will be achieved in Section 4, which is related to the questions of well-posedness and continuity of the semigroups, in Section 5, which regards dissipativity, and in Section 6, which carries the proof of existence of the attractors.

2. Preliminaries

In this section we introduce some notations and recall some preliminary machineries which are needed to state our problems in a precise way.

First of all, let us define $\Gamma := \partial\Omega$ and, for $t > 0$, $Q_t := \Omega \times (0, t)$. Then, let us set $H := L^2(\Omega)$, $V := H^1(\Omega)$, and endow both spaces with their usual scalar products. We identify H and its dual, in order that the compact inclusion $H \subset V'$ holds and (V, H, V') form a *Hilbert triplet* [18, p. 202]. We denote by $|\cdot|$, or sometimes by $|\cdot|_H$, the norm both in H and in H^d, by $\|\cdot\|$ the (usual) norm in V, by $\|\cdot\|_*$ that in V', and by $\|\cdot\|_X$ the norm in the generic Banach space X. Finally, we indicate by $(\cdot,\cdot)$, $((\cdot,\cdot))$, $((\cdot,\cdot))_*$, the scalar products in H, V, V', respectively, and by $\langle\cdot,\cdot\rangle$ the duality pairing between V' and V.

Next, for any $\zeta \in V'$ we set

$$\zeta_\Omega := \frac{1}{|\Omega|}\langle\zeta, 1\rangle, \tag{2.1}$$

$$V_0' := \{\zeta \in V' : \zeta_\Omega = 0\}, \qquad V_0 := V \cap V_0'. \tag{2.2}$$

The above notation V_0' is suggested just by the sake of convenience; indeed, we mainly see V_0, V_0' as (closed) subspaces of V, V', inheriting their norms, rather than as a couple of spaces in duality.

We introduce the realization of the Laplace operator with homogeneous Neumann boundary conditions as

$$B : V \to V', \qquad \langle Bu, v\rangle := \int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x \quad \text{for } u, v \in V. \tag{2.3}$$

Clearly, B maps V onto V_0' and its restriction to V_0 is an isomorphism of V_0 onto V_0'. We denote by $\mathcal{N} : V_0' \to V_0$ the inverse of B, so that for any $u \in V$ and $\zeta \in V_0'$ there holds

$$\langle Bu, \mathcal{N}\zeta\rangle = \langle B\mathcal{N}\zeta, u\rangle = \langle\zeta, u\rangle. \tag{2.4}$$

We also define

$$W := \{v \in H^2(\Omega) : \partial_{\mathbf{n}} v = 0 \text{ on } \Gamma\}, \tag{2.5}$$

which is a closed subspace of $H^2(\Omega)$ by continuity of the trace operator.

By using the Poincaré–Wirtinger inequality we can easily see that the norm

$$\left(\int_\Omega |\nabla(\mathcal{N}\zeta)|^2\right)^{1/2} = \langle\zeta, \mathcal{N}\zeta\rangle^{1/2} \quad \text{for } \zeta \in V_0' \tag{2.6}$$

is equivalent to the norm $\|\zeta\|_*$ and we will use it, when it is convenient.

Let us now recall some notation and basic concepts from convex analysis. Let $\hat\pi : \mathbb{R} \to (-\infty, +\infty]$ a convex, l.s.c. (i.e. lower semicontinuous), and *proper* function. The latter means that the *domain*

$$D(\hat\pi) := \{r \in \mathbb{R} : \hat\pi(r) \neq +\infty\} \tag{2.7}$$

is not empty. We note that $D(\hat\pi)$ has to be a convex set, of course. Then, it is well known (cf. [5, p. 43]) that the subdifferential $\pi := \partial\hat\pi$ is a *maximal monotone graph* in $\mathbb{R} \times \mathbb{R}$. The *domain* of π is defined by

$$D(\pi) := \{r \in \mathbb{R} : \pi(r) \neq \emptyset\}. \tag{2.8}$$

In this situation we will say that $\hat\pi$ is a *convex primitive* of π.

In the sequel we will also use some spaces of L^p_{loc}-*translation bounded* functions; actually, as X is a Banach space and $p \in [1, +\infty)$ we set

$$\mathcal{T}^p(X) := \left\{ v \in L^p_{\mathrm{loc}}(0, +\infty; X) : \sup_{t\geq 0} \int_t^{t+1} \|v(s)\|_X^p < +\infty \right\}, \tag{2.9}$$

which is a Banach space with respect to the natural (graph) norm

$$\|v\|^p_{\mathcal{T}^p(X)} := \sup_{t\geq 0} \int_t^{t+1} \|v(s)\|_X^p. \tag{2.10}$$

As a generalization of the above definition, we also set, for $\tau > 0$,

$$\mathcal{T}^p_\tau(X) := \left\{ v \in L^p_{\mathrm{loc}}(0, +\infty; X) : \sup_{t\geq \tau} \int_t^{t+1} \|v(s)\|_X^p < +\infty \right\}, \tag{2.11}$$

and a seminorm for $\mathcal{T}^p_\tau(X)$ is defined by merely mimicking (2.10).

We now recall the statement of the so-called *uniform Gronwall's lemma* (see, e.g., [26, Lemma III.1.1]).

Lemma 2.1. *Let $y, a, b \in L^1_{\mathrm{loc}}(0, +\infty)$ three non-negative functions such that $y' \in L^1_{\mathrm{loc}}(0, +\infty)$ and*

$$y'(t) \leq a(t)y(t) + b(t) \quad \textit{for a.e. } t > 0, \tag{2.12}$$

and let a_1, a_2, a_3 three non-negative constants such that

$$\|a\|_{\mathcal{T}^1(\mathbb{R})} \leq a_1, \qquad \|b\|_{\mathcal{T}^1(\mathbb{R})} \leq a_2, \qquad \|y\|_{\mathcal{T}^1(\mathbb{R})} \leq a_3. \tag{2.13}$$

Then, we have that

$$y(t+1) \leq (a_2 + a_3)e^{a_1} \quad \textit{for all } t > 0. \tag{2.14}$$

Now, let us recall some basic notions on absorbing sets and attractors. Given a strongly continuous semigroup $S(t)$ on a complete metric space (X, d_X), we say that $\mathcal{B}_0$ is an *absorbing set* for $S(t)$ iff:

- $\mathcal{B}_0$ is bounded;
- for any bounded set $\mathcal{B} \subset X$, there exists a time $T_{\mathcal{B}} \geq 0$ such that

$$S(t)\mathcal{B} \subset \mathcal{B}_0 \quad \forall t \geq T_{\mathcal{B}}. \tag{2.15}$$

Next, a set $\mathcal{K} \subset X$ is said to be *uniformly attracting* for the semigroup $S(t)$ iff for any bounded set $\mathcal{B} \subset X$, we have

$$\lim_{t\to+\infty} \mathfrak{d}(S(t)\mathcal{B}, \mathcal{K}) = 0, \tag{2.16}$$

where $\mathfrak{d}$ denotes the *unilateral* Hausdorff distance of the set $S(t)\mathcal{B}$ from $\mathcal{K}$, with respect to the metric of X, i.e.

$$\mathfrak{d}(S(t)\mathcal{B}, \mathcal{K}) := \sup_{y\in S(t)\mathcal{B}} \inf_{k\in\mathcal{K}} d_X(y, k). \tag{2.17}$$

Finally, a set $\mathcal{K}$ is the *universal attractor* of the semigroup $S(t)$ iff:

- $\mathcal{K}$ is attracting and compact in X;
- $\mathcal{K}$ is fully invariant with respect to $S(t)$, i.e. $S(t)\mathcal{K} = \mathcal{K}$ for all $t \geq 0$.

We remark that the universal attractor, if it exists, is certainly unique (cf. [26, Section I.1.3]); moreover, it is a connected set. Let us finally report the statement of a general abstract criterion [26, Theorem I.1.1] providing a sufficient condition for the existence of the attractor.

Theorem 2.2. *Let $S(t)$ be a strongly continuous semigroup on the complete metric space (X, d_X). Let us assume that*:

- *$S(t)$ admits an absorbing set $\mathcal{B}_0$ (dissipativity)*;
- *for any bounded set $\mathcal{B} \subset X$, there exists $t_{\mathcal{B}} > 0$ such that*

$$\bigcup_{t \geq t_{\mathcal{B}}} S(t)\mathcal{B} \quad \text{is compact in } X \quad (\text{uniform compactness}). \tag{2.18}$$

Then, $S(t)$ admits the universal attractor $\mathcal{K}$ which is given by

$$\mathcal{K} = \omega - \lim(S(t)\mathcal{B}_0) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} S(t)\mathcal{B}_0}. \tag{2.19}$$

3. Main results

We start by stating the precise mathematical formulations of systems (1.1) and (1.3) and (1.1) and (1.4) and presenting the related well-posedness results. In the sequel we partly follow [7,9,22]. First, let us give our basic assumptions on data, covering both the nonconserved and the conserved case:

(A1) $\alpha \in C^1((0, +\infty); \mathbb{R})$ is increasing and concave and fulfills $\alpha(1) = 0$; moreover, there exist $c_0, c_\infty > 0$, $\ell \in C^1((0, +\infty); \mathbb{R})$ such that

$$\alpha(r) = -\frac{c_0}{r} + \ell(r), \qquad \ell' \in L^\infty(0, +\infty), \qquad \lim_{r \nearrow +\infty} \ell'(r) = c_\infty; \tag{3.1}$$

(A2) there exists $b \in \mathbb{R}$ such that $\lambda(\chi) = b\chi$ for all $r \in \mathbb{R}$;
(A3) $\gamma \in C^1(\mathbb{R})$, $\gamma' \in L^\infty(\mathbb{R})$, and we set $L := \|\gamma'\|_{L^\infty(\mathbb{R})}$;
(A4) β is a *maximal monotone graph* in $\mathbb{R} \times \mathbb{R}$ such that $0 \in \operatorname{int} D(\beta)$ and $0 \in \beta(0)$;
(A5) $g \in L^2(\Omega)$, $h \in L^2(\Gamma)$;
(A6) $\vartheta_0 \in H$, $\vartheta_0 > 0$ a.e. in Ω, and $\log \vartheta_0 \in L^1(\Omega)$;
(A7) $\chi_0 \in V$, $\hat{\beta}(\chi_0) \in L^1(\Omega)$.

In (A7), $\hat{\beta} : \mathbb{R} \to [0, +\infty]$ is the convex primitive of β fulfilling $\hat{\beta}(0) = 0$.

Remark 3.1. Assumption (A5) could be generalized in several directions; for instance, less regular data, or even data suitably depending on time, might be considered.

It is clear from (A1) that α satisfies the following additional properties. First,

$$\alpha'(r) \geq c_\infty \quad \forall r \in (0, +\infty), \quad \lim_{r \to 0} r^2 \alpha'(r) = c_0. \tag{3.2}$$

Moreover, there exists $c_0' > 0$ such that

$$\alpha'(r) \geq c_0' r^{-2} \quad \forall r \in (0, +\infty). \tag{3.3}$$

Let us specify the precise form of the third type boundary conditions complementing (1.1): we assume that for some $n_0 > 0$ and a.e. $t > 0$ it is

$$-\partial_{\mathbf{n}}\alpha(\vartheta) = n_0(\alpha(\vartheta) - h) \quad \text{on } \Gamma. \tag{3.4}$$

Consequently, we introduce the operator

$$J : V \to V', \qquad \langle Jv, z\rangle := \int_\Omega \nabla v \cdot \nabla z + n_0 \int_\Gamma vz \quad \text{for } v, z \in V, \tag{3.5}$$

this is indeed the Riesz mapping associated to the norm $\|v\|_J^2 := \langle Jv, v\rangle$ on V, which is equivalent to the standard one and will be used in place of it, when it is convenient. Finally, we define the generalized heat source term as

$$\langle f, v\rangle := (g, v) + \int_\Gamma hv \quad \forall v \in V, \tag{3.6}$$

indeed, we remark that (A5) entails $f \in V'$.

Now, we are ready to recall the result [7, Theorem 2.3] related to global existence and regularity in the nonconserved case.

Theorem 3.2. *Let us assume* (A1)–(A7) *and take any* $T > 0$. *Then, there exists at least one triplet* (ϑ, χ, ξ) *such that*

$$\vartheta \in H^1(0, T; V') \cap C^0([0, T]; H) \cap L^2(0, T; V), \quad \vartheta > 0 \; a.e. \; in \; Q_T, \tag{3.7}$$

$$\frac{1}{\vartheta} \in L^2(0, T; V), \tag{3.8}$$

$$\chi \in H^1(0, T; H) \cap C^0([0, T]; V) \cap L^2(0, T; W), \tag{3.9}$$

$$\xi \in L^2(0, T; H). \tag{3.10}$$

The triplet (ϑ, χ, ξ) *satisfies*

$$\partial_t(\vartheta + b\chi) + J(\alpha(\vartheta)) = f \quad in \; V' \; a.e. \; in \; (0, T), \tag{3.11}$$

$$\partial_t\chi + B\chi + \xi + \gamma(\chi) = -\frac{b}{\vartheta} \quad a.e. \; in \; Q_T, \tag{3.12}$$

$$\chi \in D(\beta) \quad and \quad \xi \in \beta(\chi) \quad a.e. \; in \; Q_T, \tag{3.13}$$

$$\vartheta(0) = \vartheta_0, \qquad \chi(0) = \chi_0 \quad a.e. \; in \; \Omega. \tag{3.14}$$

Remark 3.3. Relation (3.12) can be formulated "a.e. in Q_T" rather than "in V' a.e. in $(0, T)$" thanks to (3.8) and (3.9). Moreover, we note that (3.7), (3.8), and (A1) entail that

$$\alpha(\vartheta) \in L^2(0, T; V'). \tag{3.15}$$

Now, let us come to the fourth-order case; the following result can be proved similarly as in [22, Theorem 2.1], where a slightly different problem is addressed.

Theorem 3.4. *Let us assume* (A1)–(A7) *and*

(A8) $$\chi_\Omega := (\chi_0)_\Omega = \frac{1}{|\Omega|}\int_\Omega \chi_0 \in \operatorname{int} D(\beta).$$

Take any $T > 0$. *Then, there exists at least one quadruple* $(\vartheta, \chi, w, \xi)$ *satisfying* (3.7), (3.8) *and* (3.10), *and*

$$w \in L^2(0, T; V), \tag{3.16}$$

$$\chi \in H^1(0, T; V') \cap L^\infty(0, T; V) \cap L^2(0, T; W). \tag{3.17}$$

The quadruple $(\vartheta, \chi, w, \xi)$ *fulfills*

$$\partial_t(\vartheta + b\chi) + J(\alpha(\vartheta)) = f \quad \text{in } V' \text{ a.e. in } (0, T), \tag{3.18}$$

$$\partial_t \chi + Bw = 0 \quad \text{in } V' \text{ a.e. in } (0, T), \tag{3.19}$$

$$w = B\chi + \xi + \gamma(\chi) + \frac{b}{\vartheta} \quad \text{in } V' \text{ a.e. in } (0, T), \tag{3.20}$$

$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \quad \text{a.e. in } Q_T, \tag{3.21}$$

$$\vartheta(0) = \vartheta_0, \qquad \chi(0) = \chi_0 \quad \text{a.e. in } \Omega. \tag{3.22}$$

Finally (*cf.* (2.1)), *we have that*

$$\frac{1}{|\Omega|}\int_\Omega \chi(t) = \chi_\Omega \quad \forall t \in [0, T]. \tag{3.23}$$

We remark that the above results neither include uniqueness nor continuous dependence. Actually, uniqueness has been proved in [9, Theorem 1] (nonconserved case) and in [22, Theorem 2.2] (conserved case). However, the results of Colli et al. [9,22], which can deal with a nonlinear (but Lipschitz) λ, hold just under stronger regularity assumptions on the initial and source data. Thus, we have to provide a generalization suitable to our less regular setting. In this direction, we present a number of theorems which should make clear the structure of the semigroups associated to the systems above.

First of all, we state two Lipschitz continuity results, entailing in particular uniqueness of the solution, and holding for $\lambda(\chi) = b\chi$ in a regularity setting compatible with (A1)–(A7). We first address the nonconserved case, where we can prove the following theorem in the spirit of [14, Theorem 3.1] which should hold, with the proper modifications, also for more general (e.g., quadratic) functions λ.

Theorem 3.5. *Assume* (A1)–(A7). *Then, the solution* (ϑ, χ, ξ) *provided by* Theorem 3.2 *is unique. More precisely, there exists* $C > 0$ *depending only on* Ω, T, c_0, c_∞, *and* L *such that for any two couples of initial data* $(\vartheta_{0,1}, \chi_{0,1})$ *and* $(\vartheta_{0,2}, \chi_{0,2})$, *denoting by* $(\vartheta_1, \chi_1, \xi_1)$ *and* $(\vartheta_2, \chi_2, \xi_2)$ *two corresponding solutions to* (3.7)–(3.14), *for any* $t \in [0, T]$ *we have*

$$\begin{aligned} &\|(\vartheta_1 - \vartheta_2)(t) + b(\chi_1 - \chi_2)(t)\|_*^2 + \|\vartheta_1 - \vartheta_2\|^2_{L^2(0,t;H)} + |(\chi_1 - \chi_2)(t)|^2 + \|\nabla(\chi_1 - \chi_2)\|^2_{L^2(0,t;H)} \\ &\quad \le C(\|(\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2})\|_*^2 + |\chi_{0,1} - \chi_{0,2}|^2). \end{aligned} \tag{3.24}$$

In the conserved setting the analog of the above theorem holds just for a more restricted class of graphs β:

(A9) Let $\beta \in C^1(\mathbb{R})$ and let us assume that there exists $c_\beta > 0$, $p \le 7$, such that

$$\beta'(r) \le c_\beta(1 + |r|^p) \quad \forall r \in \mathbb{R}. \tag{3.25}$$

Of course, if (A9) holds, then (A8) is automatically satisfied for all χ_0 fulfilling (A7). We have the following theorem.

Theorem 3.6. *Assume* (A1)–(A7) *and* (A9) *and consider, with the same notation as above, two solutions to the conserved system. Then, for any* $t \in [0, T]$ *we have*

$$\begin{aligned} &\|(\vartheta_1 - \vartheta_2)(t) + b(\chi_1 - \chi_2)(t)\|_*^2 + \|\vartheta_1 - \vartheta_2\|^2_{L^2(0,t;H)} + \|(\chi_1 - \chi_2)(t)\|_*^2 + \|\nabla(\chi_1 - \chi_2)\|^2_{L^2(0,t;H)} \\ &\quad \le C(\|(\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2})\|_*^2 + \|\chi_{0,1} - \chi_{0,2}\|_*^2 + |(\chi_{0,1})_\Omega - (\chi_{0,2})_\Omega|^2), \end{aligned} \tag{3.26}$$

where C is allowed to depend also on the norms of χ_1 *and* χ_2 *in* (3.17).

For a general graph β (i.e. without (A9)) in the conserved setting we just have Hölder continuity, as in [4, Theorem 3.1].

Theorem 3.7. *Assume* (A1)–(A7) *and consider, with the notation above, two solutions to the conserved system, where both the initial data* $\chi_{0,1}$, $\chi_{0,2}$ *fulfill* (A8). *Then, for any* $t \in [0, T]$ *we have*

$$\begin{aligned} &\|(\vartheta_1 - \vartheta_2)(t) + b(\chi_1 - \chi_2)(t)\|_*^2 + \|\vartheta_1 - \vartheta_2\|^2_{L^2(0,t;H)} + \|(\chi_1 - \chi_2)(t)\|_*^2 + \|\nabla(\chi_1 - \chi_2)\|^2_{L^2(0,t;H)} \\ &\quad \le C(\|(\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2})\|_*^2 + \|\chi_{0,1} - \chi_{0,2}\|_*^2 + |(\chi_{0,1})_\Omega - (\chi_{0,2})_\Omega| + |(\chi_{0,1})_\Omega - (\chi_{0,2})_\Omega|^2), \end{aligned} \tag{3.27}$$

where C is now allowed to depend also on the norms of χ_1 *and* χ_2 *in* (3.17) *and on the norms of* ξ_1 *and* ξ_2 *in* (3.10).

Looking back to the results stated up to this point, we notice that, in order to have existence, we choose rather regular initial data (cf. (A6) and (A7)); conversely, the continuous dependence theorems hold with respect to much weaker norms. Thus, in order to define the phase space for the asymptotic analysis, we have to make a choice between the less and the more regular setting. Actually, we decide to work with the stronger norms and take

$$\mathcal{H} := L^2(\Omega) \times H^1(\Omega), \tag{3.28}$$

which is endowed with the natural norm; moreover, we put

$$\mathcal{X} := \{(u, v) \in \mathcal{H} : u > 0 \text{ a.e. in } \Omega, \log^- u + \hat{\beta}(v) \in L^1(\Omega)\} \tag{3.29}$$

(here and in the sequel $(\cdot)^- := \max\{-(\cdot), 0\}$ denotes the *negative part* function). Let us note that $(\vartheta_0, \chi_0) \in \mathcal{X}$ if and only if it satisfies (A6) and (A7).

In the analysis of the conserved case we will also consider the following family of subsets of $\mathcal{X}$:

$$\mathcal{X}_{\boldsymbol{\eta}} := \{(u, v) \in \mathcal{X} : \eta_1 \le v_\Omega \le \eta_2\}, \tag{3.30}$$

where

$$\eta_1, \eta_2 \in \operatorname{int} D(\beta) \quad \text{with } \eta_1 < 0, \quad \eta_2 > 0. \tag{3.31}$$

Here and below, the notation $\boldsymbol{\eta}$ stands for the couple (η_1, η_2). The following simple property will allow us to use $\mathcal{X}$ and $\mathcal{X}_{\boldsymbol{\eta}}$ as phase spaces for our systems.

Lemma 3.8. *The sets $\mathcal{X}$ and $\mathcal{X}_{\boldsymbol{\eta}}$, $\forall\boldsymbol{\eta}$ as in (3.31), are complete metric spaces with respect to the distance*

$$d_{\mathcal{X}}((u_1, v_1), (u_2, v_2)) := |u_1 - u_2| + \int_\Omega |\log^- u_1 - \log^- u_2| + \|v_1 - v_2\| + \int_\Omega |\hat\beta(v_1) - \hat\beta(v_2)|. \tag{3.32}$$

Proof. We give the proof for $\mathcal{X}$; clearly $\mathcal{X}_{\boldsymbol{\eta}}$ is a closed subset of its. Let (u_n, v_n) be a Cauchy sequence in $\mathcal{X}$. Then, of course there exists $(u, v) \in \mathcal{H}$ such that $(u_n, v_n) \to (u, v)$ in $\mathcal{H}$. Furthermore, we can assume that, at least for a subsequence (which is not relabeled) such a convergence holds also pointwise in Ω. Then, by Fatou's lemma,

$$\int_\Omega \log^- u \le \liminf \int_\Omega \log^- u_n < +\infty, \tag{3.33}$$

which shows that $u > 0$ almost everywhere and $\log^- u \in L^1(\Omega)$. Let us see that the same procedure applies to $\hat\beta(v_n)$. Actually, by convexity and lower semicontinuity of $\hat\beta$, we readily have that $\hat\beta(v) \in L^1(\Omega)$. Moreover, for a subsequence, $\hat\beta(v_n) \to \hat\beta(v)$ a.e. in Ω. Indeed, since $\hat\beta$ is convex and l.s.c., its restriction to the domain $D(\hat\beta)$ is clearly continuous. Then, if we denote by A_n (resp., A) the set of the points $x \in \Omega$ such that $v_n(x) \notin D(\hat\beta)$ (resp., $v(x) \notin D(\hat\beta)$), we have that

$$|A_n| = 0 \quad \forall n \quad \text{and} \quad |A| = 0, \quad \text{so that}\, |\cup_n A_n \cup A| = 0. \tag{3.34}$$

From the above, it is clear that $\hat\beta(v_n) \to \hat\beta(v)$ a.e. in $\Omega \setminus (\cup_n A_n \cup A)$ and then a.e. in Ω. Finally, since $\log^- u_n$, $\hat\beta(v_n)$ are Cauchy sequences in L^1, it is easy to conclude that they (the whole sequences) converge to the respective limits $\hat\beta(v)$, $\log^- u$ in $L^1(\Omega)$, as desired. □

The next step consists in proving the continuity of the map

$$S(t) : (\vartheta_0, \chi_0) \mapsto (\vartheta(t), \chi(t)) \tag{3.35}$$

in the nonconserved case and $\forall\boldsymbol{\eta}$ as in (3.31), the continuity of the (identically defined) maps $S_{\boldsymbol{\eta}}(t)$ in the conserved case. We first state a regularity property as follows.

Lemma 3.9. *If χ is as in Theorem 3.2 (or as in Theorem 3.4), then we additionally have*

$$\chi \in C^0([0, T]; V) \quad \text{and} \quad \hat\beta(\chi(t)) \in L^1(\Omega) \quad \text{for all}\, t \in [0, T]. \tag{3.36}$$

Moreover, for all $s, t \in [0, T]$, there holds the integration by parts formula

$$\int_s^t \langle \chi_t, B\chi + \xi\rangle = \int_\Omega \left[\frac{|\nabla\chi(t)|^2}{2} + \hat\beta(\chi(t))\right] - \int_\Omega \left[\frac{|\nabla\chi(s)|^2}{2} + \hat\beta(\chi(s))\right]. \tag{3.37}$$

Remark 3.10. The first property in (3.36) is trivial in the nonconserved case, since the space $H^1(0, T; H) \cap L^2(0, T; W)$ turns out to be continuously embedded into $C^0([0, T]; V)$ (see, e.g., [1, Lemma 6.3]).

Lemma 3.9 is the fundamental tool for proving that (cf. Theorem 3.11), starting from $(\vartheta_0, \chi_0) \in \mathcal{X}$, $S(t)(\vartheta_0, \chi_0)$ belongs to $\mathcal{X}$ (resp., $S_{\boldsymbol{\eta}}(t)(\vartheta_0, \chi_0)$ belongs to $\mathcal{X}_{\boldsymbol{\eta}}$) for *every* (and not just a.e.) $t \ge 0$. Then, the proof of continuity of the map $S(t)$, still provided by the following theorem, requires an additional hypothesis in the conserved case.

Theorem 3.11. *Assume* (A1)–(A7). *Then, the map* $S(t)$ *defines a strongly continuous semigroup on* $\mathcal{X}$ *for the system* (3.11)–(3.14). *Analogously, let us assume* (A1)–(A7) *and*

(A10) $\quad \ell(\vartheta) = c_\infty \vartheta, \quad$ *so that* $\alpha(\vartheta) = -\dfrac{c_0}{\vartheta} + c_\infty \vartheta.$

Then, for any $\boldsymbol{\eta}$ *as in* (3.31) (*cf.* (A8)), $S_{\boldsymbol{\eta}}(t)$ *is a strongly continuous semigroup on* $\mathcal{X}_{\boldsymbol{\eta}}$ *for the system* (3.18)–(3.22).

Remark 3.12. Lemma 3.9 and Theorem 3.11 will be proved in the next section. A modification of the argument that will be used in the proofs yields a further noteworthy consequence, i.e. it can be shown that, for any $(u, v) \in \mathcal{X}$, the solution $S(t)(u, v)$ belongs to the metric space $C^0([0, T]; \mathcal{X})$, and the same holds in the conserved case. We will omit, for brevity, the simple details of the proof of this property.

We now have the basis for discussing the existence of absorbing sets for $S(t)$ and $S_{\boldsymbol{\eta}}(t)$. First of all, we have to reinforce (very slightly, indeed) our assumptions on β:

(A11) Assume that there exist $\kappa_1, \kappa_2 > 0$ such that:

$$s \geq \kappa_1 r^3 - \kappa_2 \quad \forall r \in D(\beta) \ \ \forall s \in \beta(r). \tag{3.38}$$

Of course, (A11) holds if either β is a polynomial of at least degree 3 or β is a *constraint* (i.e. it has a bounded domain). Assumption (A11) might be further relaxed; however, we keep it in this form since it covers all the physically meaningful cases.

Let us also note that, by monotonicity of β, (A11) entails that, for some $\kappa_3, \kappa_4 > 0$,

$$sr \geq \hat{\beta}(r) \geq \kappa_3 r^4 - \kappa_4 \quad \forall r \in D(\beta) \ \ \forall s \in \beta(r). \tag{3.39}$$

We can now state our results concerning the dissipativity of the system.

Theorem 3.13. *Assume* (A1)–(A7) *and* (A11). *Then, the semigroup* $S(t)$ *possesses an absorbing set* $\mathcal{B}_0$ *which is bounded in the metric* $d_{\mathcal{X}}$.

Analogously, we have the following theorem.

Theorem 3.14. *Assume* (A1)–(A7), (A10), *and* (A11). *Then, for any* $\boldsymbol{\eta}$ *as in* (3.31), *the semigroup* $S_{\boldsymbol{\eta}}(t)$ *admits an absorbing set* $\mathcal{B}_{0,\boldsymbol{\eta}}$ *which is bounded in the metric* $d_{\mathcal{X}}$ (*with the bound depending on* $\boldsymbol{\eta}$).

In the proofs of these theorems, which are presented in Section 5, we will better describe the $d_{\mathcal{X}}$-bounded sets.

As a next step, in order to address the problem of existence of the attractor, we have to introduce two further spaces (cf. (2.5)) by setting

$$\mathcal{V} := \left\{ (u, v) \in \mathcal{X} : u \in V, \frac{1}{u} \in V, v \in W \right\}, \qquad \mathcal{V}_{\boldsymbol{\eta}} := \mathcal{V} \cap \mathcal{X}_{\boldsymbol{\eta}}. \tag{3.40}$$

We note that $\mathcal{V}$ and $\mathcal{V}_{\boldsymbol{\eta}}$, endowed with the natural distance

$$d_{\mathcal{V}}((u_1, v_1), (u_2, v_2)) := \|u_1 - u_2\| + \left\| \frac{1}{u_1} - \frac{1}{u_2} \right\| + \|v_1 - v_2\|_W \tag{3.41}$$

are *metric spaces*. Moreover, the following property holds.

Proposition 3.15. *If $D(\hat{\beta})$ is closed, then $\mathcal{V} \subset \mathcal{X}$ with compact immersion; namely, if (u_n, v_n) is a $d_{\mathcal{V}}$-bounded sequence in $\mathcal{V}$, then there exists $(u, v) \in \mathcal{X}$ and a subsequence of (u_n, v_n) converging to (u, v) in $d_{\mathcal{X}}$. If $D(\hat{\beta})$ is not closed, then the same is true provided that there exists $c > 0$ such that*

$$\int_{\Omega} |(\beta^0)^2(v_n)| \le c \quad \forall n \in \mathbb{N}, \tag{3.42}$$

where for $r \in D(\beta)$, $\beta^0(r)$ denotes the element of minimum modulus in $\beta(r)$.

Proof. Let (u_n, v_n) be $d_{\mathcal{V}}$-bounded. We first note that, by the standard embeddings between Sobolev's spaces,

$$u_n \to u \quad \text{strongly in } H \text{ and a.e. in } \Omega, \qquad v_n \to v \quad \text{strongly in } V \cap C(\bar{\Omega}) \tag{3.43}$$

at least for a subsequence (not relabeled). Then, by a.e. convergence and Lebesgue's theorem,

$$\frac{1}{u_n} \to \frac{1}{u} \quad \text{strongly in } H \text{ and a.e. in } \Omega. \tag{3.44}$$

Next, by a.e. convergence, (3.44), the fact that $\log^- r \le 1/r$ for all $r \in (0, \infty)$, and Lebesgue's theorem again, we have also that

$$\log^- u_n \to \log^- u \quad \text{strongly in } L^1(\Omega) \tag{3.45}$$

(actually, much more is true). Next, let us treat the term with $\hat{\beta}$, starting from the first case (i.e., $D(\hat{\beta})$ closed). Then, we have already remarked that the restriction of $\hat{\beta}$ to $D(\hat{\beta})$ is continuous. As $\hat{\beta}(v_n) \in L^1(\Omega)$ for all n and v_n is continuous, it follows that $v_n(x) \in D(\hat{\beta})$ for all $x \in \Omega$ and for all $n \in \mathbb{N}$. Then, by the last of (3.43), $v(x) \in D(\hat{\beta})$ for all $x \in \Omega$; moreover, the sequence $\hat{\beta}(v_n)$ is uniformly bounded and tends to $\hat{\beta}(v)$ uniformly in Ω, which, of course, is enough to conclude.

Let us now assume that $D(\hat{\beta})$ is not closed, together with condition (3.42). Just for simplicity, assume that $D(\hat{\beta})$ is an open bounded interval; the other cases are treated similarly. Since β^0 is a monotone function, we have

$$\hat{\beta}(r) = \int_0^r \beta^0(s)\,\mathrm{d}s \quad \forall r \in D(\beta) = D(\hat{\beta}). \tag{3.46}$$

Then, for any $p \in [1, 2)$ and, e.g., $r > 0$, we have that

$$\frac{\hat{\beta}^p(r)}{(\beta^0)^2(r)} = \frac{1}{(\beta^0)^2(r)} \left| \int_0^r \beta^0(s)\,\mathrm{d}s \right|^p \le \left[\frac{1}{(\beta^0)^{(2-p)/p}(r)} \int_0^r \frac{\beta^0(s)}{\beta^0(r)}\,\mathrm{d}s \right]^p \le \frac{r^p}{(\beta^0)^{(2-p)}(r)} \tag{3.47}$$

and the latter quantity is clearly bounded away from zero and uniformly in r. Then, using again that $\hat{\beta}(v_n(x))$ tends to $\hat{\beta}(v(x))$ for all $x \in \Omega$ (just pointwise, since now both $\hat{\beta}(v_n(x))$ and $\hat{\beta}(v(x))$ might be $+\infty$ for some $x \in \Omega$), (3.47) and (3.42) easily yield that the sequence $\hat{\beta}(v_n)$ is bounded in $L^p(\Omega)$ for all $p \in [1, 2)$, so that we conclude again by a.e. convergence and Lebesgue's theorem. □

Let us note that the result above can be extended in the obvious way to the spaces $\mathcal{V}_{\boldsymbol{\eta}}$, $\mathcal{X}_{\boldsymbol{\eta}}$. We finally state our existence results for the universal attractor, which will be shown in Section 6.

Theorem 3.16. *Let* (A1)–(A7) *and* (A11) *hold. Then, the semigroup $S(t)$ possesses a compact attractor which is bounded in the metric $d_{\mathcal{V}}$.*

Theorem 3.17. *Let* (A1)–(A7), (A10), *and* (A11) *hold. Then, for any $\boldsymbol{\eta}$ as in* (3.31), *the semigroup $S_{\boldsymbol{\eta}}(t)$ possesses a compact attractor which is bounded in the metric $d_{\mathcal{V}}$, again with the bound depending on $\boldsymbol{\eta}$.*

4. Continuous dependence

In this section we address the questions of well-posedness and continuous dependence for our systems (3.11)–(3.14) and (3.18)–(3.22). From now on and up to the end of this note, the symbol c will be used to indicate the possibly different strictly positive constants appearing in the computations and assumed to depend only on $\Omega, \ell, L, b, c_0, c_\infty$ and, in particular, not on the time variable. When we need to denote some constant (depending on the same parameters as above) that plays a specific role, a notation like $c_1, c_2, \ldots$ will be used, instead. A constant noted, e.g., as c_σ will be allowed to depend by an additional (small) positive parameter (here σ). For instance, in the sequel we will repeatedly use the elementary Young inequality, holding for all positive σ, in the form

$$ab \le \sigma a^2 + c_\sigma b^2 \quad \forall a, b \in \mathbb{R}. \tag{4.1}$$

We start by addressing the nonconserved system.

Proof of Theorem 3.5. Let us set $(\vartheta, \chi) := (\vartheta_1, \chi_1) - (\vartheta_2, \chi_2)$. Writing (3.11) firstly for (ϑ_1, χ_1), then for (ϑ_2, χ_2), taking the difference, and integrating over $(0, t)$, we get

$$(\vartheta + b\chi) + J(1 * (\alpha(\vartheta_1) - \alpha(\vartheta_2)) = (\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2}), \tag{4.2}$$

where $*$ denotes the usual convolution operator on $(0, t)$. Let us multiply this relation by $\alpha(\vartheta_1) - \alpha(\vartheta_2)$ and integrate over Ω. Then, we get, for *every* $t > 0$,

$$(\vartheta + b\chi, \alpha(\vartheta_1) - \alpha(\vartheta_2)) + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|1 * (\alpha(\vartheta_1) - \alpha(\vartheta_2))\|_J^2 = ((\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2}), \alpha(\vartheta_1) - \alpha(\vartheta_2)). \tag{4.3}$$

Moreover, recalling (A1) and (3.2) and using the mean value theorem and (4.1), we readily get

$$(\vartheta + b\chi, \alpha(\vartheta_1) - \alpha(\vartheta_2)) \ge \tfrac{1}{2}c_\infty|\vartheta|^2 - c|\chi|^2 - c_0 b\chi\left(\frac{1}{\vartheta_1} - \frac{1}{\vartheta_2}\right). \tag{4.4}$$

Next, taking the difference of (3.12), multiplying it by χ, and taking the integral over Ω, by (A3) and (A4) we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\chi|^2 + |\nabla\chi|^2 \le L|\chi|^2 - b\chi\left(\frac{1}{\vartheta_1} - \frac{1}{\vartheta_2}\right). \tag{4.5}$$

Thus, multiplying (4.5) by c_0 and summing the result to (4.3), we note that by (4.4) two terms cancel and get

$$\frac{c_\infty}{2}|\vartheta|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|1 * (\alpha(\vartheta_1) - \alpha(\vartheta_2))\|_J^2 + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\chi|^2 + c_0|\nabla\chi|^2 \le c|\chi|^2 + ((\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2}), \alpha(\vartheta_1) - \alpha(\vartheta_2)), \tag{4.6}$$

whence, integrating over $(0, t)$ for $t \le T$, observing that $(\vartheta_{0,1} - \vartheta_{0,2}) + b(\chi_{0,1} - \chi_{0,2})$ is independent of time, and splitting the last term with respect to the duality between V' and V, we see that the standard Gronwall's lemma applies. Then, to get (3.24) it is sufficient to notice that a comparison in relation (3.11) (integrated in time) gives, for some $c > 0$,

$$c\|1 * (\alpha(\vartheta_1) - \alpha(\vartheta_2))(t)\|_J^2 \ge \|(\vartheta + b\chi)(t)\|_*^2 - 2\|\vartheta_0 + b\chi_0\|_*^2. \tag{4.7}$$

□

Proof of Theorem 3.6. Let us define (ϑ, χ) as in the proof of Theorem 3.5 and also set, with obvious meaning, $w := w_1 - w_2$. Then, we work on (3.18) as in the nonconserved case and get again (4.3). Next, we put

$$m_0 := \frac{1}{|\Omega|}\int_\Omega (\chi_{0,1} - \chi_{0,2}), \qquad \bar\chi(t) := \chi(t) - m_0 \tag{4.8}$$

and note that by (3.36) (which is shown below), it is $\bar\chi(t) \in V_0$ for *all* $t \in [0, T]$, so that it is possible to test (3.19) by $\mathcal{N}\bar\chi$ and (3.20) by $\bar\chi$ and take the difference. Noting that by (2.4) the terms involving w cancel together and using (3.23) and (2.6) it is easy to obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\bar\chi\|_*^2 + |\nabla\chi|^2 = -\int_\Omega (\xi_1 - \xi_2)\bar\chi - \int_\Omega (\gamma(\chi_1) - \gamma(\chi_2))\bar\chi - \int_\Omega b\bar\chi\left(\frac{1}{\vartheta_1} - \frac{1}{\vartheta_2}\right) \tag{4.9}$$

and we have to provide a bound for the right-hand side. Let us start by noticing that, by the compactness of the embedding $V \subset H$, it holds

$$|v|^2 \le \sigma|\nabla v|^2 + c_\sigma\|v\|_*^2 \quad \forall v \in V. \tag{4.10}$$

Furthermore, we have that

$$|m_0| \le |\Omega|^{-1/2}\|\chi_{0,1} - \chi_{0,2}\|_*. \tag{4.11}$$

Let us finally observe that, for $N \le 3$, by (3.17) it follows:

$$\chi_i \in L^{10}(Q_T) \quad \text{for } i = 1, 2. \tag{4.12}$$

To prove this property, let us consider the family of s-spaces $(\cdot,\cdot)_s$ introduced, e.g., in [19, Definition 1.1, p. 27]. Then, we look for the largest exponent q such that

$$(L^2(0, T; H^2(\Omega)), L^\infty(0, T; V))_s \subset L^q(Q_T) \quad \text{for some } s \in [0, 1].$$

Actually, we note that

$$(L^2(0, T; H^2(\Omega)), L^\infty(0, T; V))_s = L^{2/(1-s)}(0, T; H^{2-s}(\Omega))$$

and that

$$H^{2-s}(\Omega) \subset L^{6/(2s-1)}(\Omega),$$

thus,

$$\frac{2}{1-s} = \frac{6}{2s-1} \Rightarrow s = \frac{4}{5}, \qquad \frac{2}{1-s} = 10.$$

Then, by monotonicity of β and relation (A9), we have

$$\begin{aligned}
-\int_\Omega (\xi_1 - \xi_2)\bar\chi &\le c_\beta\int_\Omega |\chi||m_0|(1 + |\chi_1|^p + |\chi_2|^p) \le c_\beta\int_\Omega (|\bar\chi| + |m_0|)|m_0|(1 + |\chi_1|^p + |\chi_2|^p)\\
&\le c_\beta\int_\Omega |m_0|^2(1 + |\chi_1|^p + |\chi_2|^p) + \sigma_1\int_\Omega |\bar\chi|^2(1 + |\chi_1|^4 + |\chi_2|^4)\\
&\quad + c_{\sigma_1}\int_\Omega |m_0|^2(1 + |\chi_1|^{2p-4} + |\chi_2|^{2p-4}) =: I_1 + I_2 + I_3.
\end{aligned} \tag{4.13}$$

Next, by (3.17) and the continuity of the embedding $V \subset L^6(\Omega)$ we have

$$I_2 \le \sigma_1\|\bar\chi\|^2(c + \|\chi_1\|^4_{L^\infty(0,T;V)} + \|\chi_2\|^4_{L^\infty(0,T;V)}) \le c_1\sigma_1|\nabla\chi|^2, \tag{4.14}$$

furthermore,

$$\int_0^t (I_1(s)+I_3(s))\,\mathrm{d}s \le c_{\sigma_1}|m_0|^2. \tag{4.15}$$

Indeed, we used that, setting $q=\max\{p,2p-4\}$, it is $q\le 10$ by (A9), so that the inequality above is a consequence of (4.12). Of course, the constant c_{σ_1} in (4.15) depends on the L^{10}-norms of χ_1 and χ_2, in addition.

As for the γ-term, by (4.10), (4.11) and (4.1) it is

$$-\int_\Omega (\gamma(\chi_1)-\gamma(\chi_2))\bar\chi \le L|\chi|_H|\bar\chi|_H \le c_{\sigma_2}|\bar\chi|_*^2+\sigma_2|\nabla\chi|^2+c\|\chi_{0,1}-\chi_{0,2}\|_*^2. \tag{4.16}$$

Next,

$$-\int_\Omega b\bar\chi\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right) = -\int_\Omega b\chi\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right)+bm_0\int_\Omega\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right). \tag{4.17}$$

Furthermore, by (A1),

$$\begin{aligned} bm_0\int_\Omega\left(\frac{1}{\vartheta_1}-\frac{1}{\vartheta_2}\right) &= -\frac{b}{c_0}m_0\int_\Omega(\alpha(\vartheta_1)-\alpha(\vartheta_2)-\ell(\vartheta_1)+\ell(\vartheta_2)) \le \sigma_3|\vartheta|^2+c_{\sigma_3}|m_0|^2 \\ &\quad -\frac{b}{c_0}m_0\int_\Omega(\alpha(\vartheta_1)-\alpha(\vartheta_2)). \end{aligned} \tag{4.18}$$

Now, let us sum together (4.3) and c_0 times (4.9) and integrate in time between 0 and $t\le T$. Repeating (4.4) and recalling (4.17), we see that two terms cancel. Moreover, c_0 times the last term on the right-hand side of (4.18), after integration, is controlled by

$$-bm_0\int_0^t\int_\Omega(\alpha(\vartheta_1)-\alpha(\vartheta_2)) \le c_{\sigma_4}|m_0|^2+\sigma_4|1*(\alpha(\vartheta_1)-\alpha(\vartheta_2))|^2. \tag{4.19}$$

Then, taking the various σ_i sufficiently small (in particular we need that $c_1\sigma_1\le 1/2$), we finally deduce

$$\begin{aligned} &\|1*(\alpha(\vartheta_1)-\alpha(\vartheta_2))(t)\|^2+\int_0^t|\vartheta|^2+\|\bar\chi(t)\|_*^2+\int_0^t|\nabla\chi|^2 \le c\|(\vartheta_{0,1}-\vartheta_{0,2})+b(\chi_{0,1}-\chi_{0,2})\|_*^2 \\ &\quad +c\int_0^t\|\bar\chi\|_*^2+c\|\chi_{0,1}-\chi_{0,2}\|_*^2+c|m_0|^2, \end{aligned} \tag{4.20}$$

whence, recalling (4.7), a further application of Gronwall's lemma readily yields (3.26). □

Proof of Theorem 3.7. We operate as before on equation (3.18) and get (4.3). Next, with the same notation as above, we test again (3.19) by $\mathcal{N}\bar\chi$, (3.20) by $\bar\chi$, and take the difference, getting (4.9). Instead of repeating (4.13), now the terms with β are simply treated as follows:

$$-\int_\Omega(\xi_1-\xi_2)\bar\chi \le |m_0|\int_\Omega(|\xi_1|+|\xi_2|). \tag{4.21}$$

Repeating the rest of the procedure as in the previous case, instead of (4.20) we obtain

$$\begin{aligned} &\|1*(\alpha(\vartheta_1)-\alpha(\vartheta_2))(t)\|^2+\int_0^t|\vartheta|^2+\|\bar\chi(t)\|_*^2+\int_0^t|\nabla\chi|^2 \le c\|(\vartheta_{0,1}-\vartheta_{0,2})+b(\chi_{0,1}-\chi_{0,2})\|_*^2 \\ &+c\int_0^t\|\bar\chi\|_*^2+c\|\chi_{0,1}-\chi_{0,2}\|_*^2+c|m_0|^2+c|m_0|(\|\xi_1\|_{L^1(Q_T)}+\|\xi_2\|_{L^1(Q_T)}), \end{aligned} \tag{4.22}$$

whence (3.27) is once more a straightforward consequence of (4.7) and Gronwall's lemma. □

Proof of Lemma 3.9. We start by recalling, together with its proof, a simple preliminary result from convex analysis.

Lemma 4.1. *Let $T > 0$, let $\mathcal{J} : H \to (-\infty, +\infty]$ be a convex, l.s.c., and proper functional, and let $u \in H^1(0, T; V') \cap L^2(0, T; V)$, $\eta \in L^2(0, T; V)$, $\mathcal{A} = \partial\mathcal{J}$. Let also $\eta(t) \in \mathcal{A}u(t)$ for a.e. $t \in (0, T)$. Moreover, let us suppose that there exist $k_1, k_2 > 0$ such that*

$$\mathcal{J}(v) \geq k_1|v|^2 - k_2 \quad \text{for all } v \in H. \tag{4.23}$$

Then, the function $t \mapsto \mathcal{J}(u(t))$ is absolutely continuous in $[0, T]$ and

$$\int_s^t \langle \partial_t u(r), \eta(r) \rangle \, \mathrm{d}r = \mathcal{J}(u(t)) - \mathcal{J}(u(s)) \quad \forall s, t \in [0, T]. \tag{4.24}$$

Proof. To prove the above lemma, let us extend the functional $\mathcal{J}$ to the space V' by setting, for $v \in V'$,

$$\mathcal{J}_* : V' \to (-\infty, +\infty], \qquad \mathcal{J}_*(v) := \begin{cases} \mathcal{J}(v) & \text{if } v \in H, \\ +\infty & \text{otherwise.} \end{cases} \tag{4.25}$$

It is easy to see that, thanks to (4.23), the functional $\mathcal{J}_*$ is still convex, lower semicontinuous, and proper on V'. Let us now note by Id the identity operator on V and observe that $\mathrm{Id} + B : V \to V'$ is the Riesz mapping associated to the standard scalar product of V. Then, by assumption we have

$$(z - u, \eta) \leq \mathcal{J}(z) - \mathcal{J}(u) \quad \text{a.e. in } (0, T) \tag{4.26}$$

for every $z \in H$. Thus, since $\eta \in V$ a.e. in $(0, T)$, we immediately deduce that

$$((z - u, \eta + B\eta))_* \leq \mathcal{J}_*(z) - \mathcal{J}_*(u) \quad \text{a.e. in } (0, T) \tag{4.27}$$

for any $z \in V'$. Indeed, $\mathcal{J}_*$ is identically $+\infty$ in $V' \setminus H$. Since $\eta + B\eta \in L^2(0, T; V')$, the above relation can be restated by saying that

$$\eta + B\eta \in \mathcal{A}_*(u) \quad \text{a.e. in } (0, T), \tag{4.28}$$

where $\mathcal{A}_*$ denotes the subdifferential of $\mathcal{J}_*$ with respect to the scalar product of V'. Then, the assumptions of [5, Lemma 3.3, p. 73] are fulfilled in the space V'. By that result, $\mathcal{J}_*(u) = \mathcal{J}(u) \in AC([0, T])$ and formula (4.24) holds, so concluding the proof of Lemma 4.1. □

Let us refer to the (more difficult) conserved case (cf. Theorem 3.4). We aim to apply Lemma 4.1 to the functional

$$\mathcal{J} = G_H : H \to [0, +\infty], \qquad G_H(v) := \int_\Omega \left[\frac{\|v\|^2}{2} + \hat{\beta}(v) \right]. \tag{4.29}$$

Actually, if we set $\varphi := \gamma(\chi) + b/\theta$, then, observing that $\varphi \in L^2(0, T; V')$ by (3.8), (3.17), and (A3), and recalling relation (2.4), it is not difficult to show that χ, ξ, φ satisfy

$$\eta := -\mathcal{N}\chi_t + \xi_\Omega + \varphi_\Omega - \varphi + \chi \in \partial G_H(\chi) \quad \text{a.e. in } (0, T), \tag{4.30}$$

which allows to apply Lemma 4.1 with obvious choices of the other data. This gives the second of (3.36) and formula (3.37). It remains to show the continuity property of χ in (3.36). Indeed, as G_H is the sum of two convex functionals (cf. (4.29)), we deduce that both summands are continuous, so that this holds in particular for the map $t \mapsto \|\chi(t)\|^2$. Finally, recalling that $\chi \in C_w(0, T; V)$ by (3.17) and, e.g., [26, Lemma 3.3, p. 72], this is indeed enough to have also the first of (3.36), so that the proof of Lemma 3.9 is now complete.

Proof of Theorem 3.11. We start dealing with the nonconserved case. First of all, let us show that, as $(\vartheta_0, \chi_0) \in \mathcal{X}$, it follows that the corresponding solution $(\vartheta(t), \chi(t)) = S(t)(\vartheta_0, \chi_0)$ belongs to $\mathcal{X}$ for *every* $t > 0$.

With this aim, let us define

$$\hat{\alpha}(r) := \int_1^r \alpha(s)\,\mathrm{d}s \quad \text{for } r \in (0, +\infty) \tag{4.31}$$

and observe that, by (3.1)–(3.3), there exist a constant $\nu > 0$ and a *convex and non-negative* function $\hat{\alpha}_{\mathrm{rest}} : (0, +\infty) \to \mathbb{R}$ such that

$$\hat{\alpha}(r) = \nu \log^- r + \nu r^2 + \hat{\alpha}_{\mathrm{rest}}(r) \quad \forall r \in (0, +\infty). \tag{4.32}$$

Then, using the fact that $\vartheta > 0$ a.e. in Q_T (cf. (3.7)) together with Lemma 4.1 (applied with the choices of $u = \vartheta$, $\eta = \alpha(\vartheta)$, and $\mathcal{J}$ given by the convex functional induced on H by $\hat{\alpha}$), we obtain that the function

$$t \mapsto \int_\Omega \hat{\alpha}(\vartheta(t)) \tag{4.33}$$

is absolutely continuous in $[0, T]$. By the decomposition (4.32), it then follows that $\log^- \vartheta(t) \in L^1(\Omega)$ for *every* $t > 0$. Since $\hat{\beta}(\chi(t)) \in L^1(\Omega)$ for every $t > 0$ by Lemma 3.9, this concludes the proof that $S(t)$ maps $\mathcal{X}$ into itself $\forall t > 0$.

Next, let us prove the continuity of the map $S(t) : \mathcal{X} \to \mathcal{X}$ for any $t > 0$. This will show that $S(\cdot)$ is a strongly continuous semigroup on $\mathcal{X}$, as desired. Assume that $\{(\vartheta_{0,n}, \chi_{0,n})\} \subset \mathcal{X}$ is a sequence of initial data converging to $(\vartheta_0, \chi_0) \in \mathcal{X}$ in the metric of $\mathcal{X}$. Moreover, let us fix $t > 0$ and name (ϑ_n, χ_n) (resp., (ϑ, χ)) the solution emanating from $(\vartheta_{0,n}, \chi_{0,n})$ (resp., (ϑ_0, χ_0)), whose existence and uniqueness are guaranteed by Theorems 3.2 and 3.5. Then, let us notice that we are in the position of applying Theorem 3.5 with the choices of $(\vartheta_{0,1}, \chi_{0,1}) = (\vartheta_{0,n}, \chi_{0,n})$ and $(\vartheta_{0,2}, \chi_{0,2}) = (\vartheta_0, \chi_0)$. Thus, by (3.24) we deduce that

$$(\vartheta_n, \chi_n) \to (\vartheta, \chi) \quad \text{strongly in } L^\infty(0, T; V') \times L^\infty(0, T; H) \tag{4.34}$$

(and, in particular, the limit of the *whole* sequence is identified).

However, this is not sufficient to prove the convergence of $(\vartheta_n(t), \chi_n(t))$ to $(\vartheta(t), \chi(t))$ in the (stronger) metric of $\mathcal{X}$. Thus, to proceed, we have to repeat the energy estimates (cf. [7, Lemmas 4.1 and 4.2]) on the sequence (ϑ_n, χ_n). In this way, we get uniform bounds, independent of n, of the norms appearing in (3.7)–(3.10). By the standard compactness arguments, we also have as a byproduct the corresponding weak or weak–$*$ convergence properties, holding for the whole sequence thanks to (4.34). In particular, by standard continuous embedding results, we have that

$$\vartheta_n(t) \to \vartheta(t) \quad \text{weakly in } H, \qquad \chi_n(t) \to \chi(t) \quad \text{weakly in } V \tag{4.35}$$

for all $t > 0$. We also note that (see [9, Section 5] for the details in an even more general setting), additionally, this procedure yields

$$\alpha(\vartheta_n) \to \alpha(\vartheta) \quad \text{weakly in } L^2(0, T; V), \tag{4.36}$$

again for the whole sequence $\alpha(\vartheta_n)$. Moreover, a strong $L^2(Q_T)$-convergence $\gamma(\chi_n) \to \gamma(\chi)$ can be standardly proved.

To conclude, we use a semicontinuity argument in order to show the convergence with respect to $d_{\mathcal{X}}$. Writing (3.11) at the step n, testing it by $\alpha(\vartheta_n)$, and integrating over $[0, t]$, thanks to the integration by parts formula (4.24) we deduce

$$\int_\Omega \hat{\alpha}(\vartheta_n(t)) \le \int_\Omega \hat{\alpha}(\vartheta_{0,n}) - \int_0^t \|\alpha(\vartheta_n)\|_J^2 + \int_0^t \langle f, \alpha(\vartheta_n)\rangle - b \int_0^t \int_\Omega \partial_t \chi_n \alpha(\vartheta_n), \tag{4.37}$$

which holds for *every* time $t > 0$.

Next, testing (3.12) at the step n by $\partial_t\chi_n$, integrating over $[0,t]$, and using (3.37), for *every* $t>0$ we get

$$\frac{1}{2}|\nabla\chi_n(t)|^2+\int_\Omega\hat\beta(\chi_n(t))\le\frac{1}{2}|\nabla\chi_{0,n}|^2+\int_\Omega\hat\beta(\chi_{0,n})-\int_0^t|\partial_t\chi_n|_H^2-\int_0^t\int_\Omega\gamma(\chi_n)\partial_t\chi_n-\int_0^t\int_\Omega\frac{b}{\vartheta_n}\partial_t\chi_n. \tag{4.38}$$

Let us sum (4.37) and c_0 times (4.38) and note that two terms cancel. Then, let us take the lim sup, as $n\nearrow\infty$, of the resulting relation. Of course, our aim is letting the terms on the right-hand sides of (4.37) and (4.38) pass to the limit.

First, let us observe that the three terms related to the initial values pass to the limit since the initial data are assumed to converge in $d_\mathcal{X}$ (cf. (3.32)). Next, using the energy estimates, the standard compact embedding theorems, and (A1), (A3) and (3.6), we easily derive that

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\left[\langle f,\alpha(\vartheta_n)\rangle-c_0\int_\Omega\gamma(\chi_n)\partial_t\chi_n-b\int_\Omega\partial_t\chi_n\ell(\vartheta_n)\right]\\&\quad=\int_0^t\left[\langle f,\alpha(\vartheta)\rangle-c_0\int_\Omega\gamma(\chi)\partial_t\chi-b\int_\Omega\partial_t\chi\ell(\vartheta)\right].\end{aligned}\tag{4.39}$$

Furthermore, by semicontinuity of norms with respect to weak convergence,

$$\limsup_{n\to\infty}\int_0^t[-\|\alpha(\vartheta_n)\|_J^2-c_0|\partial_t\chi_n|_H^2]\le\int_0^t\left[-\|\alpha(\vartheta)\|_J^2-c_0|\partial_t\chi|_H^2\right].\tag{4.40}$$

Then, testing the limit relation (3.11) by $\alpha(\vartheta)$ and the limit (3.12) by $c_0\partial_t\chi$, summing, and integrating over $[0,t]$, a comparison with the result of the preceding computations yields, for every $t>0$,

$$\limsup_{n\to\infty}\int_\Omega\left[\hat\alpha(\vartheta_n(t))+\frac{c_0}{2}|\nabla\chi_n(t)|^2+c_0\hat\beta(\chi_n(t))\right]\le\int_\Omega\left[\hat\alpha(\vartheta(t))+\frac{c_0}{2}|\nabla\chi(t)|^2+c_0\hat\beta(\chi(t))\right].\tag{4.41}$$

Moreover, we remark that the converse lim inf-inequality holds by weak convergence of ϑ_n, χ_n and convexity and lower semicontinuity of $\hat\alpha$, $\hat\beta$, and of the squared modulus. This gives, $\forall t>0$,

$$\int_\Omega\hat\alpha(\vartheta_n(t)),\qquad\frac{1}{2}|\nabla\chi_n(t)|_H^2,\qquad\int_\Omega\hat\beta(\chi_n(t))\to\int_\Omega\hat\alpha(\vartheta(t)),\qquad\frac{1}{2}|\nabla\chi(t)|_H^2,\qquad\int_\Omega\hat\beta(\chi(t)),\tag{4.42}$$

respectively (and not only the convergence of the sum). Then, using the decomposition in (4.32) (and in particular the convexity of $\hat\alpha_{\text{rest}}$), we also get

$$|\vartheta_n(t)|_H^2,\qquad\int_\Omega\log^-(\vartheta_n(t))\to|\vartheta(t)|_H^2,\qquad\int_\Omega\log^-(\vartheta(t)),\tag{4.43}$$

still $\forall t>0$.

At this point, using (4.42), (4.43) and (4.35), we obtain that

$$\vartheta_n(t)\to\vartheta(t)\quad\text{strongly in } H,\qquad\chi_n(t)\to\chi(t)\quad\text{strongly in } V,\tag{4.44}$$

but not yet the convergence in $d_\mathcal{X}$. Actually, it remains to show that

$$\log^-(\vartheta_n(t)),\ \hat\beta(\chi_n(t))\to\log^-(\vartheta(t)),\ \hat\beta(\chi(t))\tag{4.45}$$

strongly in $L^1(\Omega)$. Both terms are treated with the aid of a simple lemma.

Lemma 4.2. *Let $\{u_n\}$ be a sequence of non-negative functions in $L^1(\Omega)$ converging almost everywhere to a function $u \in L^1(\Omega)$. Moreover, let*

$$\int_\Omega u_n \to \int_\Omega u \quad \text{as } n \nearrow \infty. \tag{4.46}$$

Then, $u_n \to u$ strongly in $L^1(\Omega)$.

Proof. Let $v_n := u_n - u$, which tends to 0 a.e. in Ω. Moreover, $\int_\Omega v_n \to 0$ by assumption. Since $-u \le -v_n^- \le 0$ a.e. in Ω and $u \in L^1(\Omega)$, we can apply Lebesgue's theorem to v_n^- and conclude that $\int_\Omega v_n^- \to 0$. Thus, by comparison, $\int_\Omega v_n^+ \to 0$ and $\int_\Omega |v_n| \to 0$, i.e. the assert. □

To apply the lemma, anyway, we need the a.e.-Ω convergence of the whole sequence $\{\vartheta_n(t)\}$ to $\vartheta(t)$, but we just know (by (4.44)) that this holds up to the extraction of a subsequence $\{\vartheta_{n_k}\}$. Thus, as a first step, we only obtain that

$$\log^-(\vartheta_{n_k}(t)) \to \log^-(\vartheta(t)) \quad \text{strongly in } L^1(\Omega). \tag{4.47}$$

However, since (4.44) holds for the whole sequences, then the same is true for (4.47). The same argument works also for the term $\hat\beta(\chi_n(t))$, with some small additional complication due to the fact that $\hat\beta$ is not necessarily continuous on the border of its domain (cf. the discussion leading to (3.34) in the proof of Lemma 3.8 to overcome this difficulty). This concludes the proof of Theorem 3.11 in the nonconserved case.

The proof in the conserved case is analogous. The main differences are the following: first, when repeating (4.38), we now have to test (3.19) by $\mathcal{N}\partial_t\chi_n$, (3.20) by $\partial_t\chi_n$, and take the difference. This gives $\|\partial_t\chi_n\|_*$ whenever we have had $|\partial_t\chi_n|$ before.

Consequently, some further care is required also as we perform the semicontinuity argument. Indeed, the energy estimates now just yield

$$\chi_n \to \chi \quad \text{weakly in } H^1(0,T;V') \cap L^2(0,T;W) \tag{4.48}$$

and by Aubin's lemma this still gives

$$\chi_n \to \chi \quad \text{strongly in } L^2(0,T;V). \tag{4.49}$$

However, it is not obvious how to treat the latter two terms in (4.39). As for the first, denoting by $\hat\gamma$ a primitive of γ, we have

$$\int_t^0 \langle\partial_t\chi_n, \gamma(\chi_n)\rangle = \int_\Omega \hat\gamma(\chi_n(t)) - \int_\Omega \hat\gamma(\chi_{0,n}) \to \int_\Omega \hat\gamma(\chi(t)) - \int_\Omega \hat\gamma(\chi_0) = \int_t^0 \langle\partial_t\chi, \gamma(\chi)\rangle, \tag{4.50}$$

where the chain rule used in the integrations in time is just formal in this setting, but could be made rigorous through an approximation procedure; moreover, the convergence holds since from (4.48), interpolation, and (A3), we know that

$$\hat\gamma(\chi_n) \to \hat\gamma(\chi) \quad \text{strongly in } C^0([0,T];H). \tag{4.51}$$

Finally, let us integrate by parts in time the latter term in (4.39). Thanks to (A10), we have

$$\begin{aligned} -b\int_0^t\int_\Omega \partial_t\chi_n\ell(\vartheta_n) &= bc_\infty\int_t^0\langle\partial_t\vartheta_n, \chi_n\rangle - bc_\infty\int_\Omega \chi_n(t)\vartheta_n(t) + bc_\infty\int_\Omega \chi_{0,n}\vartheta_{0,n} \to bc_\infty\int_t^0\langle\partial_t\vartheta, \chi\rangle \\ &\quad - bc_\infty\int_\Omega \chi(t)\vartheta(t) + bc_\infty\int_\Omega \chi_0\vartheta_0 = b\int_0^t\int_\Omega \partial_t\chi\ell(\vartheta), \end{aligned} \tag{4.52}$$

where we have used (4.49) and that $\vartheta_n \to \vartheta$ weakly in $H^1(0,T;V')$, which follows from (3.7). Then, the rest of the proof works exactly as in the nonconserved case. □

5. Dissipativity

Proof of Theorem 3.13. The proof is reached via a number of a priori estimates. Let us detail them.

Estimate 1. Test (3.11) by $-1/\vartheta$, (3.12) by χ_t and take the sum. Using (3.3) and noting that two terms cancel together, the procedure gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega -\log\vartheta + c_0'\int_\Omega\left|\nabla\frac{1}{\vartheta}\right|^2 + c_0 n_0\int_\Gamma\frac{1}{\vartheta^2} + |\chi_t|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[\frac{|\nabla\chi|^2}{2}+\hat\beta(\chi)\right] \le n_0\int_\Gamma\frac{\ell(\vartheta)}{\vartheta} - \left\langle f,\frac{1}{\vartheta}\right\rangle - \int_\Omega\gamma(\chi)+\chi_t. \tag{5.1}$$

Next, let us treat the three terms on the right-hand side: first, by (A1), (4.1), and continuity of the trace operator from V to $L^2(\Gamma)$

$$n_0\int_\Gamma\frac{\ell(\vartheta)}{\vartheta} \le \sigma_5\left\|\frac{1}{\vartheta}\right\|^2 + c_{\sigma_5}, \tag{5.2}$$

next,

$$-\left\langle f,\frac{1}{\vartheta}\right\rangle \le \sigma_6\left\|\frac{1}{\vartheta}\right\|^2 + c_{\sigma_6}\|f\|_*^2, \tag{5.3}$$

finally,

$$-\int_\Omega\gamma(\chi)\chi_t \le \sigma_7|\chi_t|_H^2 + \sigma_7\|\chi\|_{L^4(\Omega)}^4 + c_{\sigma_7}. \tag{5.4}$$

Putting the above computations together and choosing σ_5, σ_6, σ_7 sufficiently small, we readily have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left[-\log\vartheta + \frac{|\nabla\chi|^2}{2} + \hat\beta(\chi)\right] + c_3\left[\left\|\frac{1}{\vartheta}\right\|^2 + |\chi_t|_H^2\right] \le \sigma_7\|\chi\|_{L^4(\Omega)}^4 + c_{\sigma_5,\sigma_6,\sigma_7}, \tag{5.5}$$

where the constant $c_3 > 0$ also depends on σ_5, σ_6, and the constant $c_{\sigma_5,\sigma_6,\sigma_7}$ also depends on c_0, c_0', n_0, f, L, of course.

Estimate 2. Test now (3.12) by χ, getting

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\chi^2}{2} + \int_\Omega|\nabla\chi|^2 + \int_\Omega\xi\chi \le -\int_\Omega\gamma(\chi)\chi - \int_\Omega\frac{b\chi}{\vartheta}. \tag{5.6}$$

By (A3), the right-hand side is simply treated this way:

$$-\int_\Omega\gamma(\chi)\chi - \int_\Omega\frac{b\chi}{\vartheta} \le \sigma_8\|\chi\|_{L^4(\Omega)}^4 + \sigma_8\left|\frac{1}{\vartheta}\right|_H^2 + c_{\sigma_8}. \tag{5.7}$$

Estimate 3. Finally, test (3.11) by ϑ. By (3.2), this gives

$$\frac{d}{dt}\int_\Omega \frac{\vartheta^2}{2} + c_\infty \int_\Omega |\nabla\vartheta|^2 + n_0 \int_\Gamma \alpha(\vartheta)\vartheta \le \langle f, \vartheta\rangle - \int_\Omega b\chi_t\vartheta. \tag{5.8}$$

Let us now note that, by (A1),

$$\alpha(\vartheta)\vartheta = (\alpha(\vartheta)-\alpha(1))(\vartheta-1)+\alpha(\vartheta) \ge c_\infty(\vartheta-1)^2+\alpha(\vartheta) \ge \tfrac{1}{2}c_\infty\vartheta^2 - c_\infty + \alpha(\vartheta). \tag{5.9}$$

Furthermore, it is clear that

$$|\alpha(\vartheta)| = \left|-\frac{c_0}{\vartheta}+\ell(\vartheta)\right| \le \sigma_9\vartheta^2 + \sigma_9\frac{1}{\vartheta^2} + c_{\sigma_9}. \tag{5.10}$$

Next, also on account of (3.6), we have

$$\langle f, \vartheta\rangle - \int_\Omega b\chi_t\vartheta \le \sigma_{10}\|\vartheta\|^2 + c_{\sigma_{10}}\|\chi_t\|_*^2 + c_{\sigma_{10}}\|f\|_*^2. \tag{5.11}$$

Thus, taking σ_9, σ_{10} sufficiently small (a good choice for σ_{10} is $c_\infty/8$), (5.8) becomes

$$\frac{d}{dt}\int_\Omega \frac{\vartheta^2}{2} + c_4\|\vartheta\|^2 \le c_{\sigma_{10}}\|\chi_t\|_*^2 + \sigma_9\int_\Omega \frac{1}{\vartheta^2} + c_{\sigma_9,\sigma_{10}}, \tag{5.12}$$

where the constant $c_4 > 0$ on the left-hand side also depends on $\sigma_9, \sigma_{10}, c_\infty, n_0$, while the constant $c_{\sigma_9,\sigma_{10}}$ on the right-hand side also depends on c_0, c_∞, L, and on the V'-norm of f, of course.

Conclusion of the proof. Let us now note that, by (A11) (cf. also (3.39)), there exist constants $\kappa_5, \kappa_6 > 0$ such that

$$\xi\chi \ge \kappa_5[\hat\beta(\chi)+\chi^4+\chi^2] - \kappa_6 \quad \text{a.e. in } \Omega. \tag{5.13}$$

Thus, let us sum together (5.5), (5.6) and (5.12) multiplied by a constant $\varepsilon > 0$ to be chosen later. Using (5.7), (5.13), and the continuity of the embedding $H \subset V'$, this procedure gives

$$\begin{aligned}\frac{d}{dt}&\left[-\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2 + \frac{1}{2}\|\chi\|^2 + \int_\Omega \hat\beta(\chi)\right] + c_3\left[\left\|\frac{1}{\vartheta}\right\|^2 + |\chi_t|_H^2\right] + \kappa_5\left[\int_\Omega \hat\beta(\chi) + \|\chi\|_{L^4(\Omega)}^4 + |\chi|_H^2\right]\\ &+ |\nabla\chi|_H^2 + \varepsilon c_4\|\vartheta\|^2 \le (\sigma_8+\varepsilon\sigma_9)\left|\frac{1}{\vartheta}\right|^2 + (\sigma_7+\sigma_8)\|\chi\|_{L^4(\Omega)}^4 + \varepsilon c_{\sigma_{10}}|\chi_t|_H^2 + c_5,\end{aligned} \tag{5.14}$$

where the constant c_5 collects all the constants introduced on the right-hand sides of the computations above and depends on the various σ_i's.

Then, choosing

$$\sigma_7+\sigma_8 \le \frac{1}{\kappa_5}, \qquad \sigma_8+\varepsilon\sigma_9 \le \frac{1}{c_3}, \qquad \varepsilon \le \min\left\{\frac{c_3}{2c_{\sigma_{10}}}, 1\right\}, \tag{5.15}$$

all the terms on the right-hand side are controlled by the corresponding terms on the left-hand side. Furthermore, it is clear that there exists $c_6 > 0$, also depending on ε, such that

$$\frac{\varepsilon c_4}{2}\|\vartheta\|^2 + \frac{c_3}{4}\left\|\frac{1}{\vartheta}\right\|^2 \ge c_6\left[-\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2\right]. \tag{5.16}$$

Next, let us observe that

$$-\log r + \varepsilon\tfrac{1}{2}(r^2) \ge \log^- r + \tfrac{1}{4}\varepsilon r^2 - c_\varepsilon \quad \forall r \in (0,+\infty) \tag{5.17}$$

and for some constant $c_\varepsilon > 0$ not depending on r. We can now define

$$\Phi_\varepsilon(\vartheta,\chi) := -\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2 + \frac{1}{2}\|\chi\|^2 + \int_\Omega \hat\beta(\chi) + c_\varepsilon \tag{5.18}$$

and note that, by (5.17), Φ_ε is a non-negative functional. More precisely, we have that

$$\Phi_\varepsilon(\vartheta,\chi) \geq \int_\Omega \log^-\vartheta + \frac{\varepsilon}{4}|\vartheta|_H^2 + \frac{1}{2}\|\chi\|^2 + \int_\Omega \hat\beta(\chi). \tag{5.19}$$

Now, also by (5.16), (5.14) yields, for some $c_7, c_8, c_9 > 0$ depending on ε and on the other constants,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Phi_\varepsilon(\vartheta,\chi) + c_7\Phi_\varepsilon(\vartheta,\chi) + c_8\left[\|\chi\|^4_{L^4(\Omega)} + \left\|\frac{1}{\vartheta}\right\|^2 + \|\vartheta\|^2 + |\chi_t|_H^2\right] \leq c_9. \tag{5.20}$$

Thus, using Gronwall's lemma in the differential form (cf., e.g., [10, Lemma 2.5]), we obtain that for every $t > 0$ it holds

$$\Phi_\varepsilon(\vartheta(t),\chi(t)) \leq \Phi_\varepsilon(\vartheta_0,\chi_0)\exp(-c_7 t) + \frac{c_9}{c_7}. \tag{5.21}$$

Let us finally show that this estimate entails the existence of a set $\mathcal{B}_0$, bounded in the metric $d_\mathcal{X}$, which *absorbs* all metric bounded set $\mathcal{M}$ in a finite time $T_\mathcal{M}$ (cf. (2.15)). As a first step, we note that a set $\mathcal{M}$ of $\mathcal{X}$ is bounded with respect to $d_\mathcal{X}$ if and only if

$$\exists R_\mathcal{M} > 0 : d_\mathcal{X}((u,v),(1,0)) \leq R_\mathcal{M} \quad \forall (u,v) \in \mathcal{M}. \tag{5.22}$$

Thus, we can define our candidate set $\mathcal{B}_0$ as

$$\mathcal{B}_0 := \{(u,v) \in \mathcal{X} : d_\mathcal{X}((u,v),(1,0)) \leq R_{\mathcal{B}_0}\}, \tag{5.23}$$

where the radius $R_{\mathcal{B}_0}$ is introduced by

$$R_{\mathcal{B}_0} := \frac{16c_9}{\varepsilon c_7} + 2|\Omega| + 1 + \frac{4}{\varepsilon}. \tag{5.24}$$

The absorbing character of $\mathcal{B}_0$ is provided by the following couple of lemmas.

Lemma 5.1. *The set*

$$\tilde{\mathcal{B}}_0 := \left\{(u,v) \in \mathcal{X} : \Phi_\varepsilon(u,v) \leq \frac{2c_9}{c_7}\right\}$$

is contained into $\mathcal{B}_0$.

Proof. Let $(u,v) \in \tilde{\mathcal{B}}_0$. Then, since $\varepsilon \leq 1$ (cf. (5.15)) and by (5.17),

$$\begin{aligned}\frac{2c_9}{c_7} &\geq \Phi_\varepsilon(u,v) = \int_\Omega -\log u + \frac{\varepsilon}{2}|u|_H^2 + \frac{1}{2}\|v\|^2 + \int_\Omega \hat\beta(v) + c_\varepsilon \\ &\geq \int_\Omega \log^- u + \frac{\varepsilon}{4}|u|_H^2 + \frac{1}{2}\|v\|^2 + \int_\Omega \hat\beta(v) \\ &\geq \int_\Omega \log^- u + \frac{\varepsilon}{8}|u-1|_H^2 - \frac{\varepsilon}{4}|\Omega| + \frac{1}{2}\|v\|^2 + \int_\Omega \hat\beta(v) \\ &\geq \int_\Omega \log^- u + \frac{\varepsilon}{8}|u-1|_H - \frac{\varepsilon}{8} - \frac{\varepsilon}{4}|\Omega| + \frac{1}{2}\|v\| - \frac{1}{2} + \int_\Omega \hat\beta(v) \\ &\geq \frac{\varepsilon}{8}d_\mathcal{X}((u,v),(1,0)) - \frac{\varepsilon}{4}|\Omega| - \frac{\varepsilon}{8} - \frac{1}{2}. \qquad \square\end{aligned} \tag{5.25}$$

Lemma 5.2. *Let $\mathcal{M}$ be metric bounded in $\mathcal{X}$, i.e. let it satisfy (5.22) for some $R_{\mathcal{M}} > 0$. Then, there exists $\tilde{R}_{\mathcal{M}} > 0$ depending only on $R_{\mathcal{M}}$, ε, c_ε, and Ω, and such that*

$$\Phi_\varepsilon(u, v) \le \tilde{R}_{\mathcal{M}} \quad \forall (u, v) \in \mathcal{M}. \tag{5.26}$$

Proof. By (5.18) and (5.15), for all $(u, v) \in \mathcal{M}$ we have

$$\Phi_\varepsilon(u, v) = -\int_\Omega \log u + \frac{\varepsilon}{2}|u|_H^2 + \frac{1}{2}\|v\|^2 + \int_\Omega \hat{\beta}(v) + c_\varepsilon \le \int_\Omega |\log^- u - \log^- 1| + \varepsilon|u-1|_H^2 + \varepsilon|\Omega| + \frac{1}{2}\|v\|^2 + \int_\Omega \hat{\beta}(v) + c_\varepsilon \le d_{\mathcal{X}}((u, v), (1, 0)) + \frac{3}{2} d_{\mathcal{X}}((u, v), (1, 0))^2 + \varepsilon|\Omega| + c_\varepsilon. \tag{5.27}$$

Thus, the assert follows with the choice of:

$$\tilde{R}_{\mathcal{M}} := R_{\mathcal{M}} + \tfrac{3}{2} R_{\mathcal{M}}^2 + \varepsilon|\Omega| + c_\varepsilon. \tag{5.28}$$

□

It is now clear that, as $\mathcal{M}$ satisfies (5.22), it follows from Lemma 5.2 and (5.21) that:

$$\forall t \ge T_{\mathcal{M}} := \frac{1}{c_7} \log \frac{\tilde{R}_{\mathcal{M}} c_7}{c_9}, \qquad \Phi_\varepsilon(S(t)(u, v)) \le \frac{2c_9}{c_7} \quad \forall (u, v) \in \mathcal{M}. \tag{5.29}$$

The thesis of Theorem 3.13 is now a straightforward consequence of (5.23) and Lemma 5.1. □

Corollary 5.3. *There exists a (monotone increasing) function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that, if $\mathcal{M}$ is $d_{\mathcal{X}}$-bounded in $\mathcal{X}$ (i.e. it fulfills (5.22)), then*

$$\|\chi\|_{\mathcal{T}^4(L^4(\Omega))}^4 + \left\|\frac{1}{\vartheta}\right\|_{\mathcal{T}^2(V)}^2 + \|\vartheta\|_{\mathcal{T}^2(V)}^2 + \|\chi_t\|_{\mathcal{T}^2(H)}^2 \le \varphi(R_{\mathcal{M}}) \tag{5.30}$$

for all $(\vartheta, \chi) = S(t)(\vartheta_0, \chi_0)$ and for all $(\vartheta_0, \chi_0) \in \mathcal{M}$.

Proof. Take any $t > 0$ and integrate (5.20) in time between t and $t + 1$. Then, to control $\Phi(\vartheta(t), \chi(t))$, use Lemma 5.2 and relation (5.21). Finally, pass to the sup as t varies in $\mathbb{R}^+$ and recall the definition (2.9) of the norms of $\mathcal{T}$ type. □

Proof of Theorem 3.14. The outline of the procedure is similar to the nonconserved case. Hence, let us just sketch the differences. In the sequel, $c_{\boldsymbol{\eta}}$ will denote any positive constant additionally depending on $\boldsymbol{\eta}$.

In Estimate 1, instead of testing (3.12) by χ_t, we have to test (3.19) by $\mathcal{N}\chi_t$, (3.20) by χ_t, and take the difference. Then, we can proceed as before, by substituting any H-norm of χ_t with the V' one. However, we have to modify (5.4) as (cf. (4.50)) getting

$$-\langle \chi_t, \gamma(\chi) \rangle = -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \hat{\gamma}(\chi(t)). \tag{5.31}$$

Then, (5.5) takes now the form

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \left[-\log\vartheta + \frac{|\nabla\chi|^2}{2} + \hat{\beta}(\chi) + \hat{\gamma}(\chi) \right] + c_3 \left[\left\|\frac{1}{\vartheta}\right\|^2 + \|\chi_t\|_*^2 \right] \le c, \tag{5.32}$$

where of course—and the same for the sequel—the various constants need not assume the same values as before.

In the passage corresponding to Estimate 2, we set $\bar{\chi} := \chi - \chi_\Omega$ and test (3.19) by $\mathcal{N}\bar{\chi}$, (3.20) by $\bar{\chi}$, and take the difference. Integrating in time, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\|\bar{\chi}\|_*^2}{2} + |\nabla\chi|_H^2 + \int_\Omega \xi\chi \le \int_\Omega \xi\chi_\Omega - \int_\Omega \gamma(\chi)\bar{\chi} - \int_\Omega \frac{b\bar{\chi}}{\vartheta} \tag{5.33}$$

and we have to give a bound for the right-hand side. First of all, it is easy to see that, for some (new) $\sigma_8 > 0$,

$$-\int_\Omega \gamma(\chi)\bar{\chi} - \int_\Omega \frac{b\bar{\chi}}{\vartheta} \le c_{\boldsymbol{\eta},\sigma_8}(1 + |\bar{\chi}|_H^2) + \sigma_8\left|\frac{1}{\vartheta}\right|_H^2.$$

To deal with the first term on the right-hand side of (5.33), let us first assume that $D(\beta) = \mathbb{R}$. Then, it is easy to see that

$$\int_\Omega \xi\chi_\Omega \le c_{\boldsymbol{\eta}}\int_\Omega |\xi| \le c_{\boldsymbol{\eta}} + \frac{1}{2}\int_\Omega \xi\chi. \tag{5.34}$$

On the other hand, if $D(\beta)$ is bounded (we do not deal, just in order to avoid technicalities, with the case when $D(\beta)$ is a half line), then inequality (5.34) does no longer hold and we need two further estimates, provided by a suitable modification of an argument devised by Kenmochi et al. in [15] and also described, e.g., in [6, Section 4]. Firstly, we have to test (3.19) by $\mathcal{N}(\xi - \xi_\Omega)$, (3.20) by $\xi - \xi_\Omega$, and take the difference. Using the monotonicity of β, (A3), and (2.4), it is not difficult to infer

$$|\xi - \xi_\Omega|_H^2 \le c_{\boldsymbol{\eta}}\left(1 + |\bar{\chi}|^2 + \left|\frac{1}{\vartheta}\right|^2 + \|\chi_t\|_*^2\right). \tag{5.35}$$

Then, let us choose $m_1, m_2 \in \operatorname{int} D(\beta)$ (depending on $\boldsymbol{\eta}$), with $m_1 < \eta_1$, $m_2 > \eta_2$ (cf. (3.31)), and set $\delta = \delta(\boldsymbol{\eta}) := \min\{\eta_1 - m_1, m_2 - \eta_2\}$. Since $\chi_\Omega \in [\eta_1, \eta_2]$, we can proceed as in [6, third estimate] and get

$$\delta\int_\Omega |\xi| \le c_{\boldsymbol{\eta}} + \int_\Omega (\xi - \xi_\Omega)(\chi - \chi_\Omega). \tag{5.36}$$

Since $D(\beta)$ is bounded and χ satisfies (3.21), we deduce that

$$\int_\Omega |\xi| \le c_{\boldsymbol{\eta}}\left(1 + \int_\Omega |\xi - \xi_\Omega|\right). \tag{5.37}$$

Consequently, thanks to (5.35), it follows that:

$$\begin{aligned} |\xi|_H^2 &\le 2(|\xi - \xi_\Omega|_H^2 + |\xi_\Omega|_H^2) \le c\left[|\xi - \xi_\Omega|_H^2 + c_{\boldsymbol{\eta}}\left(1 + \int_\Omega |\xi - \xi_\Omega|\right)^2\right] \\ &\le c_{\boldsymbol{\eta}}[1 + |\xi - \xi_\Omega|_H^2] \le c_{\boldsymbol{\eta}}\left[1 + |\bar{\chi}|^2 + \left|\frac{1}{\vartheta}\right|^2 + \|\chi_t\|_*^2\right]. \end{aligned} \tag{5.38}$$

Finally, Estimate 3 is repeated exactly as before and we get again (5.12). Thus, we can put all the computations above together, starting from the case when $D(\beta)$ is not bounded. Then, from (5.32)–(5.34) and (5.12), using again (5.13), and taking ε as before, it follows that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2 + \frac{1}{2}\|\bar{\chi}\|^2 + \int_\Omega \hat{\beta}(\chi) + \int_\Omega \hat{\gamma}(\chi)\right] + c_3\left[\left\|\frac{1}{\vartheta}\right\|^2 + \|\chi_t\|_*^2\right] + \frac{\kappa_5}{2}\left[\int_\Omega \hat{\beta}(\chi) + \|\chi\|_{L^4(\Omega)}^4 + |\chi|_H^2\right] + |\nabla\chi|_H^2 + \varepsilon c_4\|\vartheta\|^2 \le c_{\boldsymbol{\eta}} + c_{\boldsymbol{\eta},\sigma_8}|\bar{\chi}|_H^2 + (\sigma_8 + \varepsilon\sigma_9)\left|\frac{1}{\vartheta}\right|_H^2 + \varepsilon c_{\sigma_{10}}\|\chi_t\|_*^2, \tag{5.39}$$

where we used the (equivalent) norm on V given by $\|\cdot\|^2 = |\nabla\cdot|^2 + \|\cdot\|_*^2$. Then, taking again $\varepsilon c_{\sigma_{10}} \le c_3/2$ and σ_8, σ_9 sufficiently small, and noting that

$$c_{\boldsymbol{\eta},\sigma_8}|\bar{\chi}|_H^2 \le c_{\boldsymbol{\eta},\sigma_8} + \tfrac{1}{4}\kappa_6\|\chi\|_{L^4(\Omega)}^4, \tag{5.40}$$

we see that all the terms on the right-hand side are controlled. Moreover, we observe that, by (A3) and (A11), there exists $c > 0$ such that

$$\hat{\beta}(r) + \hat{\gamma}(r) \ge \tfrac{1}{2}\hat{\beta}(\chi) - c \quad \forall r \in D(\hat{\beta}). \tag{5.41}$$

This yields that, for a (new) choice of $c_\varepsilon > 0$, the (new) functional

$$\Phi_\varepsilon(\vartheta,\chi) := -\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2 + \frac{1}{2}\|\bar{\chi}\|^2 + \int_\Omega \hat{\beta}(\chi) + \int_\Omega \hat{\gamma}(\chi) + c_\varepsilon \tag{5.42}$$

is non-negative. More precisely, we choose c_ε so that

$$\Phi_\varepsilon(\vartheta,\chi) \ge \int_\Omega \log^-\vartheta + \frac{\varepsilon}{4}|\vartheta|_H^2 + \frac{1}{2}\|\bar{\chi}\|^2 + \frac{1}{2}\int_\Omega \hat{\beta}(\chi). \tag{5.43}$$

Now, the proof of the existence of a $d_{\mathcal{X}}$-bounded absorbing set $\mathcal{B}_{0,\boldsymbol{\eta}}$ can be completed as in the nonconserved case (of course, the values of some constants will be different and they will depend, in addition, on $\boldsymbol{\eta}$). This concludes the case of unbounded $D(\beta)$.

If $D(\beta)$ is bounded, instead, we have to come back to (5.33) and note that, by monotonicity of β,

$$\xi\bar{\chi} = (\xi - \beta^0(\chi_\Omega))\bar{\chi} + \beta^0(\chi_\Omega)\bar{\chi} \ge \beta^0(\chi_\Omega)\bar{\chi} \tag{5.44}$$

and, of course,

$$\int_\Omega |\beta^0(\chi_\Omega)\bar{\chi}| \le c_{\boldsymbol{\eta}}\|\bar{\chi}\|_{L^1(\Omega)}.$$

Thus the terms with ξ in (5.33) are controlled but do no longer provide a contribution in the estimate. Thus, as we write the relation corresponding to (5.39), we now also have to add the contribution of (5.38) times a (small) $\mu > 0$. We obtain (for a new $c_3 > 0$)

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[-\int_\Omega \log\vartheta + \frac{\varepsilon}{2}|\vartheta|_H^2 + \frac{1}{2}\|\bar{\chi}\|^2 + \int_\Omega \hat{\beta}(\chi) + \int_\Omega \hat{\gamma}(\chi)\right] + c_3\left[\left\|\frac{1}{\vartheta}\right\|^2 + \|\chi_t\|_*^2\right] + \mu|\xi|_H^2 + |\nabla\chi|_H^2 + \varepsilon c_4\|\vartheta\|^2 \le c_{\boldsymbol{\eta}} + (c_{\boldsymbol{\eta},\sigma_8} + c_{\boldsymbol{\eta}}^1)|\bar{\chi}|_H^2 + c_{\boldsymbol{\eta}}\|\bar{\chi}\|_{L^1(\Omega)} + (\sigma_8 + \varepsilon\sigma_9 + \mu c_{\boldsymbol{\eta}}^1)\left|\frac{1}{\vartheta}\right|_H^2 + (\varepsilon c_{\sigma_{10}} + \mu c_{\boldsymbol{\eta}}^1)\|\chi_t\|_*^2, \tag{5.45}$$

where $c^1_{\boldsymbol{\eta}}$ is precisely the constant $c_{\boldsymbol{\eta}}$ appearing in (5.38). Now, the term $\hat{\beta}(\chi)$ appears on the left-hand side just under time derivative. However, working similarly as in the last part of the proof of Proposition 3.15 and using (A11), it is easy to see that

$$\int_\Omega \xi^2 \ge c\left(\int_\Omega \hat{\beta}(\chi) + \|\chi\|^4_{L^4(\Omega)} + |\chi|^2_H\right),$$

so that we can conclude the estimate by choosing ε and μ sufficiently small. The rest of the procedure is now as in the previous case. This concludes the proof of Lemma 3.14. □

Again, we also have an additional property, whose proof is analogous as before.

Corollary 5.4. *There exists a* (*monotone increasing*) *function* $\varphi_{\boldsymbol{\eta}} : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, if* $\mathcal{M}_{\boldsymbol{\eta}}$ *is* $d_{\mathcal{X}}$*-bounded in* $\mathcal{X}_{\boldsymbol{\eta}}$ (*i.e. it fulfills* (5.22)), *then*

$$\|\chi\|^4_{\mathcal{T}^4(L^4(\Omega))} + \left\|\frac{1}{\vartheta}\right\|^2_{\mathcal{T}^2(V)} + \|\vartheta\|^2_{\mathcal{T}^2(V)} + \|\chi_t\|^2_{\mathcal{T}^2(V')} \le \varphi_{\boldsymbol{\eta}}(R_{\mathcal{M}_{\boldsymbol{\eta}}}) \tag{5.46}$$

for all $(\vartheta, \chi) = S(t)(\vartheta_0, \chi_0)$ *and for all* $(\vartheta_0, \chi_0) \in \mathcal{M}_{\boldsymbol{\eta}}$.

6. Existence of the attractor

Proof of Theorem 3.16. Let us now perform some further estimates on the solution of system (3.11)–(3.14). We notice that some of the passages below might be formal in the present framework; indeed, we shall work in a regularity setting which is stronger with respect to the properties (3.7)–(3.10). However, the estimates might be made rigorous by effecting a regularization and then passing to the limit; one possibility could be to regularize (3.11)–(3.13) and replace it, e.g., with

$$\iota\partial_t\alpha(\vartheta) + \partial_t(\vartheta + b\chi) + J(\alpha(\vartheta)) = f \quad \text{in } V' \text{ a.e. in } (0, T), \tag{6.1}$$

$$\partial_t\chi + B\chi + \beta_\iota(\chi) + \gamma(\chi) = -\frac{b}{\vartheta} \quad \text{a.e. in } Q_T, \tag{6.2}$$

where β_ι is the Yosida approximation of β and the regularization parameter $\iota > 0$ is intended to go to the limit. It is easy to show that the solutions of such a regularized system gain (a priori just as $t > 0$—which is enough, indeed—unless the initial data are regularized too) all the regularity which is required to make the estimates rigorous. However, since this procedure is rather standard, we omit the details and go on in a formal way.

Our task is showing that $S(t)$ admits an absorbing set which is bounded with respect to the metric of $\mathcal{V}$ and, more precisely, fulfills the conditions in the statement of Proposition 3.15. With this aim, we first redefine the absorbing set $\mathcal{B}_0$ provided by Theorem 3.13 by setting

$$\mathcal{C}_0 := \bigcup_{t\ge 0} S(t)\mathcal{B}_0. \tag{6.3}$$

It is a standard matter to show that $\mathcal{C}_0$ is still an absorbing set for $S(t)$, which is bounded in $d_{\mathcal{X}}$ by an absolute constant $R_{\mathcal{C}_0}$ (cf. (5.24)). Furthermore, by construction $\mathcal{C}_0$ is positively invariant, i.e. $S(t)\mathcal{C}_0 \subset \mathcal{C}_0$ for all $t \ge 0$.

First step. Let us assign an initial datum $(\vartheta_0, \chi_0) \in \mathcal{C}_0$ and let the system evolve from this datum. Let us test (3.11) by the time derivative of $\alpha(\vartheta)$. Then, let us differentiate (3.12) in time and test the result by $c_0\chi_t$. Taking the sum, noting that two terms cancel, and using the monotonicity of β together with properties (3.2) and (3.3) we infer

$$\frac{c_\infty}{2}|\vartheta_t|_H^2 + \frac{c_0'}{2}|\partial_t(\log\vartheta)|_H^2 + \frac{\mathrm{d}}{\mathrm{d}t}\|\alpha(\vartheta)\|_J^2 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{c_0}{2}|\chi_t|_H^2\right) + c_0|\nabla\chi_t|_H^2 \le \frac{\mathrm{d}}{\mathrm{d}t}\langle f, \alpha(\vartheta)\rangle - c_0\int_\Omega \gamma'(\chi)\chi_t^2 - b\int_\Omega \ell'(\vartheta)\vartheta_t\chi_t \tag{6.4}$$

and again we have to control the right-hand side. First, we notice that the latter two terms, owing to (A1) and (A3), are bounded as follows:

$$-c_0\int_\Omega \gamma'(\chi)\chi_t^2 - b\int_\Omega \ell'(\vartheta)\vartheta_t\chi_t \le \frac{c_\infty}{4}|\vartheta_t|_H^2 + c|\chi_t|_H^2. \tag{6.5}$$

Next, we define the functional

$$\Psi(\vartheta,\chi) := \|\alpha(\vartheta)\|_J^2 + \frac{c_0}{2}|\chi_t|_H^2 - \langle f, \alpha(\vartheta)\rangle + C(f), \tag{6.6}$$

where we have set $C(f) := \|f\|_{*,J}^2/2$ (with obvious notation), so that

$$\Psi(\vartheta,\chi) \ge \tfrac{1}{2}c_0|\chi_t|_H^2 + \tfrac{1}{2}\|\alpha(\vartheta)\|_J^2 \ge 0. \tag{6.7}$$

Thus, relation (6.4) can be clearly rewritten as

$$\frac{\mathrm{d}}{\mathrm{d}t}\Psi(\vartheta,\chi) + c_{10}[|\vartheta_t|_H^2 + |\partial_t(\log\vartheta)|_H^2 + |\nabla\chi_t|_H^2] \le c_{11}(1 + |\chi_t|_H^2) \tag{6.8}$$

for all $t \in (0, +\infty)$ and for some $c_{10}, c_{11} > 0$ also depending on f. Then, the uniform Gronwall's lemma yields

$$\Psi(\vartheta(t+1), \chi(t+1)) \le c_{11}(1 + \|\chi_t\|_{\mathcal{T}^2(H)}) + |\Psi|_{\mathcal{T}^1(\mathbb{R})} \quad \forall t \ge 0, \tag{6.9}$$

which is a bounded quantity since

$$|\Psi|_{\mathcal{T}^1(\mathbb{R})} \le c\|\alpha(\vartheta)\|_{\mathcal{T}^2(V)} + \tfrac{1}{2}c_0\|\chi_t\|_{\mathcal{T}^2(H)} + \|f\|_{*,J} \le c\varphi(R_{\mathcal{C}_0}) + \|f\|_{*,J}^2 \tag{6.10}$$

by Corollary 5.3, (A1), (A5), and (3.6).

Finally, by (A1), (3.2) and (6.7), this clearly entails

$$\|\vartheta(t)\|^2 + \left\|\frac{1}{\vartheta(t)}\right\|^2 + |\chi_t(t)|_H^2 \le c(R_{\mathcal{C}_0}) \quad \forall t \ge 1. \tag{6.11}$$

Second step. The forthcoming procedure can be applied for any β, but is needed only as $D(\hat\beta)$ is not closed, in order to show (3.42) (with $\chi(t)$ in place of v_n). Let us test (3.12) by $B\chi + \xi$. By monotonicity of β, it is then clear that

$$|\xi(t)|_H^2 + |B\chi(t)|_H^2 \le c\left(1 + \|\chi(t)\|^2 + |\chi_t(t)|_H^2 + \left|\frac{1}{\vartheta(t)}\right|_H^2\right). \tag{6.12}$$

Thus, since $(\vartheta_0, \chi_0) \in \mathcal{C}_0$, by (6.11), it follows that:

$$|\xi(t)|_H^2 + \|\chi(t)\|_W^2 \le c(R_{\mathcal{C}_0}) \quad \text{for a.e. } t \ge 1. \tag{6.13}$$

We actually remark that the bound above holds for *all* $t \ge 1$ as far as the W-norm of χ is concerned. Indeed, (6.13), together with (3.9), yields that $\chi \in C_w([0, +\infty); W)$. This, a priori, is not true for ξ which is not known to be time-continuous with values in any space; indeed, the procedure just yields $\xi \in L^\infty(0,+\infty;H)$. However, we claim that this entails that there exists $c(R_{\mathcal{C}_0}) > 0$ such that

$$|\beta^0(\chi(t))|_H^2 \le c(R_{\mathcal{C}_0}) \quad \textit{for all } t \ge 1. \tag{6.14}$$

Actually, let us prove the above for a generic $\bar{t} \in [1,+\infty)$. Approximate $\bar{t}$ by a sequence $\{t_n\} \subset [1,+\infty)$, $t_n \to \bar{t}$, of times such that the inequality in (6.14) holds for $t = t_n$. Let us set

$$\chi_n, \xi_n : \Omega \to \mathbb{R}, \qquad \chi_n(x) := \chi(x,t_n), \qquad \xi_n(x) := \xi(x,t_n).$$

Then, as

$$\chi \in H^1(0,+\infty;V') \cap L^\infty(1,+\infty;W) \subset C(\bar{\Omega} \times [1,+\infty)),$$

we have in particular that $\chi_n \to \chi(\bar{t})$ uniformly in Ω. Moreover, we can assume that $\xi_n \in \beta(\chi_n)$ a.e. in Ω.

Thus, since β^0 is a monotone function, it is easy to see that

$$|\beta^0(\chi(x,\bar{t}))| \le \liminf_{n\to\infty} |\xi_n(x)| \quad \text{a.e. in } \Omega, \tag{6.15}$$

whence (6.14), written for $t = \bar{t}$, follows by squaring, integrating over Ω, and applying Fatou's lemma.

This actually concludes the proof of Theorem 3.16; indeed, if $D(\beta)$ is closed, then, by (6.11) and the first case of Proposition 3.15, we easily see that condition (2.18) holds. Otherwise, the same is true by (6.14) and the second case of Proposition 3.15. □

Proof of Theorem 3.17. The procedure is very similar as before; thus, we just outline the differences. The computation in Step 1 is modified as in the previous cases and this leads to replace

$$\tfrac{1}{2}c_0|\chi_t|_H^2 \quad \text{with } \tfrac{1}{2}c_0\|\chi_t\|_*^2$$

in the left-hand side of (6.4) as well as in the definition of Ψ (cf. (6.6) and (6.7)). Moreover, we have to notice that the latter term in (6.5), by the compact immersion $V \subset H$ (cf. (4.10)), has to be controlled this way

$$c|\chi_t|_H^2 \le \tfrac{1}{2}c_0|\nabla\chi_t|_H^2 + c\|\chi_t\|_*^2.$$

Then, Step 1 is completed as before if we substitute the H-norms of χ_t with V'-norms in (6.8)–(6.10), and we use Corollary 5.4 instead of Corollary 5.3. Thus, in place of (6.11), we now get

$$\|\vartheta(t)\|^2 + \left\|\frac{1}{\vartheta(t)}\right\|^2 + \|\chi_t(t)\|_*^2 \le c(R_{\mathcal{C}_0}) \quad \forall t \ge 1, \tag{6.16}$$

where the absorbing set $\mathcal{C}_0$ is now intended to depend also on $\boldsymbol{\eta}$, of course.

We finally have to modify Step 2. For simplicity, we just consider the case when $D(\beta)$ is open and bounded. Then, as we test (3.20) by $B\chi$, we can proceed similarly as before and get the inequality for χ in (6.13). To bound the norm of $\xi(t)$, we have to repeat precisely the argument of [15] already used in the previous section; thus we get again (5.38) at the time $t \ge 1$. On account of (6.16), this gives the second bound in (6.13), a priori just for *a.e.* $t \ge 1$. However, proceeding as before we can show again (6.14) for *all* $t \ge 1$ and conclude the proof. □

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References

[1] C. Baiocchi, Sulle equazioni differenziali astratte lineari del primo e del secondo ordine negli spazi di Hilbert, Ann. Mat. Pura Appl. IV 76 (1967) 233–304.

[2] P.W. Bates, S. Zheng, Inertial manifolds and inertial sets for the phase-field equations, J. Dyn. Diff. Eqs. 4 (1992) 375–398.

[3] D. Brochet, D. Hilhorst, Universal attractor and inertial sets for the phase field model, Appl. Math. Lett. 4 (1991) 59–62.

[4] D. Brochet, D. Hilhorst, A. Novick-Cohen, Maximal attractor and inertial sets for a conserved phase field model, Adv. Diff. Eqs. 1 (1996) 547–578.

[5] H. Brezis, Opérateurs Maximaux Monotones et Sémi-groupes de Contractions dans les Espaces de Hilbert, North-Holland Mathematical Studies, vol. 5, North-Holland, Amsterdam, 1973.

[6] P. Colli, G. Gilardi, M. Grasselli, G. Schimperna, The conserved phase-field system with memory, Adv. Math. Sci. Appl. 11 (2001) 265–291.

[7] P. Colli, Ph. Laurençot, Weak solutions to the Penrose–Fife phase field model for a class of admissible heat flux laws, Physica D 111 (1998) 311–334.

[8] G. Caginalp, An analysis of a phase field model of a free boundary, Arch. Rat. Mech. Anal. 92 (1986) 205–245.

[9] P. Colli, Ph. Laurençot, J. Sprekels, Global solution to the Penrose–Fife phase field model with special heat flux law, in: Variation of Domains and Free-boundary Problems in Solid Mechanics (Paris, 1997), Solid Mechanics and Application, vol. 66, Kluwer Academic Publishers, Dordrecht, 1999, pp. 181–188.

[10] C. Giorgi, M. Grasselli, V. Pata, Uniform attractors for a phase-field model with memory and quadratic nonlinearity, Indiana Univ. Math. J. 48 (1999) 1395–1445.

[11] C. Giorgi, M. Grasselli, V. Pata, Well-posedness and longtime behavior of the phase-field model with memory in a history space setting, Quart. Appl. Math. 59 (2001) 701–736.

[12] M. Grasselli, V. Pata, Existence of a universal attractor for a parabolic–hyperbolic phase-field system, Adv. Math. Sci. Appl. 13 (2003) 443–459.

[13] A. Ito, N. Kenmochi, Inertial set for a phase transition model of Penrose–Fife type, Adv. Math. Sci. Appl. 10 (2000) 353–374; correction: Adv. Math. Sci. Appl. 11 (2001) 481.

[14] N. Kenmochi, M. Kubo, Weak solutions of nonlinear systems for non-isothermal phase transitions, Adv. Math. Sci. Appl. 9 (1999) 499–521.

[15] N. Kenmochi, M. Niezgódka, I. Pawlow, Subdifferential operator approach to the Cahn–Hilliard equation with constraint, J. Diff. Eqs. 117 (1995) 320–356.

[16] Ph. Laurençot, Solutions to a Penrose–Fife model of phase-field type, J. Math. Anal. Appl. 185 (1994) 262–274.

[17] Ph. Laurençot, Long-time behaviour for a model of phase-field type, Proc. R. Soc. Edinburgh, Sect. A 126 (1996) 167–185.

[18] J.L. Lions, E. Magenes, Non-homogeneous Boundary Value Problems and Applications, vol. I, Springer-Verlag, Berlin, 1972.

[19] J.L. Lions, J. Peetre, Sur une classe d'espaces d'interpolation, Inst. Hautes Études Sci. Publ. Math. 19 (1964) 5–68.

[20] O. Penrose, P.C. Fife, Thermodynamically consistent models of phase-field type for the kinetics of phase transitions, Physica D 43 (1990) 44–62.

[21] O. Penrose, P.C. Fife, On the relation between the standard phase-field model and a "thermodynamically consistent" phase-field model, Physica D 69 (1993) 107–113.

[22] E. Rocca, The conserved Penrose–Fife phase field model with special heat flux laws and memory effects, J. Integral Eqs. Appl. 14 (2002) 425–466.

[23] W. Shen, S. Zheng, Maximal attractor for the coupled Cahn–Hilliard equations, Nonlinear Anal. Ser. A: Theory Meth. 49 (2002) 21–34.

[24] W. Shen, S. Zheng, Maximal attractors for the phase-field equations of Penrose–Fife type, Appl. Math. Lett. 15 (2002) 1019–1023.

[25] R. Temam, Infinite-dimensional Dynamical Systems in Mechanics and Physics, Springer-Verlag, New York, 1988.