

Structural Stability of Nonlinear Flows

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Structural Stability:

$$(u_n, u_n^*) \in \text{graph}(\alpha_n) \quad \forall n, \quad (u_n, u_n^*) \rightharpoonup (u, u^*) \text{ in } V \times V', \quad \alpha_n \xrightarrow{\tau} \alpha \quad (2)$$

$$\Rightarrow (u, u^*) \in \text{graph}(\alpha). \quad (3)$$

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3. This also applies e.g. to $D_t u + \alpha(u) \ni u^*$ (α : max-monotone).
4. This may be extended to α **pseudo-monotone** (in the sense of Brezis).

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In the space $L^2(0, T)$, the ordinary derivative D_t with domain $V := \{v \in H^1(0, T) : v(0) = 0\}$ is monotone, i.e.,

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May $D_t u = f$ be represented as a minimum problem?

More generally:

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Conversely, (6) entails that α is monotone (possibly not maximal).

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The vanishing of the minimum cannot be dispensed with !

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indeed $f_{\alpha+L}(v, v^*) = \langle v^*, v \rangle \Leftrightarrow [\alpha + L](v) \ni v^*$.

For instance, if

$$\begin{aligned}\beta : \mathbf{R}^N &\rightarrow \mathcal{P}(\mathbf{R}^N) \text{ is max-monotone,} \\ h &\in L^2(0, T; H^{-1}(\Omega)),\end{aligned}\tag{13}$$

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$$\limsup_{n \rightarrow \infty} \text{graph}(\alpha_n) \subset \text{graph}(\alpha) \quad (19)$$

in the sense of **Kuratowski** w.r.t. the topology $\tilde{\pi}$.

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Approximation of the operator is often mandatory, e.g. in numerical analysis.

This may raise the issues of operator convergence and structural stability.

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However this implication holds under (mild) regularity assumptions.

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(the operator A has the form of a time-integral, see e.g. [Tartar].)

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The classical Trotter-Kato theorem actually applies to (40) and not to (41).

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