

Global attractor for reaction-diffusion equations

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$$u_t - \Delta u + f(u) = g \quad \text{in } \Omega \times (0, +\infty)$$

$$u = 0 \quad \text{on } \Gamma \times (0, +\infty)$$

$$u(0) = u_0 \quad \text{in } \Omega$$

- $\Omega \subset \mathbb{R}^n$ bdd with smooth bdry Γ (e.g. of class $C^{1,1}$)
- $f \in C^1(\mathbb{R})$
- $\exists c_1, c_2, c_3 \geq 0$ s.t.

$$c_1|y|^6 - c_2 \leq f(y)y \leq c_3(|y|^6 + 1) \quad \forall y \in \mathbb{R}$$

- $f'(y) \geq \gamma \quad \forall y \in \mathbb{R}, \quad \text{for some } \gamma \in \mathbb{R}$
- $A = A_D (-\Delta + \text{homogeneous Dirichlet bdry condition})$

Theorem

Let $T > 0$ and $g \in H^{-1}(\Omega)$.

For any $u_0 \in L^2(\Omega)$, $\exists ! u \in C([0, T]; L^2(\Omega))$ (weak solution) s.t.

$$u \in L^2(0, T; H_0^1(\Omega)) \cap L^6(\Omega \times (0, T))$$

$$u_t \in L^2(0, T; H^{-1}(\Omega)) + L^{6/5}(\Omega \times (0, T)) \subset L^{6/5}(0, T; H^{-1}(\Omega))$$

$$u_t + Au + f(u) = g \quad \text{in } H^{-1}(\Omega), \quad \text{a.e. in } (0, T)$$

$$u(0) = u_0$$

Moreover, the map $u_0 \mapsto u(t)$ is Lipschitz continuous from $L^2(\Omega)$ to itself for any fixed $t \geq 0$, that is

$$\|u_1(t) - u_2(t)\|_{L^2} \leq C(T, f, \Omega) \|u_{01} - u_{02}\|_{L^2}, \quad \forall t \geq 0$$

Remark

On account of the previous theorem, we can now define the semigroup

$$S(t)u_0 = u(t) \quad \forall t \geq 0$$

and we have that $(L^2(\Omega), S(t))$ is a dynamical system

Theorem

The following *dissipative estimate* holds

$$(1) \quad \|u(t)\|_{L^2}^2 \leq \|u_0\|_{L^2}^2 e^{-c_P t} + \frac{2c_2}{c_P} + \frac{1}{c_P} \|g\|_{H^{-1}}^2 \quad \forall t \geq 0$$

where c_P is a positive constant depending on Ω .

Moreover, we also have

$$(2) \quad \int_t^{t+1} \left(\|u(\tau)\|_{H^1}^2 + \|u(\tau)\|_{L^6}^6 \right) d\tau \leq C(f)(1 + \|u_0\|_{L^2}^2 + \|g\|_{H^{-1}}^2)$$

for all $t \geq 0$ and some $C(f) > 0$ depending also on Ω .

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Proof. Take $v = u(t) \in L^2(\Omega)$ as test function. Then we have

$$\langle u_t(t), u(t) \rangle + \langle A(u)(t), u(t) \rangle + (f(u(t)), u(t)) = \langle g, u(t) \rangle$$

from which

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2\|\nabla u(t)\|_{L^2}^2 + 2(f(u(t)), u(t)) = 2\langle g, u(t) \rangle$$

Recall that, if Ω is bdd, it holds $|\langle F, v \rangle_{H_0^1}| \leq \|F\|_{H^{-1}} \|\nabla v\|_{L^2}$.

Thus we deduce (using also the lower bound on $f(y)y$)

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + 2\|\nabla u(t)\|_{L^2}^2 + 2c_1 \|u(t)\|_{L^6}^6 \leq 2c_2 + 2\|g\|_{H^{-1}} \|\nabla u(t)\|_{L^2}$$

and an application of the Young's inequality gives

$$(*) \quad \frac{d}{dt} \|u(t)\|_{L^2}^2 + \|\nabla u(t)\|_{L^2}^2 + 2c_1 \|u(t)\|_{L^6}^6 \leq 2c_2 + \|g\|_{H^{-1}}^2$$

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Using Poincaré's inequality, we get

$$\frac{d}{dt} \|u(t)\|_{L^2}^2 + c_P \|u(t)\|_{L^2}^2 \leq 2c_2 + \|g\|_{H^{-1}}^2 \quad \forall t \geq 0$$

and standard Gronwall's lemma yields estimate (1):

$$\begin{aligned} \|u(t)\|_{L^2}^2 &\leq \|u_0\|_{L^2}^2 e^{-c_P t} + (2c_2 + \|g\|_{H^{-1}}^2) \int_0^t e^{-c_P(t-s)} ds \\ &\leq \|u_0\|_{L^2}^2 e^{-c_P t} + \frac{1}{c_P} (2c_2 + \|g\|_{H^{-1}}^2) \quad \forall t \geq 0 \end{aligned}$$

Then, integrating (*) on $(t, t+1)$, we also recover estimate (2):

$$\begin{aligned} \int_t^{t+1} \left(\|\nabla u(\tau)\|_{L^2}^2 + 2c_1 \|u(\tau)\|_{L^6}^6 \right) d\tau &\leq \|u(t)\|_{L^2}^2 + 2c_2 + \|g\|_{H^{-1}}^2 \\ &\leq \|u_0\|_{L^2}^2 e^{-c_P t} + (1/c_P + 1)(2c_2 + \|g\|_{H^{-1}}^2) \end{aligned}$$

Corollary

$(L^2(\Omega), S(t))$ has a bounded (in $L^2(\Omega)$) absorbing set.

Proof. Denote by $B(\rho) \subset L^2(\Omega)$ a generic ball with radius $\rho > 0$. If $u_0 \in B(\rho)$ then, on account of (1), we have

$$\|u(t)\|_{L^2}^2 \leq \rho e^{-c_P t} + \frac{1}{c_P} (2c_2 + \|g\|_{H^{-1}}^2) \quad \forall t \geq 0$$

Choosing any $t_{B(\rho)} \geq 0$ such that

$$\rho e^{-c_P t_{B(\rho)}} \leq 1/c_P (2c_2 + \|g\|_{H^{-1}}^2)$$

then we easily get

$$\|u(t)\|_{L^2}^2 \leq 2/\lambda_1 (2c_2 + \|g\|_{H^{-1}}^2) =: \rho_0 \quad \forall t \geq t_{B(\rho)}$$

We conclude that $B(\rho_0)$ is a bounded absorbing set.

Preliminary result: Uniform Gronwall's lemma

Lemma

Let η be an absolutely continuous nonnegative function on $[t_0, \infty)$ and $\phi, \psi \in L^1_{loc}([t_0, +\infty))$ two nonnegative functions (a.e.) such that

- $\int_t^{t+r} \phi(s) ds \leq a_1, \quad \int_t^{t+r} \psi(s) ds \leq a_2, \quad \int_t^{t+r} \eta(s) ds \leq a_3$
 $\forall t \geq t_0$, for some positive constants r and a_j
- $\eta'(t) \leq \phi(t)\eta(t) + \psi(t) \quad \text{a.e. in } [t_0, \infty)$

Then

$$\eta(t+r) \leq \left(\frac{a_3}{r} + a_2 \right) e^{a_1} \quad \forall t \geq t_0$$

Proof of uniform Gronwall's lemma

Proof. Let $t_0 \leq t \leq s \leq t + r$. Then

$$\begin{aligned}\frac{d}{ds} \left(\eta(s) e^{-\int_t^s \phi(\sigma) d\sigma} \right) &= e^{-\int_t^s \phi(\sigma) d\sigma} \left(\eta'(s) - \phi(s) \eta(s) \right) \\ &\leq e^{-\int_t^s \phi(\sigma) d\sigma} \psi(s) \leq \psi(s).\end{aligned}$$

Integrating in s on $[z, t + r]$, where $z : t < z < t + r$, we obtain

$$\eta(t + r) e^{-\int_t^{t+r} \phi(\sigma) d\sigma} \leq a_2 + \eta(z) e^{-\int_t^z \phi(\sigma) d\sigma}$$

from which

$$\eta(t + r) \leq (\eta(z) + a_2) e^{a_1}.$$

Finally, if we integrate in z on $[t, t + r]$ we get the wanted estimate.

Theorem

$(L^2(\Omega), S(t))$ has an absorbing set B_1 that is bounded in $H_0^1(\Omega)$

Proof. We argue formally by taking $v = u_t(t)$ as test function (to be rigorous we should build a Faedo-Galerkin scheme). This gives

$$(u_t(t), u_t(t)) + \langle A(u)(t), u_t(t) \rangle + (f(u(t)), u_t(t)) = \langle g, u_t(t) \rangle$$

from which $(u = 0 \text{ on } \Gamma \times (0, T) \Rightarrow u_t = 0 \text{ on } \Gamma \times (0, T))$

$$\|u_t(t)\|_{L^2}^2 + (\nabla u(t), \nabla u_t(t)) + (f(u(t)), u_t(t)) = \langle g, u_t(t) \rangle$$

Setting $F(y) = \int_0^y f(z)dz$, then we get

$$(a) \quad \frac{d}{dt} \left(\|\nabla u(t)\|_{L^2}^2 + 2(F(u(t)), 1) - 2\langle g, u(t) \rangle \right) + 2\|u_t(t)\|_{L^2}^2 = 0$$

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On account of assumptions on f , it can be shown that there exist $c'_i \geq 0$ such that (see notes on reaction-diffusion eqs)

$$(b) \quad c'_1|y|^6 - c'_2 \leq F(y) \leq c'_3(|y|^6 + 1) \quad \forall y \in \mathbb{R}.$$

Let us set

$$(*) \quad E(u(t)) = \|\nabla u(t)\|_{L^2}^2 + 2(F(u(t)), 1) - 2\langle g, u(t) \rangle + 2c'_2|\Omega| + 2\|g\|_{H^{-1}}^2$$

Using Young's inequality ($|\langle g, u(t) \rangle| \leq \|g\|_{H^{-1}}^2 + \frac{1}{4}\|\nabla u(t)\|_{L^2}^2$) and (b), we obtain

$$(c) \quad E(u(t)) \geq \frac{1}{2}\|\nabla u(t)\|_{L^2}^2 + 2c'_1\|u(t)\|_{L^6}^6 \geq 0$$

$$(d) \quad E(u(t)) \leq \frac{3}{2}\|\nabla u(t)\|_{L^2}^2 + 2c'_3\|u(t)\|_{L^6}^6 + 4\|g\|_{H^{-1}}^2 + 2(c'_2 + c'_3)|\Omega|$$

Remark

Let $\mathcal{B}$ be any bounded set in $L^2(\Omega)$. Thanks to the previous corollary we know that after a time $t_{\mathcal{B}} \geq 0$ all the trajectories starting from $u_0 \in \mathcal{B}$ enter in $B_0 = B(\rho_0)$, $B(\rho_0) \subset L^2(\Omega)$ being an absorbing set.

Then we can always write, $\forall t \geq t_{\mathcal{B}}, \forall \tau \geq 0$,

$$S(t)u_0 = S(t - t_{\mathcal{B}} + t_{\mathcal{B}})u_0 = S(t - t_{\mathcal{B}})S(t_{\mathcal{B}})u_0 = S(\tau)u_1$$

where $u_1 \in B_0$.

Hence, for the sake of simplicity, in the following part of the proof we will take $t \geq 0$ and $u_0 \in B_0$.

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Going back to (d), taking $u_0 \in B_0$ and applying inequality (2) of the previous theorem, we deduce

$$\sup_{t \geq 0} \int_t^{t+1} E(u(\tau)) d\tau \leq C(\rho_0)$$

where $C(\rho_0)$ is a positive constant depending also on f, g, Ω . Moreover, recalling (a) and definition (*) one has

$$\frac{d}{dt} (E(u(t))) \leq 0$$

Then, an application of the **uniform Gronwall's lemma** (with $\eta = E(u)$, $\phi = 0$, $\psi = 0$, $r = 1$, $t_0 = 0$) gives

$$E(u(t)) \leq C(\rho_0) \quad \forall t \geq 1$$

Recalling (c) and Poincaré's inequality, from the previous equation we conclude

$$\|u(t)\|_{H^1} \leq \tilde{C}(\rho_0) \quad \forall t \geq 1$$

Finally, on account of the previous remark, for any bounded set $\mathcal{B} \subset L^2(\Omega)$ we have

$$(e) \quad \|S(t)\mathcal{B}\|_{H^1} \leq \tilde{C}(\rho_0) \quad \forall t \geq t_{\mathcal{B}} + 1$$

Hence

$$B_1 = \{x \in H_0^1(\Omega) : \|x\|_{H^1} \leq \tilde{C}(\rho_0)\}$$

is an absorbing set in $L^2(\Omega)$ that is bounded in $H_0^1(\Omega)$.

Remark

As by-product we deduce

$$\int_{t^*}^{+\infty} \|u_t(\tau)\|_{L^2}^2 d\tau \leq C(\rho_0), \text{ where } t^* = t_B + 1$$

Proof. Recall definition (*) and (a). Then we have

$$\frac{1}{2} \frac{d}{dt} (E(u(t))) + \|u_t(t)\|_{L^2}^2 = 0$$

Integrating on (t^*, t) and using (c) and (d) we get

$$\begin{aligned} \int_{t^*}^t \|u_t(\tau)\|_{L^2}^2 d\tau &\leq \frac{E(u(t^*))}{2} \\ &\leq C(f, \Omega) (\|\nabla u(t^*)\|_{L^2}^2 + \|u(t^*)\|_{L^6}^6 + \|g\|_{H^{-1}}^2 + 1) \end{aligned}$$

Passing to the limit for $t \rightarrow +\infty$ and on account of (e) we get the estimate.

Corollary

$(L^2(\Omega), S(t))$ is a compact dynamical system which has a (connected) global attractor $\mathcal{A}$ that is bounded in $H_0^1(\Omega)$. Moreover, for any bounded absorbing set B_0 it holds

$$\mathcal{A} = \omega(B_0)$$

Proof. In the previous theorem we proved the existence of a bounded absorbing set $B_1 \subset L^2(\Omega)$ that is closed and bounded in $H_0^1(\Omega)$. Hence, B_1 is compact in $L^2(\Omega)$.

Using corollary 3.34 (see notes on attractors) we get the existence of the global attractor $\mathcal{A} \subseteq B_1 \subset H_0^1(\Omega)$.

On account of Theorem 3.23, $\mathcal{A}$ is also connected (as $L^2(\Omega)$).

Finally, an application of theorem 3.33 gives $\mathcal{A} = \omega(B_0)$, for any bounded absorbing set B_0 .