

Existence results for a PDE system modeling damage, in nonsmooth domains

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joint work with
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Diffuse Interface MOdels, Levico,
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A rate-independent model for damage

- $\Omega \subset \mathbb{R}^d$ bounded

The PDE system

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell \quad \text{in } \Omega \times (0, T)$$

$$\partial\mathcal{R}_1(z') + A_q z + f'(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) \quad \text{in } \Omega \times (0, T)$$

- damage parameter z

$$z(\cdot, x) = 0 \text{ (1)} = \text{full (no) damage "around" } x$$

- small strains, quadratic elastic energy $\frac{1}{2} \int_{\Omega} \mathbb{C}(x)g(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) \, dx$;
- $f \in C^1(\mathbb{R})$ (e.g., $f(z) = (1 - z)^2$)

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- $f \in C^1(\mathbb{R})$ (e.g., $f(z) = (1 - z)^2$)
- gradient theory

$$A_q z = -\operatorname{div} \left((1 + |\nabla z|^2)^{(q-2)/2} \nabla z \right), \quad q \geq 2$$

- **rate-independent** and **unidirectional**:

$$\mathcal{R}_1(\dot{z}) = \begin{cases} \int_{\Omega} \kappa |\dot{z}| & \text{if } \dot{z} \leq 0 \text{ a.e. in } \Omega, \\ \infty & \text{else.} \end{cases}$$

A rate-independent model for damage: results

$$\begin{aligned}
 -\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) &= \ell && \text{in } \Omega \times (0, T) \\
 \partial\mathcal{R}_1(z') + A_q z + f'(z) + \partial I_{[0,+\infty)}(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) && \text{in } \Omega \times (0, T)
 \end{aligned}$$

Energetic solutions

- ▶ existence results for (non-quadratic elastic energy and) **I**Ncomplete damage $g(z) \geq c > 0$:

[Mielke-Roubíček'06] with $-\Delta_q$, $q > d$; [Mielke-Thomas'10] with $q > 1$; [Thomas'12] with $q = 1$

- ▶ existence results for **C**OMplete damage $g(z) = z$:

[Bouchitté-Mielke-Roubíček'09], [Mielke'11],
[Mielke-Roubíček-Zeman'10] with viscosity+inertia for u

A rate-independent model for damage: our goals

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+ homog. (for simplicity) Dir. b.c. for u on $\Gamma_D \subset \partial\Omega$ + no-flux b.c. for z

BUT: energetic solutions jump 'too early' and 'too long'

A rate-independent model for damage: our goals

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell \quad \text{in } \Omega \times (0, T)$$

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The vanishing viscosity approach

- ♣ Consider solutions of the rate-independent system arising in the limit $\epsilon \downarrow 0$ of the **viscous** approximation: this provides
 - ▶ selection criterion for mechanically feasible solutions
 - ▶ description of energetic behaviour of solutions at jumps

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 - ▶ description of energetic behaviour of solutions at jumps
- ♣ preliminary: existence result for the viscous system
(NON-trivial, focus TODAY)

A viscous model for damage: analytical difficulties

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Analytical difficulties

- complete damage $g(z) = z \rightsquigarrow$ elliptic degeneracy for u
 BUT, here INcomplete damage: $g \in C^1(\mathbb{R})$, $g(z) \geq c > 0$

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- ▶ quadratic term $g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) \rightsquigarrow$ need for **enhanced estimates on u**
 $\rightsquigarrow A_q(z)$ with $q > d$ needed
- ▶ $A_q + \partial\mathcal{R}_1 \rightsquigarrow$ **doubly nonlinear** character in z -eq.

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 $\rightsquigarrow A_q(z)$ with $q > d$ needed
- ▶ $A_q + \partial\mathcal{R}_1 \rightsquigarrow$ **doubly nonlinear** character in z-eq.
- ▶ Unidirectionality = $\partial\mathcal{R}_1$ **unbounded** operator $\rightsquigarrow$ no good comparison estimates in z-eq.

A viscous model for damage: literature

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∃ of viscous solutions

- ▶ **[Bonetti-Schimperna'04] (Frémond's model)**: $q = 2$, with $-\Delta z_t$,
COMplete damage $\Rightarrow$ **local** ∃
- ▶ **[Bonetti-Schimperna-Segatti'05]**: $q = 2$, viscosity+inertia for $u \Rightarrow$
enhanced regularity estimates for u based on Ω **smooth**, **COM**plete
damage $\Rightarrow$ **local** ∃

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- ▶ **[Bonetti-Schimperna-Segatti'05]**: $q = 2$, viscosity+inertia for $u \Rightarrow$ enhanced regularity estimates for u based on Ω **smooth**, COMplete damage $\Rightarrow$ **local** ∃
- ▶ **[Bonetti-Bonfanti'08]**: $q = 2$, with temperature, Ω **smooth**, **local** ∃
- ▶ **[Heinemann-Kraus (+Bonetti-Segatti)'10-'13]** (coupling w. phase separation): global ∃ for COMplete damage, Ω **NONsmooth** $\Rightarrow$ NO elliptic regularity estimates but **weak formulation** for the z-eq. ($q > d$), also in **[Rocca-R.'12]** (with viscosity+inertia & temperature)

A viscous model for damage: our results

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell \quad \text{in } \Omega \times (0, T)$$

$$\partial\mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) \quad \text{in } \Omega \times (0, T)$$

◆ **Our goal:** existence of global-in-time-solutions, based on careful higher integrability estimates for u , under

“minimal” regularity requirements on the domain Ω

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♦ **Our goal:** existence of global-in-time-solutions, based on careful higher integrability estimates for u , under

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♣ From now on, assume $\boxed{q > d} \Rightarrow W^{1,q}(\Omega) \subset C^0(\overline{\Omega})$, and then

MORE regularity of $\Omega \rightarrow$ high **ER** integrab. for u

↓

ENHANCED estimates for z

↓

BETTER existence results

Outlook

- ▶ An **abstract** approach to $\exists$ for the viscous problem $\rightsquigarrow$ the role of the **chain rule**
- ▶ Analysis for '**smooth**' domains
- ▶ Analysis for '**less smooth**' domains

The PDE system rewritten (I)

$$-\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell \quad \text{in } \Omega \times (0, T)$$

$$\partial\mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) \quad \text{in } \Omega \times (0, T)$$

◆ $\mathcal{Z} := W^{1,q}(\Omega) \rightsquigarrow$ state space for z

◆ Consider the minimum problem, for fixed $z \in \mathcal{Z}$

$$\operatorname{Min}_{\tilde{u} \in \mathcal{U} := W_{\Gamma_D}^{1,2}(\Omega; \mathbb{R}^d)} \mathcal{E}(t, \tilde{u}, z) \quad \text{with}$$

$$\mathcal{E}(t, \tilde{u}, z) = \int_{\Omega} \frac{1}{q} (1 + |\nabla z|^2)^{q/2} + f(z) + \frac{1}{2} g(z) \mathbb{C} \varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, dx - {}_{\mathcal{U}^*} \langle \ell(t), \tilde{u} \rangle_{\mathcal{U}},$$

Then,

$$-\operatorname{div}(g(z(t))\mathbb{C}\varepsilon(u(t))) = \ell(t) \quad \text{in } \mathcal{U}^*, \quad \text{for a.a. } t \in (0, T)$$

↗ **Euler-Lagrange eq. for $\operatorname{Min}_{\tilde{u} \in \mathcal{U}}$**

The PDE system rewritten (II)

$$\mathcal{Z} := W^{1,q}(\Omega)$$

♣ Introduce the **reduced energy** $\mathcal{J} : [0, T] \times \mathcal{Z} \rightarrow \mathbb{R}$

$$\begin{aligned} \mathcal{J}(t, z) = \text{Min}_{\tilde{u} \in \mathcal{U}} \bigg(& \int_{\Omega} \frac{1}{q} (1 + |\nabla z|^2)^{q/2} + f(z) + \frac{1}{2} g(z) \mathbb{C} \varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, dx \\ & - {}_{\mathcal{U}} \langle \ell(t), \tilde{u} \rangle_{\mathcal{U}} \bigg) \end{aligned}$$

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Then

$$\begin{cases} -\text{div}(g(z) \mathbb{C} \varepsilon(u)) = \ell & \text{in } \Omega \times (0, T) \\ \partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) & \text{in } \Omega \times (0, T) \end{cases}$$

$\Updownarrow$

$$\partial \mathcal{R}_1(z'(t)) + \epsilon z'(t) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T)$$

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Then

$$\begin{cases} -\text{div}(g(z) \mathbb{C} \varepsilon(u)) = \ell & \text{in } \Omega \times (0, T) \\ \partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) + \frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) \ni 0 & \text{in } \Omega \times (0, T) \end{cases}$$

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with $\partial \mathcal{R}_1 : \mathcal{Z} \rightrightarrows \mathcal{Z}^*$ convex anal. subdiff. while

quadratic elastic energy $\implies$

unique minimizer for $\mathcal{E}(t, \cdot, z)$ $\implies$

$z \mapsto \mathcal{J}(t, z)$ **Gâteaux-differentiable but nonconvex**

The PDE system rewritten (III)

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♣ Introduce the **reduced energy** $\mathcal{J} : [0, T] \times \mathcal{Z} \rightarrow \mathbb{R}$

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Then

$$\begin{cases} -\text{div}(g(z) \mathbb{C} \varepsilon(u)) = \ell & \text{in } \Omega \times (0, T) \\ \partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) & \text{in } \Omega \times (0, T) \end{cases}$$

$\Updownarrow$

$$\partial \mathcal{R}_{\epsilon}(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T)$$

with

$$\mathcal{R}_{\epsilon}(\dot{z}) := \mathcal{R}_1(\dot{z}) + \frac{\epsilon}{2} \|\dot{z}\|_{L^2(\Omega)}^2$$

... And now, let's go abstract

- ▶ $\mathcal{Z}$ reflexive Banach space
- ▶ $\mathcal{R} : \mathcal{Z} \rightarrow [0, +\infty]$ (convex) **dissipation** potential
- ▶ $\mathcal{J} : [0, T] \times \mathcal{Z} \rightarrow \mathbb{R}$ **energy** s.t. $z \mapsto \mathcal{J}(t, z)$ **Gâteaux-diff.** but **nonconvex**
& $t \mapsto \mathcal{J}(t, z)$ **smooth**

The Cauchy problem

$$\partial \mathcal{R}(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^*, \quad t \in (0, T), \quad (\text{DNE})$$

with $z(0) = z_0 \in \mathcal{Z}$

♣ **Aim:** enucleate **(abstract) conditions on $\mathcal{J} \Rightarrow \exists$** for (DNE)

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♦ **Bonus:**

- ▶ $\exists$ theory OK also for $z \mapsto \mathcal{J}(t, z)$ **NON**smooth, cf. **variational theory for gradient flows & rate-independent evolution [De Giorgi, Ambrosio-Gigli-Savaré, Mielke....]**
- ▶ abstract procedure highlights **WHY** certain properties of $\mathcal{J}$ are needed $\Rightarrow$ **guidelines** for “concrete” analysis

Time discretization of $\partial\mathcal{R}(z'(t)) + D_z\mathcal{J}(t, z(t)) \ni 0$

Fixed time-step $\tau > 0 \rightsquigarrow 0 = t_0 < t_1 < \dots < t_n < \dots < t_N = T$

- **Discrete solutions** $z_\tau^0, z_\tau^1, \dots, z_\tau^N$: solve recursively

$$z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ \tau \mathcal{R} \left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{J}(t_n, z) \right\}, \quad z_\tau^0 := z_0$$

- **Euler-Lagrange equation**

$$\partial\mathcal{R} \left(\frac{z_\tau^n - z_\tau^{n-1}}{\tau} \right) + D_z\mathcal{J}(t_n, z_\tau^n) \ni 0$$

- **Approximate solutions**: interpolants on $(0, T)$ of $\{z_\tau^k\}_{k=1}^n$: $(\bar{z}_\tau)_\tau$ (pcw. constant) & $(\hat{z}_\tau)_\tau$ (pcw. linear) satisfy

$$\partial\mathcal{R}(\hat{z}'_\tau(t)) + D_z\mathcal{J}(t, \bar{z}_\tau(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T),$$

The variational interpolant (I)

- First idea: in addition to $\bar{z}_\tau$ & $\widehat{z}_\tau$, consider

The variational interpolant $\widetilde{z}_\tau$ [E. DeGiorgi, theory of Minimizing Movements, Gradient Flows in Metric Spaces]

Defined by $\widetilde{z}_\tau(0) := z_\tau^0 = z_0$ and

$$\widetilde{z}_\tau(t) \in \operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ (t - t_{n-1}) \mathcal{R} \left(\frac{z - z_\tau^{n-1}}{(t - t_{n-1})} \right) + \mathcal{J}(t, z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

- Compare with **time-incremental minimization**

$$\operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ \tau \mathcal{R} \left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{J}(t_n, z) \right\}$$

The discrete energy inequality

$$\tilde{z}_\tau(t) \in \operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ (t - t_{n-1}) \mathcal{R} \left(\frac{z - z_\tau^{n-1}}{(t - t_{n-1})} \right) + \mathcal{J}(t, z) \right\} \text{ for } t \in (t_{n-1}, t_n]$$

satisfies the **discrete energy** inequality

$$\int_0^t \mathcal{R}(\tilde{z}'_\tau(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(t, \tilde{z}_\tau(t))) \, ds + \mathcal{J}(t, \tilde{z}_\tau(t)) \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds$$

with $\mathcal{R}^*$ Fenchel-Moreau conjugate of $\mathcal{R}$

$$\mathcal{R}^*(\xi) := \sup_{v \in \operatorname{dom}(\mathcal{R})} (\langle \xi, v \rangle - \mathcal{R}(v))$$

in some duality pairing $\langle \cdot, \cdot \rangle$.

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$\Rightarrow$ A priori estimates for $(\bar{z}_\tau)_\tau$, $(\hat{z}_\tau)_\tau$, $(\tilde{z}_\tau)_\tau$

$\Rightarrow$ compactness for $(\bar{z}_\tau)_\tau$, $(\hat{z}_\tau)_\tau$, $(\tilde{z}_\tau)_\tau$

$\Rightarrow \exists$ curve z s.t. $\bar{z}_\tau, \hat{z}_\tau, \tilde{z}_\tau \rightarrow z$ as $\tau \downarrow 0$

Passage to the limit: the upper energy estimate

♣ Idea: instead of taking the limit as $\tau \downarrow 0$ of

$$\partial \mathcal{R}(\tilde{z}'_\tau(t)) + D_z \mathcal{J}(t, \tilde{z}_\tau(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T),$$

do it in the **approximate energy inequality**

$$\liminf_{\tau \downarrow 0} \int_0^t \mathcal{R}(\tilde{z}'_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, \tilde{z}_\tau(s))) \, ds + \liminf_{\tau \downarrow 0} \mathcal{J}(t, \tilde{z}_\tau(t))$$

$$\leq \mathcal{J}(0, z_0) + \liminf_{\tau \downarrow 0} \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds$$

↓ (via **LOWER SEMICONTINUITY**)

$$\int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds$$

HERE: needed that

$$\left(z_n \rightarrow z, \sup_n \mathcal{J}(z_n) < +\infty, \right) \Rightarrow D_z \mathcal{J}(t, z_n) \rightharpoonup D_z \mathcal{J}(t, z)$$

Passage to the limit: lower energy estimate

IF $\mathcal{J}$ fulfills the **chain rule** along the limit curve z , i.e.

$$\frac{d}{dt}\mathcal{J}(t, z(t)) = \partial_t \mathcal{J}(t, z(t)) + \langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle \quad \text{for a.a. } t \in (0, T),$$

then

$$\begin{aligned} \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) &\leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds \\ &= \mathcal{J}(t, z(t)) - \int_0^t \langle D_z \mathcal{J}(s, z(s)), z'(s) \rangle \, ds \end{aligned}$$

Passage to the limit: lower energy estimate

IF $\mathcal{J}$ fulfills the **chain rule** along the limit curve z , i.e.

$$\frac{d}{dt}\mathcal{J}(t, z(t)) = \partial_t \mathcal{J}(t, z(t)) + \langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle \quad \text{for a.a. } t \in (0, T),$$

then

$$\begin{aligned} \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) &\leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds \\ &= \mathcal{J}(t, z(t)) - \int_0^t \langle D_z \mathcal{J}(s, z(s)), z'(s) \rangle \, ds \\ &\leq \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \end{aligned}$$

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$$\begin{aligned} \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) &\leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds \\ &= \mathcal{J}(t, z(t)) - \int_0^t \langle D_z \mathcal{J}(s, z(s)), z'(s) \rangle \, ds \\ &\leq \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \\ \implies \int_0^t \left(\mathcal{R}(z'(s)) + \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) - \langle -D_z \mathcal{J}(s, z(s)), z'(s) \rangle \right) ds &= 0 \end{aligned}$$

Passage to the limit: lower energy estimate

IF $\mathcal{J}$ fulfills the **chain rule** along the limit curve z , i.e.

$$\frac{d}{dt}\mathcal{J}(t, z(t)) = \partial_t \mathcal{J}(t, z(t)) + \langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle \quad \text{for a.a. } t \in (0, T),$$

then

$$\begin{aligned} \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) &\leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds \\ &= \mathcal{J}(t, z(t)) - \int_0^t \langle D_z \mathcal{J}(s, z(s)), z'(s) \rangle \, ds \\ &\leq \int_0^t \mathcal{R}(z'(s)) \, ds + \int_0^t \mathcal{R}^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \\ \implies \mathcal{R}(z'(t)) + \mathcal{R}^*(-D_z \mathcal{J}(t, z(t))) - \langle -D_z \mathcal{J}(t, z(t)), z'(t) \rangle &= 0 \\ \implies -D_z \mathcal{J}(t, z(t)) &\in \partial \mathcal{R}(z'(t)) \end{aligned}$$

i.e. z solves

$$\partial \mathcal{R}(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$$

Conclusion

Theorem [Mielke-R.-Savaré'13]

Under

- ♦ Weak continuity of $D_z \mathcal{J}(t, \cdot)$
- ♦ Chain rule

$\exists$ solution $z : [0, T] \rightarrow \mathcal{Z}$ to

$$\partial \mathcal{R}(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad + \text{Cauchy condition,}$$

fulfilling the **energy identity**

$$\int_s^t \mathcal{R}(z'(r)) \, dr + \int_s^t \mathcal{R}^*(-D_z \mathcal{J}(r, z(r))) \, dr + \mathcal{J}(t, z(t)) = \mathcal{J}(s, z(s)) + \int_s^t \partial_t \mathcal{J}(r, z(r)) \, dr$$

for all $0 \leq s \leq t \leq T$.

Chain rule argument vs. Minty's trick (I)

Energy-identity argument requires chain rule

$$\frac{d}{dt} \mathcal{J}(t, z(t)) = \partial_t \mathcal{J}(t, z(t)) + \langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle$$

◆ Alternatively, direct passage to the limit in $\partial \mathcal{R}(\widehat{z}'_\tau(t)) + D_z \mathcal{J}(t, \bar{z}_\tau(t)) \ni 0$, viz.

$$\begin{cases} \bar{\omega}_\tau(t) + D_z \mathcal{J}(t, \bar{z}_\tau(t)) \ni 0 \\ \bar{\omega}_\tau(t) \in \partial \mathcal{R}(\widehat{z}'_\tau(t)) \end{cases} \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T),$$

For this, continuity of $D_z \mathcal{J}(t, \cdot)$ & to identify weak limit of $(\bar{\omega}_\tau)_\tau$ use **Minty's trick**

$$\begin{cases} \widehat{z}'_\tau \rightharpoonup z', \\ \bar{\omega}_\tau \rightharpoonup \omega \end{cases} \Rightarrow \omega(t) \in \partial \mathcal{R}(z'(t)) \text{ for a.a. } t \in (0, T)$$

$$\limsup_\tau \int_0^T \langle \bar{\omega}_\tau, \widehat{z}'_\tau \rangle \, dt \leq \int_0^T \langle \omega, z' \rangle \, dt$$

Chain rule argument vs. Minty's trick (II)

♣ Minty's trick in detail

$$\begin{aligned} & \limsup_{\tau} \int_0^T \langle \bar{\omega}_{\tau}(t), \widehat{z}'_{\tau}(t) \rangle \, dt \\ &= \limsup_{\tau} \int_0^T \langle -D_z \mathcal{J}(t, \bar{z}_{\tau}(t)), \widehat{z}'_{\tau}(t) \rangle \, dt \\ &\leq \limsup_{\tau} \int_0^T \langle -D_z \mathcal{J}(t, z(t)), z'(t) \rangle \, dt \\ &= \int_0^T \langle \omega(t), z'(t) \rangle \, dt \end{aligned}$$

For this, needed that $\langle -D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ well defined!!

Chain rule argument vs. Minty's trick (II)

♣ Minty's trick in detail

$$\begin{aligned}
 & \limsup_{\tau} \int_0^T \langle \bar{\omega}_{\tau}(t), \widehat{z}'_{\tau}(t) \rangle \, dt \\
 &= \limsup_{\tau} \int_0^T \langle -D_z \mathcal{J}(t, \bar{z}_{\tau}(t)), \widehat{z}'_{\tau}(t) \rangle \, dt \\
 &\leq \limsup_{\tau} \int_0^T \langle -D_z \mathcal{J}(t, z(t)), z'(t) \rangle \, dt \\
 &= \int_0^T \langle \omega(t), z'(t) \rangle \, dt
 \end{aligned}$$

For this, needed that $\langle -D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ well defined!!

♦ The property

$z'(t)$ and $D_z \mathcal{J}(t, z(t))$ are **in duality** for almost all $t \in (0, T)$

is at the core of **chain rule** & **Minty's trick**!!

Back to our problem....

$$\mathcal{Z} = W^{1,q}(\Omega)$$

$$\mathcal{R}(\dot{z}) = \mathcal{R}_\epsilon(\dot{z}) = \mathcal{R}_1(\dot{z}) + \frac{\epsilon}{2} \|\dot{z}\|_{L^2(\Omega)}^2$$

$$\mathcal{J}(t, z) = \text{Min}_{\tilde{u} \in \mathcal{U}} \left(\int_{\Omega} \frac{1}{q} (1 + |\nabla z|^2)^{q/2} + f(z) + \frac{1}{2} g(z) \mathbb{C} \varepsilon(\tilde{u}) : \varepsilon(\tilde{u}) \, dx - {}_{\mathcal{U}^*} \langle \ell(t), \tilde{u} \rangle_{\mathcal{U}} \right)$$

$$\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad \text{in } \mathcal{Z}^* \quad \text{for a.a. } t \in (0, T)$$

- $\hookrightarrow$ **Abstract conditions for $\exists$** satisfied?

$\hookrightarrow$ **continuity** of $z \mapsto D_z \mathcal{J}(t, z)$

$\hookrightarrow \hookrightarrow$ **chain rule**, i.e.

$\langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ well-defined for some $\langle \cdot, \cdot \rangle$, for limit $z : [0, T] \rightarrow \mathcal{Z}??$

$\hookrightarrow$ Which regularity for z' and $D_z \mathcal{J}(\cdot, z(\cdot))$ known so far?

... back to our analytical difficulties

Discrete energy inequality

$$\int_0^t \mathcal{R}_\epsilon (\tilde{z}'_\tau(s)) \, ds + \int_0^t \mathcal{R}_\epsilon^* (-D_z \mathcal{J}(s, \tilde{z}_\tau(s))) \, ds + \mathcal{J}(t, \tilde{z}_\tau(t)) \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds$$

gives

$$\int_0^t \left(\mathcal{R}_1 (\tilde{z}'_\tau(s)) + \frac{\epsilon}{2} \|\tilde{z}'_\tau(s)\|_{L^2(\Omega)}^2 \right) \, ds < \infty \quad \Rightarrow \quad z' \in L^2(0, T; L^2(\Omega)),$$

$$\int_0^t \mathcal{R}_\epsilon^* (-D_z \mathcal{J}(s, \tilde{z}_\tau(s))) \, ds \quad \not\Rightarrow \quad D_z \mathcal{J}(t, z(t)) \in L^2(0, T; L^2(\Omega))$$

$$\text{as } \mathcal{R}_\epsilon^*(\xi) = \frac{1}{2\epsilon} \min_{w \in \partial \mathcal{R}_1(0)} \|\xi - w\|_{L^2(\Omega)}^2 \quad \text{BUT } \partial \mathcal{R}_1(0) \text{ unbounded in } L^2(\Omega)!!$$

$\nearrow \mathcal{R}_1(\dot{z}) \in [0, +\infty]$ due to **unidirectionality**

... back to our analytical difficulties

Discrete energy inequality

$$\int_0^t \mathcal{R}_\epsilon (\tilde{z}'_\tau(s)) \, ds + \int_0^t \mathcal{R}_\epsilon^* (-D_z \mathcal{J}(s, \tilde{z}_\tau(s))) \, ds + \mathcal{J}(t, \tilde{z}_\tau(t)) \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds$$

gives

$$\int_0^t \left(\mathcal{R}_1 (\tilde{z}'_\tau(s)) + \frac{\epsilon}{2} \|\tilde{z}'_\tau(s)\|_{L^2(\Omega)}^2 \right) \, ds < \infty \quad \Rightarrow \quad z' \in L^2(0, T; L^2(\Omega)),$$

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$\nearrow \mathcal{R}_1(\dot{z}) \in [0, +\infty]$ due to **unidirectionality**

♠ We only have $\mathcal{J}(t, z(t)) < \infty \Rightarrow D_z \mathcal{J}(t, z(t)) \in \mathcal{Z}^*$

Main difficulty

♣ **Estimates from energy inequality** NOT sufficient $\Rightarrow$ need additional estimates for

$$\langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle \text{ well-defined}$$

So far

- ▶ $z'(t) \in L^2(\Omega) \rightsquigarrow$ $\mathcal{H}$ estimate $D_z \mathcal{J}(t, z(t))$ in $L^2(\Omega)??$
- ▶ $D_z \mathcal{J}(t, z(t)) \in \mathcal{Z}^* \rightsquigarrow$ $\mathcal{H}$ estimate $z'(t) \in \mathcal{Z} = W^{1,q}(\Omega)?? \leftarrow$ **difficult**

Higher differentiability for u

$$\begin{cases} -\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell & \text{in } \Omega \times (0, T) \\ \partial\mathcal{R}_1(z') + \varepsilon z' + A_q z + f'(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) & \text{in } \Omega \times (0, T) \end{cases}$$

Lemma

Suppose $\Omega \in C^{1,1}$ – smooth and $\ell \in L^2(\Omega; \mathbb{R}^d)$ and $\Gamma_D = \partial\Omega$.

Then

$$\begin{cases} -\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

$$\Rightarrow u \in W^{2,2}(\Omega; \mathbb{R}^d) \text{ with}$$

$$\|u\|_{W^{2,2}(\Omega; \mathbb{R}^d)} \leq C(1 + \|z\|_{W^{1,q}(\Omega)})^\alpha (\|u\|_{W^{1,2}(\Omega; \mathbb{R}^d)} + \|\ell\|_{L^2(\Omega; \mathbb{R}^d)}) \text{ for some } \alpha > 1$$

Higher differentiability for u

$$\begin{cases} -\operatorname{div}(g(z)\mathbb{C}\varepsilon(u)) = \ell & \text{in } \Omega \times (0, T) \\ \partial\mathcal{R}_1(z') + \varepsilon z' + A_q z + f'(z) \ni -\frac{1}{2}g'(z)\mathbb{C}\varepsilon(u) : \varepsilon(u) & \text{in } \Omega \times (0, T) \end{cases}$$

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$\Rightarrow u \in W^{2,2}(\Omega; \mathbb{R}^d)$ with

$$\|u\|_{W^{2,2}(\Omega; \mathbb{R}^d)} \leq C(1 + \|z\|_{W^{1,q}(\Omega)})^\alpha (\|u\|_{W^{1,2}(\Omega; \mathbb{R}^d)} + \|\ell\|_{L^2(\Omega; \mathbb{R}^d)}) \text{ for some } \alpha > 1$$

◆ Corollary

$$\mathcal{J}(t, z(t)) \leq C \Rightarrow \begin{cases} \|u(t)\|_{W^{2,2}(\Omega; \mathbb{R}^d)} \leq C' \\ \|u'(t)\|_{W^{1,d/(d-2)}(\Omega; \mathbb{R}^d)} \leq C'(1 + \|z'(t)\|_{L^{2d/(d-2)}(\Omega)}) \end{cases}$$

Consequence: enhanced estimate

- All calculations to be done **rigorously on time-discrete** level

$$\begin{aligned}
 & \int_0^T \left(\frac{d}{dt} \left(\partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) \right) \times z' \right) dt \\
 & \Rightarrow \int_0^T \partial_t \left(\underbrace{A_q z(t)}_{= -\operatorname{div}((1 + |\nabla z(t)|^2)^{(q-2)/2} \nabla z)} \right) z'(t) dt \leq C \\
 & \Rightarrow \int_0^T \int_{\Omega} (1 + |\nabla z(t)|^2)^{(q-2)/2} |\nabla z'(t)|^2 dx dt \leq C \quad \text{(mixed estimate)}
 \end{aligned}$$

Consequence: enhanced estimate

- All calculations to be done **rigorously on time-discrete** level

$$\begin{aligned}
 & \int_0^T \left(\frac{d}{dt} \left(\partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) \right) \times z' \right) dt \\
 & \Rightarrow \int_0^T \partial_t \left(\underbrace{A_q z(t)}_{= -\operatorname{div}((1 + |\nabla z(t)|^2)^{(q-2)/2} \nabla z)} \right) z'(t) dt \leq C \\
 & \Rightarrow \int_0^T \int_{\Omega} (1 + |\nabla z(t)|^2)^{(q-2)/2} |\nabla z'(t)|^2 dx dt \leq C \quad \text{(mixed estimate)} \\
 & \Rightarrow z' \in L^2(0, T; W^{1,2}(\Omega)) \\
 & \Rightarrow u' \in L^2(0, T; W^{1,d/(d-2)}(\Omega)) \\
 & \quad \text{due to } \|u'(t)\|_{W^{1,d/(d-2)}(\Omega; \mathbb{R}^d)} \leq C'(1 + \|z'(t)\|_{L^{2d/(d-2)}(\Omega)})
 \end{aligned}$$

Further: $L^2(\Omega)$ -estimate for $D_z \mathcal{J}(t, z(t))$

$$D_z \mathcal{J}(t, z) = A_q z + \underbrace{f'(z)}_{\text{lower order}} + \underbrace{\frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u)}_{\text{boded in } L^2(\Omega) \text{ via } \|u(t)\|_{W^{2,2}(\Omega; \mathbb{R}^d)} \leq C'}$$

♣ **TO DO:** $L^2(\Omega)$ -estimate for $A_q z$

Further: $L^2(\Omega)$ -estimate for $D_z \mathcal{J}(t, z(t))$

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♣ **TO DO:** $L^2(\Omega)$ -estimate for $A_q z$

♦ “classical” test:

$$\int_0^T \left(\partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) + \frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) \ni 0 \right) \times \partial_t (A_q z) \, dt$$

✓ **OK** by estimates on z' & u'

$$\implies \sup_{t \in (0, T)} \|A_q z(t)\|_{L^2(\Omega)} \leq C$$

$$\implies \boxed{\|D_z \mathcal{J}(t, z(t))\|_{L^\infty(0, T; L^2(\Omega))} \leq C}$$

The abstract theory applies!

- $\mathfrak{J}$ **Abstract conditions** for $\exists$ satisfied? **YES**

✓ **Chain rule** OK, due to

$$_{L^2(\Omega)} \langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle_{L^2(\Omega)} \text{ well-defined}$$

✓ **continuity** of $z \mapsto D_z \mathcal{J}(t, z) = A_q + f'(z) + \frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u)$
holds:

$$\left. \begin{array}{l} z_n \rightharpoonup z \text{ in } \mathcal{Z} = W^{1,q}(\Omega) \\ \|A_q z_n\|_{L^2(\Omega)} \leq C \end{array} \right\} \Rightarrow D_z \mathcal{J}(t, z_n) \rightarrow D_z \mathcal{J}(t, z) \text{ in } L^2(\Omega)$$

A first existence result for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$

Theorem 1 [Knees-R.-Zanini'13]

Suppose $\Omega \text{ } C^{1,1} \text{ - smooth}$ + $\Gamma_D = \partial \Omega$

+ conditions on ℓ, f, g + $D_z \mathcal{J}(0, z(0)) \in L^2(\Omega)$.

Then, there exists a solution

$$z \in \underbrace{L^\infty(0, T; W^{1,q}(\Omega))}_{\text{due to } \mathcal{J}(t, z(t)) \leq C} \cap \underbrace{W^{1,2}(0, T; W^{1,2}(\Omega))}_{\text{due to mixed estimate}} \quad \& \quad \underbrace{A_q z \in L^\infty(0, T; L^2(\Omega))}_{\text{due to enhanced reg.}}$$

+ $z(0) = z_0$ + **energy identity** for all $0 \leq s \leq t \leq T$

$$\int_s^t \mathcal{R}_\epsilon(z'(r)) \, dr + \int_s^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(r, z(r))) \, dr + \mathcal{J}(t, z(t)) = \mathcal{J}(s, z(s)) + \int_s^t \partial_t \mathcal{J}(r, z(r)) \, dr$$

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Theorem 1 [Knees-R.-Zanini'13]

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+ $z(0) = z_0$ + **energy identity** for all $0 \leq s \leq t \leq T$

$$\int_s^t \mathcal{R}_\epsilon(z'(r)) \, dr + \int_s^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(r, z(r))) \, dr + \mathcal{J}(t, z(t)) = \mathcal{J}(s, z(s)) + \int_s^t \partial_t \mathcal{J}(r, z(r)) \, dr$$

Moreover, **IF** $f(0) \leq f(z)$ and $g(0) \leq g(z)$ for $z \leq 0$, then

$$0 \leq z(t, x) \leq 1 \quad \text{for a.a. } (t, x) \in \Omega \times (0, T).$$

The enhanced estimate revisited in less smooth domains

♦ **DROP** $\Omega \in C^{1,1}$ – smooth

♣ Enhanced estimate

$$\int_0^T \left(\frac{d}{dt} \left(\partial \mathcal{R}_1(z') + \epsilon z' + A_q z + f'(z) \ni -\frac{1}{2} g'(z) \mathbb{C} \epsilon(u) : \epsilon(u) \right) \times z' \right) dt$$

still doable, **IF**

$$\int_0^T \int_{\Omega} \frac{1}{2} g'(z) \mathbb{C} \epsilon(u) : \epsilon(u) z' dx dt \text{ can be estimated by}$$

$$\int_0^T \|z'\|_{W^{1,2}(\Omega)}^2 dt \text{ on the l.h.s.}$$

Displacement regularity revisited

$$\int_0^T \int_{\Omega} \frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) z' \, dx \, dt \text{ estimated by } \int_0^T \|z'\|_{W^{1,2}(\Omega)}^2 \, dt \text{ on the l.h.s.}$$

OK, via

Lemma

Under **suitable assumptions** on Ω ($\Gamma_D \neq \partial\Omega$ allowed), $\exists p_* > d$ such that

$$\begin{cases} -\operatorname{div}(g(z) \mathbb{C} \varepsilon(u)) = \ell & \text{in } \Omega \\ u = 0 & \text{in } \Gamma_D \end{cases}$$

$$\Rightarrow W^{1,p_*}(\Omega; \mathbb{R}^d) \text{ with}$$

$$\|u\|_{W^{1,p_*}(\Omega; \mathbb{R}^d)} \leq C(1 + \|z\|_{W^{1,q}(\Omega)})^\alpha (\|u\|_{W^{1,2}(\Omega; \mathbb{R}^d)} + \|\ell\|_{W^{-1,p_*}(\Omega; \mathbb{R}^d)}) \text{ for } \alpha > 1$$

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$$\int_0^T \int_{\Omega} \frac{1}{2} g'(z) \mathbb{C} \varepsilon(u) : \varepsilon(u) z' \, dx \, dt \text{ estimated by } \int_0^T \|z'\|_{W^{1,2}(\Omega)}^2 \, dt \text{ on the l.h.s.}$$

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Under **suitable assumptions** on Ω ($\Gamma_D \neq \partial\Omega$ allowed), $\exists p_* > d$ such that

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♦ Corollary

$$\exists p_* > d : \mathcal{J}(t, z(t)) \leq C \Rightarrow \begin{cases} \|u(t)\|_{W^{1,p_*}(\Omega; \mathbb{R}^d)} \leq C' \\ \|u'(t)\|_{W^{1,2p_*/(p_*+2)}(\Omega; \mathbb{R}^d)} \leq C'(1 + \|z'(t)\|_{L^{2d/(d-2)}(\Omega)}) \end{cases}$$

What smoothness of Ω needed?

◆ **Assumption:** $\Omega \subset \mathbb{R}^d$ bounded & $\Gamma_D \subset \partial\Omega$ closed fulfill

- (i) The spaces $W_{\Gamma_D}^{1,p}(\Omega)$, $p \in (1, \infty)$, form an interpolation scale
- (ii) There exists $p_* > d$ such that for all $p \in [2, p_*]$ the elasticity operator

$$L : W_{\Gamma_D}^{1,p}(\Omega) \rightarrow W_{\Gamma_D}^{-1,p}(\Omega) \quad \langle Lu, v \rangle := \int_{\Omega} \mathbb{C} \varepsilon(u) : \varepsilon(v) \, dx$$

is an isomorphism

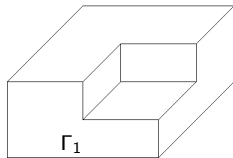


Figure: For example: (i) Dirichlet-conditions on the bottom plane and Neumann conditions on the remaining part of $\partial\Omega$ or (ii) $\Gamma_D = \Gamma_1$ and Neumann conditions on the rest.

Lingering difficulties

- ♣ For admissible domains Ω ,

enhanced estimate OK $\Rightarrow z \in W^{1,2}(0, T; W^{1,2}(\Omega))$

- ♠ BUT,

$L^2(\Omega)$ – estimate for $D_z \mathcal{J}(t, z)$ NOT possible!

- $\imath$ **Abstract conditions** for $\exists$ satisfied?

$\imath$ **continuity** of $z \mapsto D_z \mathcal{J}(t, z) \rightsquigarrow$ **PROBLEM:**

$$\sup_n \mathcal{J}(t, z_n) \leq C \Rightarrow z_n \rightharpoonup z \text{ in } W^{1,q}(\Omega) \not\Rightarrow A_q z_n \rightharpoonup A_q z$$

$\Rightarrow$ further **spatial compactness** for z needed!!

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$\imath\imath$ **chain rule**, i.e.

$\langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ well-defined

for suitable $\langle \cdot, \cdot \rangle$???

Higher differentiability for z

Lemma

In addition, suppose that

Ω fulfills **UNIFORM CONE** condition.

Then,

$$\exists C > 0 \quad \forall \beta \in \left[0, 1 - \frac{d}{q}\right) : \|z\|_{L^{2q}(0,T;W^{1+\beta,q}(\Omega))} \leq C.$$

Note that $\beta > 0$ due to $q > d$, hence $W^{1+\beta,q}(\Omega) \Subset W^{1,q}(\Omega)$

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Proof via a difference quotient argument, directly on minimizers

$$z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ \tau \mathcal{R}_\epsilon \left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{J}(t, z) \right\}$$

based on techniques by [Savaré'98], [Ebmeyer-Frehse'99], [Knees, PhD thesis'05]

Existence proof for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$ in less smooth domains (I)

► Discrete solutions

$$z_\tau^n \in \operatorname{Argmin}_{z \in \mathcal{Z}} \left\{ \tau \mathcal{R}_\epsilon \left(\frac{z - z_\tau^{n-1}}{\tau} \right) + \mathcal{J}(t_n, z) \right\}, \quad z_\tau^0 := z_0$$

► **Approximate solutions:** interpolants $(\bar{z}_\tau)_\tau$ & $(\hat{z}_\tau)_\tau$ & $(\tilde{z}_\tau)_\tau$

► **Discrete energy inequality**

$$\int_0^t (\mathcal{R}_\epsilon(\hat{z}'_\tau(s)) + \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(t, \tilde{z}_\tau(t)))) \, ds + \mathcal{J}(t, \tilde{z}_\tau(t)) \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds$$

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► ENERGY+ENHANCED+HIGHER DIFFERENTIABILITY estimates

for $(\bar{z}_\tau)_\tau$, $(\hat{z}_\tau)_\tau$, $(\tilde{z}_\tau)_\tau$

$\Rightarrow \exists$ limiting curve z

$$z \in L^{2q}(0, T; W^{1+\beta, q}(\Omega)) \cap L^\infty(0, T; W^{1, q}(\Omega)) \cap W^{1, 2}(0, T; W^{1, 2}(\Omega))$$

and convergences in

$$L^{2q}(0, T; W^{1+\beta, q}(\Omega)) \cap L^\infty(0, T; W^{1, q}(\Omega)) \cap W^{1, 2}(0, T; W^{1, 2}(\Omega)).$$

$\rightsquigarrow$ In particular, **spatial compactness** in $W^{1, q}(\Omega)$!

Existence proof for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$ in less smooth domains (II)

► Passage to the limit in energy inequality:

$$\begin{aligned}
 & \liminf_{\tau \downarrow 0} \int_0^t \mathcal{R}_\epsilon(\tilde{z}'_\tau(s)) \, ds + \liminf_{\tau \downarrow 0} \int_0^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(s, \tilde{z}_\tau(s))) \, ds + \liminf_{\tau \downarrow 0} \mathcal{J}(t, \tilde{z}_\tau(t)) \\
 & \leq \mathcal{J}(0, z_0) + \liminf_{\tau \downarrow 0} \int_0^t \partial_t \mathcal{J}(s, \tilde{z}_\tau(s)) \, ds \\
 & \quad \downarrow \text{(via LOWER SEMICONTINUITY)} \\
 & \int_0^t \mathcal{R}_\epsilon(z'(s)) \, ds + \int_0^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \\
 & \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds
 \end{aligned}$$

$\Rightarrow$ **UPPER ENERGY ESTIMATE**

?? Chain rule??

$\langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ well-defined

for suitable $\langle \cdot, \cdot \rangle$???

◆ Observe:

$$D_z \mathcal{J}(t, z(t)) = A_q z(t) + \underbrace{D_z \tilde{\mathcal{J}}(t, z(t))}_{\text{"lower order"}}$$

$\Rightarrow \int_{\Omega} D_z \tilde{\mathcal{J}}(t, z(t)) z'(t) \, dx$ well-defined

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$\Rightarrow \int_{\Omega} D_z \tilde{\mathcal{J}}(t, z(t)) z'(t) dx$ well-defined

◆ Observe

$$\left\{ \begin{array}{l} \int_0^T \int_{\Omega} (1 + |\nabla z|^2)^{(q-2)/2} |\nabla z'|^2 dx dt < \infty \\ z \in L^{\infty}(0, T; W^{1,q}(\Omega)) \\ \text{Hölder's inequality} \end{array} \right.$$

$$\Rightarrow \int_{\Omega} \int_{\Omega} (1 + |\nabla z|^2)^{(q-2)/2} \nabla z \cdot \nabla z' dx dt \text{ well-defined}$$

$\nearrow$ surrogate for $\langle A_q z, z' \rangle$

The chain rule in less smooth domains

Lemma: weak chain rule

For any $z \in L^\infty(0, T; W^{1,q}(\Omega)) \cap W^{1,2}(0, T; W^{1,2}(\Omega))$ with

$$\int_0^T \int_\Omega (1 + |\nabla z|^2)^{(q-2)/2} |\nabla z'|^2 \, dx \, dt < \infty$$

we have $t \mapsto \mathcal{J}(t, z(t))$ absolutely continuous, and for a.a. $t \in (0, T)$

$$\begin{aligned} \frac{d}{dt} \mathcal{J}(t, z(t)) &= \partial_t \mathcal{J}(t, z(t)) \\ &\quad + \int_\Omega (1 + |\nabla z(t)|^2)^{(q-2)/2} |\nabla z'(t)|^2 \, dx + \int_\Omega D_z \tilde{\mathcal{J}}(t, z(t)) z'(t) \, dx. \end{aligned}$$

↗ surrogate for $\langle A_q z, z' \rangle$

Existence proof for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$ in less smooth domains (III)

From the upper to the lower energy estimate??

$$\begin{aligned}
 & \int_0^t \mathcal{R}_\epsilon(z'(s)) \, ds + \int_0^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \\
 & \leq \mathcal{J}(0, z_0) + \int_0^t \partial_t \mathcal{J}(s, z(s)) \, ds \\
 & = \mathcal{J}(t, z(t)) - \int_\Omega (1 + |\nabla z(t)|^2)^{(q-2)/2} |\nabla z'(t)|^2 \, dx - \int_\Omega D_z \tilde{\mathcal{J}}(t, z(t)) z'(t) \, dx
 \end{aligned}$$

Existence proof for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$ in less smooth domains (III)

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 & \neq \mathcal{J}(t, z(t)) - \int_0^t \langle D_z \mathcal{J}(s, z(s)), z'(s) \rangle \, ds \quad \leftarrow \text{not defined} \\
 & \left(\leq \int_0^t \mathcal{R}_\epsilon(z'(s)) \, ds + \int_0^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(s, z(s))) \, ds + \mathcal{J}(t, z(t)) \right)
 \end{aligned}$$

$\Rightarrow$ we cannot conclude the lower energy estimate!!

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$\Rightarrow$ we cannot conclude the lower energy estimate!!

♠ With **Minty's trick** same troubles, $\langle D_z \mathcal{J}(t, z(t)), z'(t) \rangle$ NOT well-defined

Existence of weak solutions for $\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0$

Theorem 2 [Knees-R.-Zanini'13]

Suppose Ω “less smooth” & with **uniform cone** condition

+ conditions on ℓ, f, g

+ $D_z \mathcal{J}(0, z(0)) \in L^2(\Omega)$.

Then, there exists a **WEAK solution**

$$z \in L^{2q}(0, T; W^{1+\beta, q}(\Omega)) \cap L^\infty(0, T; W^{1, q}(\Omega)) \cap W^{1, 2}(0, T; W^{1, 2}(\Omega))$$

fulfilling $z(0) = z_0$ + **energy INequality** for all $0 \leq s \leq t \leq T$

$$\int_s^t \mathcal{R}_\epsilon(z'(r)) \, dr + \int_s^t \mathcal{R}_\epsilon^*(-D_z \mathcal{J}(r, z(r))) \, dr + \mathcal{J}(t, z(t)) \leq \mathcal{J}(s, z(s)) + \int_s^t \partial_t \mathcal{J}(r, z(r)) \, dr$$

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Moreover, **IF** $f(0) \leq f(z)$ and $g(0) \leq g(z)$ for $z \leq 0$, then

$$\boxed{0 \leq z(t, x) \leq 1 \quad \text{for a.a. } (t, x) \in \Omega \times (0, T).}$$

Weak solutions: “almost solutions”...

◆ **Remark:** energy inequality $\forall 0 \leq s \leq t \leq T$

$$\int_s^t \mathcal{R}_\epsilon(z'(r)) \, dr + \int_s^t \mathcal{R}_\epsilon^*(-D_z \mathcal{I}(r, z(r))) \, dr + \mathcal{I}(t, z(t)) \leq \mathcal{I}(s, z(s)) + \int_s^t \partial_t \mathcal{I}(r, z(r)) \, dr$$

is equivalent (via the **WEAK chain rule**) to

$$\begin{aligned} \mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) &\geq_{\mathcal{Z}^*} \langle -A_q z(t), w \rangle_{\mathcal{Z}} + \int_{\Omega} (1 + |\nabla z(t)|^2)^{(q-2)/2} \nabla z(t) \cdot \nabla z'(t) \, dx \\ &\quad - \int_{\Omega} D_z \tilde{\mathcal{I}}(t, z(t))(w - z'(t)) \, dx \\ &\quad \text{for all } w \in \mathcal{Z} \quad \text{for a.a. } t \in (0, T). \end{aligned}$$

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... **almost a solution:** **IF**

$$\int_{\Omega} (1 + |\nabla z(t)|^2)^{(q-2)/2} \nabla z(t) \cdot \nabla z'(t) \, dx =_{z^*} \langle A_q(z(t)), z'(t) \rangle_{\mathcal{Z}},$$

$$\text{we'd get } \mathcal{R}_\epsilon(w) - \mathcal{R}_\epsilon(z'(t)) \geq_{z^*} \langle -D_z I(t, z(t)), w - z'(t) \rangle_{\mathcal{Z}} \quad \forall w \in \mathcal{Z}$$

$$\Rightarrow -D_z I(t, z(t)) \in \partial \mathcal{R}_\epsilon(z'(t))$$

Final remarks

MORE regularity of $\Omega \rightarrow$ high**ER** integrab. for u
 $\downarrow$
ENHANCED estimates for z
 $\downarrow$
BETTER existence results

- $\Omega \in C^{1,1}$ – **smooth** $\Rightarrow \exists$ of a solution to

$$\partial \mathcal{R}_\epsilon(z'(t)) + D_z \mathcal{J}(t, z(t)) \ni 0 \quad \text{in } L^2(\Omega) \quad \text{for a.a. } t \in (0, T)$$

- Ω 'less smooth' $\Rightarrow \exists$ of weak solutions

... in both cases, vanishing viscosity analysis $\epsilon \downarrow 0$ can be done..