

Errata and additional material for
Infinite-Dimensional Dynamical Systems

Errata

(2.3) should read $T(0) = \text{id}$.

(5.1) and (5.3): all derivatives (d/dx_j) should be partial derivatives ($\partial/\partial x_j$).

Section 7.4.3: in fact it is possible to make estimates on $u_n - u_m$, following the calculations in section 7.4.2, and show directly that $\{u_n\}$ is a Cauchy sequence in $L^2(0, T; V)$ and $L^\infty(0, T; H)$, removing the need to extract subsequences. However, the presentation as it stands closely parallels the treatment in subsequent chapters when the subsequence method is necessary. (See lecture notes, chapter *).

(8.30) the expression $[1 + \|u\|_{H^1} + \|v\|_{H^1}]$ should be raised to the power of γ rather than $\frac{1}{2}$: the last line of the inequality in the proof should end $(1 + |u|_{L^{2q\gamma}}^{2\gamma} + |v|_{L^{2q\gamma}})^{2\gamma}$ rather than what is written.

Page 204: weak-* convergence of $P_n f$ to f in $L^2(0, T; V^*)$ follows using a much simpler argument. In fact, a straightforward application of Lebesgue's dominated convergence theorem (theorem 1.7 (iii)) shows that $P_n f$ converges strongly to f in $L^2(0, T; V^*)$. (See lecture notes, page *)

Page 248: we in fact need to take $w \in C^1(0, T; \mathbb{C}^1(\Omega))$ in the argument showing the convergence of $B(u_n, u_n)$ to $B(u, u)$, and then use the density of $C^1(0, T; \mathbb{C}^1(\Omega))$ in $L^q(0, T; V)$. In fact it is better to take $w = \sum_{j=1}^N c_j(t)w_j$, with $c_j(t)$ continuous: for such w it is easy to show that

$$\int_0^T \langle P_n B(u_n, u_n), w \rangle dt \rightarrow \int_0^T \langle B(u, u), w \rangle dt,$$

and then since such w are dense in $L^p(0, T; V)$ we obtain convergence of $P_n B(u_n, u_n)$ to $B(u, u)$ in one step. (See lecture notes, page *)

Page 277: the example is wrong (thanks to Prof. Grzegorz Lukaszewicz of Warsaw University for pointing this out). The calculations given are in

fact for

$$\begin{aligned} dx/dt &= -zy \\ dy/dt &= zx \\ dz/dt &= -\mu z|z|. \end{aligned}$$

However, there are some subtleties here. In fact for this example every point has a compact ω limit set, even though the attractor $z \equiv 0$ is not compact, and this is why proposition 10.14 and its corollary still apply: that problems can arise otherwise is shown in the simpler system $\dot{z} = -z^2$ with $\dot{x} = zx$ which has solution $x(t) = x_0(1 + z_0 t)$ and $z(t) = z_0/(1 + z_0 t)$: although on $z \equiv 0$ all points are stationary, the x component of every solution has constant speed $x_0 z_0$. An example that does away with the need for such subtleties – since it does in fact have a compact global attractor – is

$$\begin{aligned} dx/dt &= (1 - (x^2 + y^2))x - yz \\ dy/dt &= (1 - (x^2 + y^2))y + xz \\ dz/dt &= -\mu z|z|, \end{aligned}$$

or in polar coordinates

$$\begin{aligned} dr/dt &= r(1 - r^2) \\ d\theta/dt &= z \\ dz/dt &= -\mu z|z|. \end{aligned}$$

(11.17) the central term in the equality should be $\int_{\Omega} f(u)u_t \, dx$.

On page 354 the reference for fractal dimension is to Clark Robinson's 1995 book "Dynamical Systems".

(14.35) the term in k should be $|k|^{1/2}$.

Page 418, while checking that φ satisfies the uniqueness property, the left-hand side should be $\varphi(t; \varphi(s; x))$.

Additional material

There is a very elegant formulation of an existence result for global attractors (cf. theorem 10.5) that is due to Crauel (2001) in the case of random attractors.

Theorem 1. *There exists a global attractor $\mathcal{A}$ iff there exists a compact attracting set K , and then $\mathcal{A} = \omega(K)$.*

Note that the condition of a compact attracting set is much weaker than the existence of a compact absorbing set. The proof requires the following simple lemma:

Lemma 2. *If K is a compact set and x_n is a sequence such that*

$$\text{dist}(x_n, K) \rightarrow 0$$

then $\{x_n\}$ has a convergent subsequence whose limit lies in K .

As a first step to proving this new theorem first we reprove proposition 10.3 under the weaker condition here.

Proposition 3. *If there exists a compact attracting set K then the ω -limit set $\omega(X)$ of any bounded set X is a non-empty, invariant, closed subset of K . Furthermore $\omega(X)$ attracts X .*

Proof. To see that $\omega(X)$ is non-empty choose some point $x \in X$. Then since K is attracting

$$\text{dist}(S(n)x, K) \rightarrow 0.$$

It follows that for some sequence $n_j \rightarrow \infty$

$$S(n_j)x \rightarrow x^* \in K.$$

As the intersection of a decreasing sequence of closed sets $\omega(X)$ is clearly closed. To show that $\omega(X) \subset K$ suppose that $t_n \rightarrow \infty$, $x_n \in X$ and

$$S(t_n)x_n \rightarrow y.$$

Then since K is attracting

$$\text{dist}(S(t_n)x_n, K) \rightarrow 0,$$

implying that a subsequence of $S(t_n)x_n$ converges to a point in K . Since the sequence itself converges it follows that $y \in K$. So $\omega(X)$ is compact.

Now suppose that $\omega(X)$ does not attract X . Then there exists a $\delta > 0$ and a sequence of t_n such that

$$\text{dist}(S(t_n)X, \omega(X)) > \delta,$$

and hence $x_n \in X$ such that

$$\text{dist}(S(t_n)x_n, \omega(X)) > \delta. \tag{1}$$

However, the argument above shows that a subsequence of $\{S(t_n)x_n\}$ converges to some point z . By (1) we should have

$$\text{dist}(z, \omega(X)) \geq \delta,$$

while by definition $z \in \omega(X)$. So $\omega(X)$ attracts X . $\square$

Now observe that

$$A \subseteq B \implies \omega(A) \subseteq \omega(B), \tag{2}$$

and that since $\omega(X)$ is invariant

$$\omega[\omega(X)] = \bigcap_{t \geq 0} \overline{\bigcup_{s \geq t} S(s)\omega(X)} = \omega(X). \tag{3}$$

Proof. (Proof of theorem 1). It follows from the previous proposition that $\omega(K)$ is non-empty, compact, invariant, and attracts K . So all we have to prove is that $\omega(K)$ attracts X . Since $\omega(X)$ attracts X it suffices to show that $\omega(X) \subset \omega(K)$. But this follows immediately from (2) and (3). $\square$