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A REMARK ON THE DAMPED WAVE EQUATION

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Abstract. In this short note we present a direct method to establish the optimal regularity of the attractor for the semilinear damped wave equation with a nonlinearity of critical growth.

We consider the semilinear (weakly) damped wave equation on a bounded domain $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega$

$$\begin{cases} \partial_{tt} u + \partial_t u - \Delta u + \varphi(u) = f, \\ u(0) = u_0, \quad \partial_t u(0) = u_1, \\ u_{|\partial\Omega} = 0. \end{cases} \tag{1}$$

Here, $f \in L^2(\Omega)$ is independent of time and $\varphi \in C^2(\mathbb{R})$, with $\varphi(0) = 0$, satisfies the growth and the dissipation conditions

$$|\varphi''(u)| \leq c\big(1 + |u|\big), \tag{2}$$

$$\liminf_{|u|\to\infty} \frac{\varphi(u)}{u} > -\lambda_1, \tag{3}$$

$$\varphi'(u) \geq -\ell, \tag{4}$$

where $c, \ell \geq 0$ and $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ on $L^2(\Omega)$ with Dirichlet boundary conditions.

The asymptotic behavior of solutions to equation (1) has been the object of extensive studies (see, e.g. [1]–[5], [7], [9]–[14] and [17]–[21]). Denoting

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \qquad \text{and} \qquad \mathcal{V} = \big[H^2(\Omega) \cap H_0^1(\Omega)\big] \times H_0^1(\Omega),$$

problem (1) is known to generate a C_0-semigroup $S(t)$ on the phase space $\mathcal{H}$, and the following result holds.

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Theorem 1. *The semigroup $S(t)$ on $\mathcal{H}$ possesses a compact global attractor $\mathcal{A}$. Besides, $\mathcal{A}$ is a bounded subset of $\mathcal{V}$.*

Theorem 1 was first proved by Babin and Vishik [3]. We mention that the result is still valid if one removes condition (4), which is however very reasonable. In that case, the existence of the attractor was shown in [1], whereas its $\mathcal{V}$-regularity first appeared in the papers [9, 10, 20]. In particular, the argument presented in [20] allows also to treat the nonautonomous case.

In all the preceding works, the $\mathcal{V}$-regularity of the attractor is achieved by means of rather complicated and long procedures, requiring multiplications by fractional operators and bootstrap arguments. The aim of this note is to show how to obtain this result in a very direct way, exploiting only quite simple energy estimates. This approach can be applied to treat more complicated boundary conditions such as, for example, dynamic boundary conditions (where the use of fractional operators may be problematic), as well as to deal with stabilization problems (see the end of the paper for more details). The key step of our proof is a suitable decomposition of the solution u to (1), which has been already successfully employed in the recent works [6, 8, 21].

A new proof of Theorem 1. We denote by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ the inner product and the norm on $L^2(\Omega)$. In what follows, we will often make use without explicit mention of the Sobolev embeddings and of the Young, the Hölder and the Poincaré inequalities. As usual, we will perform formal estimates that can be justified in a proper Galerkin approximation scheme. Finally, for any function $z(t)$, we will write for short $\xi_z(t) = (z(t), \partial_t z(t))$.

We begin recalling a basic estimate.

Lemma 2. *For every $t \geq 0$,*

$$\|\xi_u(t)\|^2_{\mathcal{H}} + \int_t^\infty \|\partial_t u(\tau)\|^2 d\tau \leq Q(\|\xi_u(0)\|_{\mathcal{H}})e^{-\varepsilon t} + Q(\|f\|),$$

for some $\varepsilon > 0$ and some positive increasing function Q.

The proof may be found, for instance, in [3], and it is carried out by multiplying the equation by $\partial_t u + \varepsilon u$, for some $\varepsilon > 0$ suitably small. In particular, this result yields the existence of a bounded absorbing set $\mathbb{B}_0 \subset \mathcal{H}$ for the semigroup $S(t)$.

In view of (2), for every $z, \zeta \in H^1_0(\Omega)$ we have that

$$|\langle\varphi'(\zeta)z, z\rangle| \leq c_1\big(1 + \|\nabla\zeta\|\big)^2\|z\|\|\nabla z\| \leq \frac{1}{2}\|\nabla z\|^2 + c_2(1 + \|\nabla\zeta\|\big)^4\|z\|^2,$$

for some $c_1, c_2 \geq 0$. Consequently, by Lemma 2, we can choose $\theta \geq \ell$ large enough such that the inequality

$$\frac{1}{2}\|\nabla z\|^2 + (\theta - 2\ell)\|z\|^2 - \langle\varphi'(u(t))z, z\rangle \geq 0 \tag{5}$$

holds for every $z \in H^1_0(\Omega)$, every $t \geq 0$ and every solution $u(t)$ with $\xi_u(0) \in \mathbb{B}_0$. Then, we set

$$\psi(r) = \varphi(r) + \theta r.$$

Clearly, condition (2) still holds with ψ in place of φ. Besides, on account of (4),

$$\psi'(r) \geq 0. \tag{6}$$

We now consider initial data $\xi_u(0) \in \mathbb{B}_0$, and we decompose the solution to (1) into the sum $u = v + w$, where v and w solve the equations

$$\begin{cases} \partial_{tt} v + \partial_t v - \Delta v + \psi(u) - \psi(w) = 0, \\ \xi_v(0) = \xi_u(0), \\ v_{|\partial\Omega} = 0, \end{cases} \tag{7}$$

and

$$\begin{cases} \partial_{tt} w + \partial_t w - \Delta w + \psi(w) = \theta u + f, \\ \xi_w(0) = (0,0), \\ w_{|\partial\Omega} = 0. \end{cases} \tag{8}$$

In the following, $c \geq 0$ will stand for a generic constant depending (possibly) only on the size of $\mathbb{B}_0$ (but neither on the particular $\xi_u(0) \in \mathbb{B}_0$ nor on the time t).

Lemma 3. *For every $t \geq 0$, we have that $\|\xi_w(t)\|_{\mathcal{H}} \leq c$.*

Proof. The same argument of the proof of Lemma 2 applies to (8), since from Lemma 2 we know that the right-hand side belongs to $L^\infty(0,\infty; L^2(\Omega))$. Observe also that here the initial data are null. $\square$

Lemma 4. *For every $t \geq s \geq 0$ and every $\omega > 0$,*

$$\int_s^t \|\partial_t w(\tau)\|^2 d\tau \leq \omega(t-s) + \frac{c}{\omega}.$$

Proof. Define the functional

$$\Lambda = \|\nabla w\|^2 + \|\partial_t w\|^2 + 2\langle \Psi(w), 1\rangle - 2\theta\langle u, w\rangle - 2\langle f, w\rangle,$$

where $\Psi(w) = \int_0^w \psi(y)dy$. Note that $\Lambda \leq c$, due to (2) and Lemma 3. Thus, multiplying (8) by $\partial_t w$, and applying once more Lemma 3, we obtain

$$\frac{d}{dt}\Lambda + 2\|\partial_t w\|^2 = -2\theta\langle \partial_t u, w\rangle \leq 2\omega + \frac{c}{\omega}\|\partial_t u\|^2,$$

and the claim is proved integrating in time on (s,t), exploiting the integral estimate furnished by Lemma 2. $\square$

Collecting the above results, for all initial data $\xi_u(0) \in \mathbb{B}_0$ we have the bounds

$$\|\xi_u(t)\|_{\mathcal{H}} + \|\xi_w(t)\|_{\mathcal{H}} \leq c, \tag{9}$$

and

$$\int_s^t \big[\|\partial_t u(\tau)\|^2 + \|\partial_t w(\tau)\|^2\big] d\tau \leq \omega(t-s) + \frac{c}{\omega}, \qquad \forall \omega > 0. \tag{10}$$

In order to conclude, we need the following generalized version of the Gronwall lemma.

Lemma 5. *Let $\Lambda : \mathbb{R}^+ \to \mathbb{R}^+$ be an absolutely continuous function satisfying*

$$\frac{d}{dt}\Lambda(t) + 2\varepsilon\Lambda(t) \leq h(t)\Lambda(t) + k,$$

where $\varepsilon > 0$, $k \geq 0$ and $\int_s^t h(\tau)d\tau \leq \varepsilon(t-s) + m$, for all $t \geq s \geq 0$ and some $m \geq 0$. Then,

$$\Lambda(t) \leq \Lambda(0)e^m e^{-\varepsilon t} + \frac{ke^m}{\varepsilon}, \qquad \forall t \geq 0.$$

We are now in a position to prove

Lemma 6. *For every $t \ge 0$ and some $\nu > 0$,*

$$\|\xi_v(t)\|_{\mathcal{H}} \le ce^{-\nu t}.$$

Proof. For $\varepsilon \in (0,1)$ to be determined later, define the functional

$$\Lambda = \|\nabla v\|^2 + \|\partial_t v\|^2 + \varepsilon\|v\|^2 + 2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle + 2\varepsilon\langle \partial_t v, v\rangle.$$

Note that, from (4) and (5),

$$2\langle \psi(u) - \psi(w), v\rangle - \langle \psi'(u)v, v\rangle \ge (\theta - 2\ell)\|v\|^2 - \langle \varphi'(u)v, v\rangle \ge -\frac{1}{2}\|\nabla v\|^2.$$

Hence, on account of (2) and (9), Λ satisfies the inequalities

$$\frac{1}{4}\|\xi_v\|^2_{\mathcal{H}} \le \Lambda \le c\|\xi_v\|^2_{\mathcal{H}}, \tag{11}$$

provided that ε is small enough. Multiplying (7) by $\partial_t v + \varepsilon v$, we find the equality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla v\|^2 + \Gamma = 2\langle (\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle \psi''(u)\partial_t u, v^2\rangle,$$

where we set

$$\Gamma = \frac{\varepsilon}{2}\|\nabla v\|^2 + (2 - 3\varepsilon)\|\partial_t v\|^2 + \varepsilon\langle \psi'(u)v, v\rangle - \varepsilon^2\|v\|^2 - 2\varepsilon^2\langle \partial_t v, v\rangle.$$

Using (2), (6) and (9), it is apparent that $\Gamma \ge 0$ if ε is small enough, and

$$\begin{aligned} 2\langle (\psi'(u) - \psi'(w))\partial_t w, v\rangle - \langle \psi''(u)\partial_t u, v^2\rangle &\le c\big(\|\partial_t u\| + \|\partial_t w\|\big)\|\nabla v\|^2 \\ &\le \frac{\varepsilon}{2}\|\nabla v\|^2 + \frac{c}{\varepsilon}\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda, \end{aligned}$$

by means of (11). At this point, choosing $\varepsilon > 0$ such that the above conditions are all satisfied, we obtain the differential inequality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda \le c\big(\|\partial_t u\|^2 + \|\partial_t w\|^2\big)\Lambda. \tag{12}$$

In view of (10), the desired conclusion follows from Lemma 5 and (11). □

Lemma 7. *For every $t \ge 0$,*

$$\|\xi_w(t)\|_{\mathcal{V}} \le c.$$

Proof. Setting $q = \partial_t w$, we differentiate (8) with respect to time, so to obtain

$$\partial_{tt} q + \partial_t q - \Delta q + \psi'(w)q = \theta\partial_t u.$$

Then, for $\varepsilon > 0$, we define the functional

$$\Lambda = \|\nabla q\|^2 + \|\partial_t q\|^2 + \varepsilon\|q\|^2 + \langle \psi'(w)q, q\rangle + 2\varepsilon\langle \partial_t q, q\rangle,$$

which, similarly to the previous lemma, satisfies the inequalities

$$\frac{1}{2}\|\xi_q\|^2_{\mathcal{H}} \le \Lambda \le c\|\xi_q\|^2_{\mathcal{H}},$$

when ε is small enough. Multiplying the above equation by $\partial_t q + \varepsilon q$, we are led to

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda + \frac{\varepsilon}{2}\|\nabla q\|^2 + \|\partial_t q\|^2 + \Gamma = 2\theta\langle \partial_t u, \partial_t q\rangle + \langle \psi''(w)\partial_t w, q^2\rangle + 2\varepsilon\theta\langle \partial_t u, \partial_t w\rangle,$$

where

$$\Gamma = \frac{\varepsilon}{2}\|\nabla q\|^2 + (1 - 3\varepsilon)\|\partial_t q\|^2 + \varepsilon\langle \psi'(w)q, q\rangle - \varepsilon^2\|q\|^2 - 2\varepsilon^2\langle \partial_t q, q\rangle.$$

Again, $\Gamma \geq 0$ provided that ε is small enough, whereas the right-hand side of the above differential equality is controlled as

$$\begin{aligned}&2\theta\langle\partial_t u, \partial_t q\rangle + 2\varepsilon\theta\langle\partial_t u, \partial_t w\rangle + \langle\psi''(w)\partial_t w, q^2\rangle\\ &\leq \frac{\varepsilon}{2}\|\nabla q\|^2 + \|\partial_t q\|^2 + \frac{c}{\varepsilon}\|\partial_t w\|^2\Lambda + c.\end{aligned}$$

Hence, fixing ε small, we end up with the differential inequality

$$\frac{d}{dt}\Lambda + \varepsilon\Lambda \leq c\|\partial_t w\|^2\Lambda + c,$$

and from Lemma 5, we get the bound

$$\|\nabla\partial_t w(t)\| + \|\partial_{tt} w(t)\| \leq c.$$

With this information, we recover from (8) the further control $\|\Delta w(t)\| \leq c$. $\square$

Collecting Lemma 6 and Lemma 7, we learn that $S(t)\mathbb{B}_0$ is (exponentially) attracted by a bounded subset $\mathcal{C} \subset \mathcal{V}$. In other words, $\mathcal{C}$ is a compact attracting set. This, by standard arguments of the theory of attractors (see e.g. [3, 12, 19]), yields the existence of a compact global attractor $\mathcal{A} \subset \mathcal{C}$ for the semigroup $S(t)$. The proof of Theorem 1 is then completed.

Further remarks. The proof of Lemma 7 repeats word by word the standard argument of the dissipative estimate in the phase space $\mathcal{V}$ for the original hyperbolic problem (1). Indeed, equation (8) can be rewritten in the form

$$\partial_{tt} w + \partial_t w - \Delta w + \varphi(w) = f + \theta v, \tag{13}$$

which coincides with (1), up to the exponentially decaying (and so nonessential) external force θv. Therefore, our technique allows to reduce the proof of the existence of an exponentially attracting set in a more regular space to verifying the dissipative estimate in that space. It is worth noting that the latter problem has been always considered much simpler (usually, it does not require any fractional Sobolev spaces, bootstrapping, etc.).

Another interesting application of our method is the stabilization to a single equilibrium for the solutions to (1) as $t \to \infty$. Indeed, since the equation possesses a global Lyapunov function, the convergence to the *whole set* $\mathcal{E}$ of equilibria is immediate, but if $\mathcal{E}$ is not totally disconnected, the convergence to a *single* equilibrium may not take place in general. Nevertheless, when the nonlinearity φ is real analytic, the above convergence can be recovered by the so-called Simon-Lojashevich technique. However, this technique naturally provides the stabilization of *more regular* solutions (or it requires the nonlinearity φ to be subcritical [15, 16]), and the convergence of *weak energy* solutions in the critical case is more delicate. In contrast to that, our decomposition allows to show the convergence of the weak energy solution u by proving the convergence of the more regular solution w to problem (13). Indeed, since $u = w + v$ and v is exponentially decaying, then the convergence of w obviously implies the convergence of u. We point out that the presence of the exponentially decaying external force θv in (13) is nonessential for the Simon-Lojashevich technique.

Finally, we mention a more recent application to the hyperbolic equation with nonlinear damping in a bounded two-dimensional domain

$$\partial_{tt} u + \sigma(u)\partial_t u - \Delta u + \varphi(u) = f,$$

where the damping σ is strictly positive ($\sigma(u) > \sigma_0 > 0$) and satisfies some natural assumptions. In this situation, the standard bootstrapping methods seem to be inapplicable in order to improve the regularity of the attractor (we do not know whether or not this equation is well-posed in fractional spaces). On the contrary, the above decomposition works perfectly, and it yields the optimal regularity of the global attractor (see [18]).

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