

Global attractor for dissipative wave equations

Cecilia Cavaterra Elisabetta Rocca

Dipartimento di Matematica "F. Enriques"
Università degli Studi di Milano
cecilia.cavaterra@unimi.it elisabetta.rocca@unimi.it

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The sine-Gordon equation 1

$$u_{tt} + \alpha u_t - \Delta u + f(u) = g \quad \text{in } \Omega \times (0, +\infty)$$

$$u = 0 \quad \text{on } \Gamma \times (0, +\infty)$$

$$u(0) = u_0 \quad u_t(0) = u_1 \quad \text{in } \Omega$$

1. $\Omega \subset \mathbb{R}^n$ bdd with smooth bdry Γ (e.g. of class $C^{1,1}$)
2. $\alpha > 0$, $f \in C^1(\mathbb{R})$
3. $f(u) = \beta \sin u$, $\beta \in \mathbb{R}$
4. $A = -\Delta : D(A) = H^2(\Omega) \cap H_0^1(\Omega) \rightarrow L^2(\Omega)$

Remark

We could take in 3. f bounded with f' bounded.

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Theorem

Let $T > 0$, $\alpha = 1$, and $g \in L^2(\Omega)$. If $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ then $\exists ! u \in C^0([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega))$ s.t.

$$u_{tt} \in L^\infty(0, T; H^{-1}(\Omega))$$

$$u_{tt} + u_t + Au + f(u) = g \text{ in } H^{-1}(\Omega), \text{ a.e. in } (0, T)$$

$$u(0) = u_0, \quad u_t(0) = u_1$$

Moreover, the map $(u_0, u_1) \mapsto (u(t), u_t(t))$ is Lipschitz continuous from $H_0^1(\Omega) \times L^2(\Omega)$ to itself for any fixed $t \geq 0$, that is, for all $t \in [0, T]$, it holds true

$$\begin{aligned} \|u_1(t) - u_2(t)\|_{H_0^1} + \|(u_1)_t(t) - (u_2)_t(t)\|_{L^2} \\ \leq C(T, f, \Omega) \left(\|u_{01} - u_{02}\|_{H_0^1} + \|u_{11} - u_{12}\|_{L^2} \right) \end{aligned}$$

Remark

On account of the previous theorem, we can now define the semigroup

$$S(t)(u_0, u_1) = (u(t), u_t(t)) \quad \forall t \geq 0$$

and we have that $(H_0^1(\Omega) \times L^2(\Omega), S(t))$ is a dynamical system

Preliminary result: the Gronwall's lemma

Lemma

Let $\sigma > 0$ constant and h, y, y' be locally integrable functions on (t_0, ∞) satisfying

$$\frac{d}{dt}y(t) + \sigma y(t) \leq h(t) \quad \text{for a.e. } t \in [t_0, \infty)$$

Then, for all $t \geq t_0 > 0$, it holds true

$$y(t) \leq 2y(t_0) \exp(-\sigma(t - t_0)) + \int_{t_0}^t h(s) \exp(-\sigma(t - s)) ds$$

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Theorem

$(H_0^1(\Omega) \times L^2(\Omega), S(t))$ has a bounded absorbing set.

Let us consider the Hilbert triplet

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \equiv (L^2(\Omega))' \hookrightarrow H^{-1}(\Omega)$$

and denote by

$$|u|^2 = (u, u), \quad \|u\|^2 = (\nabla u, \nabla u) \quad \text{for all } u \in H_0^1(\Omega)$$

and by $((\cdot, \cdot))$ the scalar product in $H_0^1(\Omega)$.

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Proof. Take $v(t) = u_t + \varepsilon u \in L^2(\Omega)$ as test function. We have

$$\langle u_{tt}, u_t + \varepsilon u \rangle + (u_t, u_t + \varepsilon u) + \langle Au, u_t + \varepsilon u \rangle + (f(u), u_t + \varepsilon u) = (g, u_t + \varepsilon u)$$

from which

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\{ \|u\|^2 + |v|^2 \right\} + \varepsilon \|u\|^2 + (1 - \varepsilon) |v|^2 - \varepsilon(1 - \varepsilon)(u, v) \\ + (f(u), v) = (g, v) \end{aligned}$$

Notice that, choosing $\varepsilon \leq \min\{1/4, 1/c_p^2\}$, where c_p is the Poincaré constant such that $|u| \leq c_p \|u\|$, we get

$$\varepsilon \|u\|^2 + (1 - \varepsilon) |v|^2 - \varepsilon(1 - \varepsilon)(u, v) \geq \alpha_1 \left(\|u\|^2 + |v|^2 \right)$$

where $\alpha_1 = \varepsilon/2$. Thus we deduce (using the Young and Hölder inequality)

$$\begin{aligned} \frac{d}{dt} \left\{ \|u\|^2 + |v|^2 \right\} + 2\alpha_1 \left(\|u\|^2 + |v|^2 \right) &\leq 2(g - f(u), v) \\ &\leq 2|g||v| + 2|\beta||\Omega|^{1/2}|v| \leq \alpha_1 |v|^2 + \frac{1}{\alpha_1} \left(|g|^2 + |\beta|^2 |\Omega| \right) \quad (*) \end{aligned}$$

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Thus, we immediately get

$$\frac{d}{dt}y + \alpha_1 y \leq \alpha_2 + \alpha_3 |g|^2 \quad \forall t \geq 0 \quad (\star)$$

where

$$y(t) = \|u\|^2 + |v|^2, \quad \alpha_1 = \frac{\varepsilon}{2}, \quad \alpha_2 = \frac{1}{\alpha_1} |\beta|^2 \|\Omega\|, \quad \alpha_3 = \frac{1}{\alpha_1}$$

and the Gronwall's lemma yields estimate (1):

$$y(t) \leq 2y(0) \exp(-\alpha_1 t) + \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1} |g|^2 \right) (1 - \exp(-\alpha_1 t)), \quad \forall t \geq 0$$

and

$$\limsup_{t \rightarrow +\infty} \left(\|u(t)\|^2 + |u_t(t)|^2 \right) \leq \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1} |g|^2 \right)$$

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Denote by $B(\rho) \subset H_0^1(\Omega) \times L^2(\Omega)$ a generic ball with radius $\rho > 0$. If $(u_0, u_1) \in B(\rho)$ then, we have $y(0) \leq C_\varepsilon \rho$. On account of (1), we deduce

$$\|u(t)\|^2 + |u_t(t)|^2 \leq 2C_\varepsilon \rho \exp(-\alpha_1 t) + \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1} |g|^2 \right), \quad \forall t \geq 0$$

Choosing any $t_{B(\rho)} \geq 0$ such that

$$2C_\varepsilon \rho \exp(-\alpha_1 t_{B(\rho)}) \leq \frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1} |g|^2$$

then we easily get

$$\|u(t)\|^2 + |u_t(t)|^2 \leq 2 \left(\frac{\alpha_2}{\alpha_1} + \frac{\alpha_3}{\alpha_1} |g|^2 \right) := \rho_0 \quad \forall t \geq t_{B(\rho)}$$

We conclude that $B(\rho_0)$ is a bounded absorbing set.

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Lemma

Under the previous assumptions, we have

$$S(t) = S_1(t) + S_2(t)$$

(where $S_j(t) : H_0^1(\Omega) \times L^2(\Omega) \rightarrow H_0^1(\Omega) \times L^2(\Omega)$ are not necessarily semigroups), with

- $\|S_1(t)x\| \rightarrow 0$ as $t \rightarrow +\infty$ uniformly with respect to $x \in B(\rho_0)$
- $S_2(t)B(\rho_0)$ is uniformly compact for t large

that is: for every $\mathcal{B}$ bounded set in $H_0^1(\Omega) \times L^2(\Omega)$, there exists $t_0 = t_0(\mathcal{B})$ such that $\bigcup_{t \geq t_0} S_2(t)\mathcal{B}$ is relatively compact in $H_0^1(\Omega) \times L^2(\Omega)$.

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Let us set

$$u = u^d + u^c$$

where

$$u_{tt}^d + u_t^d + Au^d = 0$$

$$u^d(0) = u_0, \quad u_t^d(0) = u_1$$

and

$$u_{tt}^c + u_t^c + Au^c + f(u) = g$$

$$u^c(0) = 0, \quad u_t^c(0) = 0.$$

u being the solution to our problem.

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Let us test the equation for u^d with $v^d = u_t^d + \varepsilon u^d$. This gives

$$\frac{d}{dt} \left(\|u^d\|^2 + |v^d|^2 \right) + 2\varepsilon \|u^d\|^2 + 2(1-\varepsilon)|v^d|^2 = 2\varepsilon(1-\varepsilon)(u^d, v^d)$$

Observe that for ε small, it holds true

$$2\varepsilon \|u\|^2 + 2(1-\varepsilon)|v|^2 - 2\varepsilon(1-\varepsilon)(u, v) \geq \varepsilon \left(\|u\|^2 + |v|^2 \right)$$

and set

$$\mathcal{E}^d = \|u^d\|^2 + |v^d|^2$$

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If we choose ε small enough then we get (cf. slides sine-Gordon no. 5,6)

$$\frac{d}{dt}\mathcal{E}^d + \varepsilon\mathcal{E}^d \leq 0$$

and this gives

$$\mathcal{E}^d(t) \leq 2\mathcal{E}^d(0)e^{-\varepsilon t} \quad \text{for all } t \geq 0$$

that is the exponential decay of $\|(u^d(t), u_t^d(t))\|_{H_0^1(\Omega) \times L^2(\Omega)}$.

Hence

$$\|S_1(t)x\| \rightarrow 0 \quad \text{as } t \rightarrow +\infty \text{ uniformly with respect to } x \in B(\rho_0)$$

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We now test the equation for u^c with $A(v^c) = A(u_t^c + \varepsilon u^c)$, getting

$$\begin{aligned} \frac{d}{dt} \left(|Au^c|^2 + \|v^c\|^2 \right) + 2\varepsilon |Au^c|^2 + 2(1 - \varepsilon) \|v^c\|^2 - 2\varepsilon(1 - \varepsilon) ((u^c, v^c)) \\ = 2(g - f(u), Av^c) \end{aligned}$$

For $\varepsilon \leq \min\{1/4, 1/C_p^2\}$, where C_p is the Poincaré constant such that $\|u\| \leq C_p |Au|$, we get

- $\varepsilon |Au^c|^2 + (1 - \varepsilon) \|v^c\|^2 - \varepsilon(1 - \varepsilon) ((u^c, v^c)) \geq \frac{\varepsilon}{2} \left(|Au^c|^2 + \|v^c\|^2 \right)$
- $$\begin{aligned} (g, Av^c) &= \frac{d}{dt} (g, Au^c) + \varepsilon (g, Au^c) \\ &\leq \frac{d}{dt} (g, Au^c) + \frac{\varepsilon}{4} |Au^c|^2 + \frac{\varepsilon}{4} |g|^2 \end{aligned}$$
- $$-(\beta \sin u, Av^c) = -((\beta \sin u, v^c)) \leq \frac{\varepsilon}{2} \|v^c\|^2 + \frac{|\beta|^2}{\varepsilon} \|u\|^2$$

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Denoting $\alpha_1 = \varepsilon/2$, we have

$$\begin{aligned} \frac{d}{dt} \left(|Au^c|^2 + \|v^c\|^2 - 2(g, Au^c) \right) + \alpha_1 \left(|Au^c|^2 + \|v^c\|^2 \right) \\ \leq 2\varepsilon|g|^2 + \frac{2}{\varepsilon}|\beta|^2\|u\|^2 \end{aligned}$$

Setting $y = |Au^c - g|^2 + \|v^c\|^2$, we obtain

$$\frac{d}{dt}y(t) + \frac{\alpha_1}{2}y(t) \leq (3\varepsilon + \alpha_1)|g|^2 + \frac{2}{\varepsilon}|\beta|^2\|u\|^2$$

Moreover, from the estimate $(\star)$ at slide 6, leading to the existence of the absorbing set, we know that $\|u\|_{L^1_{loc}(0,+\infty)}^2 \leq c$ and hence we have

$$\int_0^t e^{-\alpha(t-s)} \|u(s)\|^2 ds \leq \frac{e^\alpha c}{1 - e^{-\alpha}} \quad \forall \alpha > 0, \quad t \geq 0$$

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Applying the Gronwall lemma, we get (for all $t \geq 0$)

$$y(t) \leq 2|g|^2 \exp\left(-\frac{\alpha_1}{2}(t)\right) + \frac{1}{\alpha_1} \left((2\varepsilon + \alpha_1)|g|^2 + \frac{2c}{\varepsilon}|\beta|^2 \frac{\exp\left(\frac{\alpha_1}{2}\right)}{1 - \exp\left(-\frac{\alpha_1}{2}\right)} \right)$$

Choosing $t_1 \geq 0$ such that

$$2|g|^2 \exp\left(-\frac{\alpha_1}{2}t\right) \leq \frac{1}{\alpha_1} \left((2\varepsilon + \alpha_1)|g|^2 + \frac{2c}{\varepsilon}|\beta|^2 \frac{\exp\left(\frac{\alpha_1}{2}\right)}{1 - \exp\left(-\frac{\alpha_1}{2}\right)} \right)$$

we get (for all $t \geq t_1$)

$$|Au^c(t)|^2 + \|u_t^c(t)\|^2 \leq \frac{2}{\alpha_1} \left((2\varepsilon + \alpha_1)|g|^2 + \frac{2c}{\varepsilon}|\beta|^2 \frac{\exp\left(\frac{\alpha_1}{2}\right)}{1 - \exp\left(-\frac{\alpha_1}{2}\right)} \right)$$

Hence, if $\mathcal{B}$ is bounded in $H_0^1(\Omega) \times L^2(\Omega)$, then $\bigcup_{t \geq t_1} S_2(t)\mathcal{B}$ is bounded in $D(A) \times H_0^1(\Omega)$, which is compact in $H_0^1(\Omega) \times L^2(\Omega)$.

This concludes the proof of the Lemma.

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As by-product of a small variant of Corollary 3.36 (see notes on attractors) we get the existence of the global attractor $\mathcal{A}$.

Corollary

$(H_0^1(\Omega) \times L^2(\Omega), S(t))$ has a (connected) global attractor $\mathcal{A}$ that is bounded in $H_0^1(\Omega) \times L^2(\Omega)$.