

# THE PENROSE-FIFE MODEL WITH DYNAMIC BOUNDARY CONDITIONS

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Kinetics of phase transitions : analysis of a free boundary problem

Different approach : phase-field methodology

A phase function (order parameter)  $\chi$  describes the state of the system

→ The free phase boundary is approximately given by the level curves of  $\chi$

A well studied model : the Caginalp phase-field model (1986)

Based on the Ginzburg-Laundau theory of phase transition and the Ginzburg-Laundau free energy

Other approaches : Alt and Pawlow, Kenmochi and Niezgodka, ...

Penrose-Fife model (1990) :

$$\frac{\partial \chi}{\partial t} = K^*(\kappa \Delta \chi + s'_0(\chi) - \frac{\lambda'(\chi)}{\theta}), \quad K^*, \kappa > 0$$

$$c_0 \frac{\partial \theta}{\partial t} = -K \Delta \frac{1}{\theta} + \lambda'(\chi) \frac{\partial \chi}{\partial t} + h, \quad c_0, K > 0$$

$\chi$  : order parameter

$\theta$  : absolute temperature

$\lambda$  : quadratic function

$-s_0$  : double-well potential

$h$  : external heat source

Aim of Penrose and Fife : derive a thermodynamically consistent phase-field model

The derivation of the model is based on an entropy functional instead of a free energy

Can also be derived based on a proper free energy

The Caginalp system can be derived from the Penrose-Fife system by considering a conserved energy

## Derivation of the model

$\Omega \subset R^3$  : domain occupied by the system (bounded and regular)

$w$  : free energy density

$s$  : entropy density

Depend on  $\chi$  and  $\theta$

Have the same behavior w.r.t.  $\theta$  and  $e$  (energy density) as in the absence of an order parameter : concave w.r.t.  $\theta$  and  $e$

$$w = e - \theta s \text{ (Gibbs' relation)}$$

$$s = -\frac{\partial w}{\partial \theta}$$

$$e = \frac{\partial(\frac{w}{\theta})}{\partial(\frac{1}{\theta})}$$

Total free energy :

$$\psi(\chi, \nabla\chi, \theta) = \int_{\Omega} \left( \frac{\kappa\theta}{2} |\nabla\chi|^2 + w(\theta, \chi) \right) dx, \quad \kappa > 0$$

Entropy functional :

$$S(\chi, \nabla\chi, e) = \int_{\Omega} \left( -\frac{\kappa}{2} |\nabla\chi|^2 + s(e, \chi) \right) dx$$

Equation for  $\chi$  : the entropy is maximal along the solution paths

$$\frac{\partial\chi}{\partial t} = K^* \frac{\delta S}{\delta\chi} = K^* \left( \kappa \Delta\chi + \frac{\partial s}{\partial\chi} \right), \quad K^* > 0$$

Equation for  $\theta$  :

$$\frac{\partial e}{\partial t} = -K\Delta \frac{\delta S}{\delta e} + h = -K\Delta \frac{\partial s}{\partial e} + h, \quad K > 0$$

$h$  : heat source

Typical choices :

$$e = c_0\theta + \lambda(\chi), \quad c_0 > 0$$

$$\lambda(r) = -ar^2 + br + c, \quad a > 0$$

We have

$$e = -\theta^2 \frac{\partial(\frac{w}{\theta})}{\partial\theta}$$

Integrate between  $\theta_0$  and  $\theta > 0$  :

$$w(\theta) = -c_0\theta \ln \theta + c_0\theta \ln \theta_0 + \lambda(\chi) - \theta s_0(\chi)$$

Entropy density :

$$s(e, \chi) = c_0 \ln \theta + s_0(\chi) + c_1, \quad c_1 = -c_0 \ln \theta_0 + c_0$$

$-s_0$  : smooth double-well potential, defined on  $R$ ,  $s_0'' \leq k$ ,  $k \geq 0$

Other possibility :

$$s_0(r) = -kr \ln r - k(1 - r) \ln(1 - r), \quad k > 0$$

Phase transitions in ferromagnetic materials :  $\chi$  is the fraction of lattice sites at which spins are pointing up

$s_0$  : configurational entropy per lattice site in the mean field theory for Ising ferromagnets

We have

$$s(e, \chi) = c_0 \ln\left(\frac{e - \lambda(\chi)}{c_0}\right) + s_0(\chi) + c_1$$

$$\rightarrow \frac{\partial s}{\partial e} = \frac{1}{\theta}, \quad \frac{\partial s}{\partial \chi} = -\frac{\lambda'(\chi)}{\theta} + s'_0(\chi)$$

Equations :

$$\begin{aligned} \frac{\partial \chi}{\partial t} &= K^*(\kappa \Delta \chi + s'_0(\chi) - \frac{\lambda'(\chi)}{\theta}) \\ c_0 \frac{\partial \theta}{\partial t} &= -K \Delta \frac{1}{\theta} - \lambda'(\chi) \frac{\partial \chi}{\partial t} + h \end{aligned}$$

Boundary conditions :

$$\begin{aligned} \frac{\partial \chi}{\partial \nu} &= 0 \text{ on } \Gamma \\ \frac{\partial \frac{1}{\theta}}{\partial \nu} &= \zeta(\theta - \theta_s) \text{ on } \Gamma, \quad \zeta, \theta_s > 0 \end{aligned}$$

$\Gamma = \partial\Omega$ ,  $\nu$  : unit outer normal

Well-posedness, regularity : P. Colli, W. Horn, A. Ito, N. Kenmochi, M. Kubo, Ph. Laurençot, J. Sprekels, S. Zheng, ...

Asymptotic behavior : E. Feireisl, A. Ito, N. Kenmochi, E. Rocca, A. Segatti, G. Schimperna, S. Zelik

Signorini boundary conditions : A. Ito, N. Kenmochi, M. Niezgodka

Dynamic boundary conditions : proposed to account for interactions with the walls in confined systems

Proposed in the context of phase separation

Also relevant in phase transition

Studied in the context of the Caginalp model : L. Cherfils, M. Conti, S. Gatti, M. Grasselli, G. Schimperna

Bulk and surface free energy densities :  $w_b$  and  $w_\Gamma$

Bulk and surface entropy densities :  $s_b$  and  $s_\Gamma$

Generalized free energy and entropy functional :

$$\psi(\chi, \nabla\chi, \nabla_\Gamma\chi, \theta) = \int_\Omega \left( \frac{\kappa_b\theta}{2} |\nabla\chi|^2 + w_b(\theta, \chi) \right) dx \\ + \int_\Gamma \left( \frac{\kappa_\Gamma\theta}{2} |\nabla_\Gamma\chi|^2 + w_\Gamma(\theta, \chi) \right) d\sigma, \quad \kappa_b, \kappa_\Gamma > 0$$

$$S(\chi, \nabla\chi, \nabla_\Gamma\chi, e) = \int_\Omega \left( -\frac{\kappa_b}{2} |\nabla\chi|^2 + s_b(e, \chi) \right) dx \\ + \int_\Gamma \left( -\frac{\kappa_\Gamma}{2} |\nabla_\Gamma\chi|^2 + s_\Gamma(e, \chi) \right) d\sigma$$

$\nabla_\Gamma$  : surface gradient

We assume that

$$\begin{aligned}w_b &= e - \theta s_b, \quad w_\Gamma = e - \theta s_\Gamma \\s_b &= -\frac{\partial w_b}{\partial \theta}, \quad s_\Gamma = -\frac{\partial w_\Gamma}{\partial \theta} \\e_b &= \frac{\partial(\frac{w_b}{\theta})}{\partial(\frac{1}{\theta})}, \quad e_\Gamma = \frac{\partial(\frac{w_\Gamma}{\theta})}{\partial(\frac{1}{\theta})}\end{aligned}$$

Equation for  $\chi$  :

$$\frac{\partial \chi}{\partial t} = K^\star \frac{\delta S}{\delta \chi}, \quad K^\star > 0$$

Now :

$$\begin{aligned}\frac{\delta S}{\delta \chi} &= \kappa_b \Delta \chi + \frac{\partial s_b}{\partial \chi} \text{ in } \Omega \\ \frac{\delta S}{\delta \chi} &= \kappa_\Gamma \Delta_\Gamma \chi + \frac{\partial s_\Gamma}{\partial \chi} - \kappa_b \frac{\partial \chi}{\partial \nu} \text{ on } \Gamma\end{aligned}$$

$\Delta_\Gamma$  : Laplace-Beltrami operator

Equations for  $\chi$  :

$$\begin{aligned}\frac{\partial \chi}{\partial t} &= K^*(\kappa_b \Delta \chi + \frac{\partial s_b}{\partial \chi}) \text{ in } \Omega \\ \frac{\partial \chi}{\partial t} &= K^*(\kappa_\Gamma \Delta_\Gamma \chi + \frac{\partial s_\Gamma}{\partial \chi} - \kappa_b \frac{\partial \chi}{\partial \nu}) \text{ on } \Gamma\end{aligned}$$

Equations for  $\theta$  :

We set, for  $U = \begin{pmatrix} u|_\Omega \\ v|_\Gamma \end{pmatrix}$

$$\mathbf{A}U = \begin{pmatrix} -\Delta u|_\Omega \\ -\Delta_\Gamma v|_\Gamma + \frac{\partial u|_\Omega}{\partial \nu}|_\Gamma \end{pmatrix}$$

We then write

$$\frac{\partial}{\partial t} \begin{pmatrix} e|_\Omega \\ e|_\Gamma \end{pmatrix} = -K\mathbf{A} \begin{pmatrix} \frac{\delta S}{\delta e}|_\Omega \\ \frac{\delta S}{\delta e}|_\Gamma \end{pmatrix} + H, \quad K > 0$$

$H = \begin{pmatrix} h_b \\ h_\Gamma \end{pmatrix}$  : external heat sources

We assume that

$$\begin{aligned}e &= c_{0,b}\theta + \lambda_b(\chi) \text{ in } \Omega, \ c_{0,b} > 0 \\e &= c_{0,\Gamma}\theta + \lambda_\Gamma(\chi) \text{ on } \Gamma, \ c_{0,\Gamma} > 0\end{aligned}$$

We proceed as in the classical derivation :

$$\begin{aligned}s_b &= c_{0,b} \ln \theta + s_{0,b}(\chi) + c_{1,b} \text{ in } \Omega \\s_\Gamma &= c_{0,\Gamma} \ln \theta + s_{0,\Gamma}(\chi) + c_{1,\Gamma} \text{ on } \Gamma\end{aligned}$$

Equations for  $\theta$  :

$$\begin{aligned}c_{0,b} \frac{\partial \theta}{\partial t} &= -K \Delta \frac{1}{\theta} - \lambda'_b(\chi) \frac{\partial \chi}{\partial t} + h_b \text{ in } \Omega \\c_{0,\Gamma} \frac{\partial \theta}{\partial t} &= -K \Delta_\Gamma \frac{1}{\theta} - \lambda'_\Gamma(\chi) \frac{\partial \chi}{\partial t} + K \frac{\partial \frac{1}{\theta}}{\partial \nu} + h_\Gamma \text{ on } \Gamma\end{aligned}$$

Full set of equations :

$$\begin{aligned}\frac{\partial \chi}{\partial t} &= \Delta \chi + s'_{0,b}(\chi) - \frac{\lambda'_b(\chi)}{\theta} \text{ in } \Omega \\ \frac{\partial \chi}{\partial t} &= \Delta_{\Gamma} \chi + s'_{0,\Gamma}(\chi) - \frac{\lambda'_{\Gamma}(\chi)}{\theta} - \frac{\partial \chi}{\partial \nu} \text{ on } \Gamma \\ \frac{\partial \theta}{\partial t} &= -\Delta \frac{1}{\theta} - \lambda'_b(\chi) \frac{\partial \chi}{\partial t} + h_b \text{ in } \Omega \\ \frac{\partial \theta}{\partial t} &= -\Delta_{\Gamma} \frac{1}{\theta} - \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial t} + \frac{\partial \frac{1}{\theta}}{\partial \nu} + h_{\Gamma} \text{ on } \Gamma\end{aligned}$$

Compact form :

$$\begin{aligned}\frac{\partial}{\partial t} \begin{pmatrix} \chi|_{\Omega} \\ \chi|_{\Gamma} \end{pmatrix} &= -\mathbf{A} \begin{pmatrix} \chi|_{\Omega} \\ \chi|_{\Gamma} \end{pmatrix} + \begin{pmatrix} s'_{0,b}(\chi) \\ s'_{0,\Gamma}(\chi) \end{pmatrix} - \begin{pmatrix} \frac{\lambda'_b(\chi)}{\theta} \\ \frac{\lambda'_{\Gamma}(\chi)}{\theta} \end{pmatrix} \\ \frac{\partial}{\partial t} \begin{pmatrix} \theta|_{\Omega} \\ \theta|_{\Gamma} \end{pmatrix} &= \mathbf{A} \begin{pmatrix} \frac{1}{\theta}|_{\Omega} \\ \frac{1}{\theta}|_{\Gamma} \end{pmatrix} + \begin{pmatrix} \lambda'_b(\chi) \frac{\partial \chi}{\partial t} \\ \lambda'_{\Gamma}(\chi) \frac{\partial \chi}{\partial t} \end{pmatrix} + H\end{aligned}$$

## Notation and assumptions

- $\Omega$  : bounded regular domain of  $R^3$ ,  $\Gamma = \partial\Omega$ ,  $\overline{\Omega} = \Omega \cup \Gamma$

## Function spaces

- $H = L^2(\Omega)$ ,  $((\cdot, \cdot))$ ,  $\|\cdot\|$
- $H_\Gamma = L^2(\Gamma)$ ,  $((\cdot, \cdot))_\Gamma$ ,  $\|\cdot\|_\Gamma$
- $V = H^1(\Omega)$ ,  $V_\Gamma = H^1(\Gamma)$
- $\|\cdot\|_X$  : norm on  $X$
- $\mathcal{H} = H \times H_\Gamma$ ,  $((k_1, \kappa_1), (k_2, \kappa_2))_{\mathcal{H}} = ((k_1, k_2)) + ((\kappa_1, \kappa_2))_\Gamma$ ,  $\|\cdot\|_{\mathcal{H}}$

- $\mathcal{V} = \{z \in V, z_\Gamma \in V_\Gamma\}, z_\Gamma : \text{trace on } \Gamma$
- $$((z, w))_{\mathcal{V}} = ((\nabla z, \nabla w)) + ((\nabla_\Gamma z_\Gamma, \nabla_\Gamma w_\Gamma))_\Gamma + ((z_\Gamma, w_\Gamma))_\Gamma, \|\cdot\|_{\mathcal{V}}$$
- $$((z, w))_V = ((\nabla z, \nabla w)) + ((z_\Gamma, w_\Gamma))_\Gamma, \|\cdot\|_V$$

### Remarks :

- $\mathcal{V}$  is dense in  $\mathcal{H}$
- The  $V$ -norm is equivalent to the usual  $H^1$ -norm

- $\mathcal{W} = \{z \in H^2(\Omega), z_\Gamma \in H^2(\Gamma)\}$

### Convention

When considering equations in  $\Omega$  : elements of  $\mathcal{V}$  are functions defined in  $\Omega$

When considering a system on  $\Omega \times \Gamma$  : pairs of functions  $(z, z_\Gamma)$  ( $\mathcal{V}$  is a closed subset of  $H^1(\Omega) \times H^1(\Gamma)$ )

$V$  can be seen as a subspace of  $H^1(\Omega) \times H^{\frac{1}{2}}(\Gamma)$

$$\rightarrow \mathcal{V} \subset V \subset \mathcal{H} = \mathcal{H}' \subset V' \subset \mathcal{V}'$$

## Elliptic operators

- $A : V \rightarrow V', \langle Az_1, z_2 \rangle = ((\nabla z_1, \nabla z_2))$
- $A_\Gamma : V_\Gamma \rightarrow V'_\Gamma, \langle A_\Gamma \kappa_1, \kappa_2 \rangle_\Gamma = ((\nabla_\Gamma \kappa_1, \nabla_\Gamma \kappa_2))_\Gamma$
- $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}', \langle \mathcal{A}z_1, z_2 \rangle_{\mathcal{V}} = \langle Az_1, z_2 \rangle + \langle A_\Gamma z_{1,\Gamma}, z_{2,\Gamma} \rangle$

**Remark :** The operator  $\mathbf{A}$  can be seen as a linear operator from  $\mathcal{W}$  onto  $\mathcal{H}$

## Other notation

- $\int_{\overline{\Omega}} z dm = \int_{\Omega} z dx + \int_{\Gamma} z_\Gamma d\sigma$

## Assumptions on the nonlinear terms

Most complicated case :  $-s_{0,b}$  singular

$\rightarrow -s_{0,\Gamma}$  singular

For simplicity :  $(-1, 1)$  domain of  $-s_{0,b}$  and  $-s_{0,\Gamma}$

- $-s'_{0,b}(r) = f_b(r) - \delta r$ ,  $-s'_{0,\Gamma}(r) = f_\Gamma(r) - \delta r$ ,  $f_b, f_\Gamma$  monotone,  $\delta \geq 0$
- $f_b, f_\Gamma \in \mathcal{C}^0(-1, 1)$ ,  $f_b(0) = f_\Gamma(0) = 0$
- $\lim_{\pm 1} f_b = \pm\infty$ ,  $\lim_{\pm 1} f_\Gamma = \pm\infty$
- Compatibility assumption :

$$f_b(r)f_\Gamma(r) \geq c|f_b(r)|^2 - c', \quad r \in (-1, 1), \quad c > 0, \quad c' \geq 0$$

We set  $F_b(r) = \int_0^r f_b(\tau) d\tau$ ,  $F_\Gamma(r) = \int_0^r f_\Gamma(\tau) d\tau$

We extend  $F_b$  and  $F_\Gamma$  to  $R$  by  $+\infty$  outside their effective domain

Relevant logarithmic potential :  $F_b$  is bounded and can be extended to  $[-1, 1]$  by continuity

We can identify  $f$  and  $f_\Gamma$  with maximal monotone graphs in  $R \times R$

$\rightarrow f_b = \partial F_b, f_\Gamma = \partial F_\Gamma$

- $\lambda_b, \lambda_\Gamma \in \mathcal{C}^2([-1, 1])$ ,  $\lambda_b'', \lambda_\Gamma'' \in L^\infty(-1, 1)$

## Assumptions on the initial data

- $(\theta_0, \eta_0) \in \mathcal{H}$ ,  $\theta_0, \eta_0 > 0$  a.e.,  $(\ln \theta_0, \ln \eta_0) \in L^1(\Omega) \times L^1(\Gamma)$

( $\eta_0$  need not coincide with  $\theta_0$ )

- $\chi_0 \in \mathcal{V}$ ,  $s_{0,b}(\chi_0) \in L^1(\Omega)$ ,  $s_{0,\Gamma}(\chi_{0,\Gamma}) \in L^1(\Gamma)$

( $\chi_0$  has a trace)

- $H = 0$

We introduce the energy

$$\begin{aligned}\mathcal{E}((\theta, \eta), (\chi, \chi_\Gamma)) & (= \mathcal{E}(\theta, \chi)) \\ &= \int_{\Omega} (\theta - \ln \theta + \lambda_b(\chi) + \frac{1}{2} |\nabla \chi|^2 - s_{0,b}(\chi)) dx \\ &+ \int_{\Gamma} (\eta - \ln \eta + \lambda_{\Gamma}(\chi_{\Gamma}) + \frac{1}{2} |\nabla_{\Gamma} \chi_{\Gamma}|^2 - s_{0,\Gamma}(\chi_{\Gamma})) d\sigma\end{aligned}$$

( $\mathcal{E}$  can take the value  $+\infty$ )

$\mathcal{E}$  is coercive :

$$\mathcal{E}(\theta, \chi) \geq c \|\chi\|_{\mathcal{V}}^2 + c' (\|\ln \theta\|_{L^1(\Omega)} + \|\ln \eta\|_{L^1(\Gamma)}) - c'', \quad c, c' > 0, c'' \geq 0$$

**Remark :** For regular potentials, we need further coercivity assumptions on  $\lambda_b$ ,  $\lambda_{\Gamma}$ ,  $s_{0,b}$  and  $s_{0,\Gamma}$

- $\mathcal{E}_0 = \mathcal{E}(\theta_0, \chi_0) < +\infty$

## Existence of weak solutions

**Definition :**  $(\theta, \eta, u, \chi)$  is a weak solution if

- (i)  $(\theta, \eta) \in H^1(0, T; \mathcal{V}') \cap L^\infty(0, T; \mathcal{H})$
  - (ii)  $(\ln \theta, \ln \eta) \in L^\infty(0, T; L^1(\Omega) \times L^1(\Gamma))$
  - (iii)  $\theta, \eta > 0$  a.e.
  - (iv)  $u \in L^2(0, T; \mathcal{V})$
  - (v)  $\chi \in H^1(0, T; \mathcal{H}) \cap L^\infty(0, T; \mathcal{V}) \cap L^2(0, T; \mathcal{W})$
  - (vi)  $f_b(\chi) \in L^2(0, T; H), f_\Gamma(\chi_\Gamma) \in L^2(0, T; H_\Gamma)$
  - (vii)  $\frac{d}{dt} \begin{pmatrix} \theta \\ \eta \end{pmatrix} = -\mathcal{A} \begin{pmatrix} u \\ u_\Gamma \end{pmatrix} - \begin{pmatrix} \lambda'_b(\chi) \frac{\partial \chi}{\partial t} \\ \lambda'_\Gamma(\chi_\Gamma) \frac{\partial \chi_\Gamma}{\partial t} \end{pmatrix} \text{ in } \mathcal{V}'$
- $u = -\frac{1}{\theta}$  a.e in  $\Omega, \eta = -\frac{1}{\eta}$  a.e. on  $\Gamma$

$$\frac{d}{dt} \begin{pmatrix} \chi \\ \chi_\Gamma \end{pmatrix} = -\mathcal{A} \begin{pmatrix} \chi \\ \chi_\Gamma \end{pmatrix} + \begin{pmatrix} s'_{0,b}(\chi) \\ s'_{0,\Gamma}(\chi_\Gamma) \end{pmatrix} + \begin{pmatrix} \lambda'_b(\chi) u \\ \lambda'_\Gamma(\chi_\Gamma) u_\Gamma \end{pmatrix} \text{ in } \mathcal{H}$$

(viii)  $(\theta, \eta)|_{t=0} = (\theta_0, \eta_0)$ ,  $(\chi, \chi_\Gamma)|_{t=0} = (\chi_0, \chi_{0,\Gamma})$  a.e.

For simplicity, we denote  $(\theta, \chi)$  a (weak) solution

$\chi$  is smooth enough to have a trace

$u$  also has a trace  $u_\Gamma$

We cannot a priori write  $\eta = \theta_\Gamma : \eta = -\frac{1}{u_\Gamma}$

**Theorem :** The problem possesses a unique weak solution s.t.

$$\mathcal{E}(\theta(t), \chi(t)) + \int_0^t (\|\nabla u\|_{\mathcal{H}}^2 + \|\frac{\partial \chi}{\partial t}\|_{\mathcal{H}}^2) ds = \mathcal{E}_0, \quad t > 0$$

Furthermore,  $\forall \tau \in (0, T)$

$$\begin{aligned} \|u\|_{L^\infty(\tau, T; V)} + \|u\|_{L^\infty((\tau, T) \times \Omega)} &\leq Q(\mathcal{E}_0, \tau^{-1}) \\ \|\chi\|_{L^\infty(\tau, T; H^2(\Omega))} + \|f_b(\chi)\|_{L^\infty(\tau, T; H)} &\leq Q(\mathcal{E}_0, \tau^{-1}) \\ \|u_\Gamma\|_{L^\infty(\tau, T; V_\Gamma)} + \|u_\Gamma\|_{L^\infty((\tau, T) \times \Gamma)} &\leq Q(\mathcal{E}_0, \tau^{-1}) \\ \|\chi_\Gamma\|_{L^\infty(\tau, T; H^2(\Gamma))} + \|f_\Gamma(\chi_\Gamma)\|_{L^\infty(\tau, T; H_\Gamma)} &\leq Q(\mathcal{E}_0, \tau^{-1}) \end{aligned}$$

**Uniqueness : Standard**

Existence :

- Approximation procedure
- One obtains the energy equation

$$\mathcal{E}(\theta(t), \chi(t)) + \int_0^t (\|\nabla u\|_{\mathcal{H}}^2 + \|\frac{\partial \chi}{\partial t}\|_{\mathcal{H}}^2) ds = \mathcal{E}_0$$

→ Not sufficient to estimate the full  $\mathcal{V}$ -norm of  $u = -\frac{1}{\theta}$

We derive a bound on the  $L^1$ -norm by using the

**Lemma :** (Schimperna, Segatti, Zelik)  $\forall v \in W^{1,1}(\Omega), v \geq 0$  a.e. in  $\Omega$

$$\|v\|_{L^1(\Omega)} \leq |\Omega| e^c \int_{\Omega} (\ln v)^+ dx + \frac{c'}{|\Omega|} \|\nabla v\|_{L^1(\Omega)}$$

Take  $v = -\frac{1}{\theta}$  :

$$\left\| \frac{1}{\theta} \right\|_{L^1(\Omega)} \leq |\Omega| e^c \int_{\Omega} (\ln \theta)^{-} dx + \frac{c'}{|\Omega|} \left\| \nabla \frac{1}{\theta} \right\|_{L^1(\Omega)}$$

Use the energy equation, the coercivity of  $\mathcal{E}$  and the trace theorem ( $\left\| \frac{1}{\eta} \right\|_{V_{\Gamma}} \leq c(\left\| \frac{1}{\theta} \right\|_V + \left\| \nabla_{\Gamma} \frac{1}{\eta} \right\|_{\Gamma})$ )

→ We obtain the full  $\mathcal{V}$ -norm

**Remark :** Formally,  $\eta = \theta_{\Gamma}$

- Estimate on the nonlinear terms : use the compatibility assumption to obtain

$$\|f_b(\chi)\|_{L^2(0,T;H)} + \|f_{\Gamma}(\chi_{\Gamma})\|_{L^2(0,T;H_{\Gamma})} \leq Q(\mathcal{E}_0, T)$$

- Passage to the limit : monotonicity arguments for  $f_b, f_{\Gamma}$  and  $v \mapsto -\frac{1}{v}$  (and its realization in  $L^2(0, T; \mathcal{H})$ )

- Further regularity : further a priori estimates and Moser iterations technique

## Additional regularity

Further compatibility assumption :

$$|f_{\Gamma}(r)| \geq c|f_b(r)| - c', \quad r \in (-1, 1), \quad c \in (0, 1), \quad c' \geq 0$$

**Theorem :**  $\forall \tau > 0, \exists \omega \in (0, 1)$ , depending on  $\tau$ , but independent of  $T$ , s.t.

$$|\chi(t, x)| \leq 1 - \omega \text{ a.e. in } (\tau, T) \times \Omega \text{ and } (\tau, T) \times \Gamma$$

Based on the comparison principle : we consider the ODE

$$y' + cf_b(y) = M + c', \quad y(\tau) = 1 - \xi, \quad \xi \in (0, 1)$$

$M$  is s.t.

$$\|\delta\chi + \lambda'_b(\chi)u\|_{L^\infty((\tau, T) \times \Omega)} + \|\delta\chi_{\Gamma} + \lambda'_{\Gamma}(\chi_{\Gamma})u_{\Gamma}\|_{L^\infty((\tau, T) \times \Gamma)} \leq M$$

We can prove that  $\exists \omega = \omega(\tau) \in (0, 1)$  s.t.

$$y(t) \leq 1 - \omega, \quad \forall t \geq 2\tau, \quad \forall \xi \in (0, 1)$$

$\omega$  independent of  $\xi$

**Theorem :** We further assume that  $(\theta_0, \eta_0) \in L^{3+\epsilon}(\Omega) \times L^{3+\epsilon}(\Gamma)$ ,  $\epsilon > 0$ .  
Then,  $\forall \tau \in (0, T)$

$$\|\theta\|_{L^\infty((\tau, T) \times \Omega)} + \|\eta\|_{L^\infty((\tau, T) \times \Gamma)} \leq Q(\mathcal{E}_0, \tau^{-1}, \|\theta_0\|_{L^{3+\epsilon}(\Omega)}, \|\eta_0\|_{L^{3+\epsilon}(\Gamma)})$$

Furthermore,  $\theta \in H^1(\tau, T; \mathcal{H})$

**Remarks :**

- a) The compatibility condition can be replaced by  $\chi_0 \in L^\infty(\overline{\Omega}, dm)$
- b) We have, a.e.  $t > 0$  and in  $\mathcal{H}$

$$\frac{d}{dt} \begin{pmatrix} \theta \\ \theta_\Gamma \end{pmatrix} = \mathbf{A} \begin{pmatrix} \frac{1}{\theta} \\ \frac{1}{\theta_\Gamma} \end{pmatrix} - \begin{pmatrix} \lambda'_b(\chi) \frac{\partial \chi}{\partial t} \\ \lambda'_\Gamma(\chi_\Gamma) \frac{\partial \chi_\Gamma}{\partial t} \end{pmatrix}$$

and  $\theta_\Gamma = \eta$  a.e.

## Asymptotic behavior

We assume that  $(\theta_0, \eta_0) \in L^{3+\epsilon}(\Omega) \times L^{3+\epsilon}(\Gamma)$ ,  $\epsilon > 0$

We have the conservation of the total internal energy :

$$\mu = \mu(\theta, \chi) = \int_{\Omega} (\theta + \lambda_b(\chi)) dx + \int_{\Gamma} (\eta + \lambda_{\Gamma}(\chi_{\Gamma})) d\sigma$$

## $\omega$ -limit sets

Stationary problem associated with the temperature :  $u = -\frac{1}{\theta} = \text{Constant}$

Stationary problem :

$$\begin{aligned} \theta_\infty &\in (0, +\infty), \quad u = -\frac{1}{\theta_\infty} \\ -\Delta \chi_\infty + f_b(\chi_\infty) - \delta \chi_\infty &= \lambda'_b(\chi_\infty) u_\infty \text{ in } \Omega \\ -\Delta_\Gamma \chi_{\infty, \Gamma} + f_\Gamma(\chi_{\infty, \Gamma}) - \delta \chi_{\infty, \Gamma} &= \lambda'_\Gamma(\chi_{\infty, \Gamma}) u_\infty - \frac{\partial \chi_\infty}{\partial \nu} \text{ on } \Gamma \\ \int_\Omega (\theta_\infty + \lambda_b(\chi_\infty)) dx + \int_\Gamma (\theta_\infty + \lambda_\Gamma(\chi_{\Gamma, \infty})) d\sigma &= \mu \end{aligned}$$

**Theorem :** Let  $(\theta, \chi)$  be a weak solution. Then, as  $t \rightarrow +\infty$ ,  $(u(t), \chi(t))$  is precompact in  $\mathcal{H} \times \mathcal{V}$ . Furthermore, any limit point  $(\theta_\infty, \chi_\infty)$  is solution to the stationary problem (for  $\mu = \mu(\theta_0, \chi_0)$ )

## Global attractors

Phase space :

$$\begin{aligned}\mathcal{X}_\mu &= \{(\theta, \chi) \in L^{3+\epsilon}(\Omega) \times L^{3+\epsilon}(\Gamma)) \times \mathcal{V}, \\ \theta &> 0 \text{ } dm - \text{a.e. in } \overline{\Omega}, \ln \theta \in L^1(\overline{\Omega}, dm), \\ F_b(\chi) &\in L^1(\Omega), F_\Gamma(\chi_\Gamma) \in L^1(\Gamma) \\ \int_\Omega (\theta_\infty + \lambda_b(\chi_\infty)) dx &+ \int_\Gamma (\theta_\infty + \lambda_\Gamma(\chi_{\Gamma,\infty})) d\sigma = \mu\}\end{aligned}$$

Distance :

$$\begin{aligned}\text{dist}_\mathcal{X}((\theta_1, \chi_1), (\theta_2, \chi_2)) &= \|\theta_1 - \theta_2\|_{L^{3+\epsilon}(\overline{\Omega}, dm)} + \|\chi_1 - \chi_2\|_\mathcal{V} \\ &+ \|\ln \theta_1 - \ln \theta_2\|_{L^1(\overline{\Omega}, dm)} + \|F_b(\chi_1) - F_b(\chi_2)\|_{L^1(\Omega)} \\ &+ \|F_\Gamma(\chi_{1,\Gamma}) - F_\Gamma(\chi_{2,\Gamma})\|_{L^1(\Gamma)}\end{aligned}$$

$(\mathcal{X}_\mu, \text{dist}_\mathcal{X})$  is a complete metric space

**Theorem :** We assume that

$$\mu > |\Omega| \max_{r \in [-1,1]} \lambda_b(r) + |\Gamma| \max_{r \in [-1,1]} \lambda_\Gamma(r)$$

Then, the dynamical system acting on  $\mathcal{X}_\mu$  possesses the global attractor  $\mathcal{A}_\mu$ .  
Furthermore,  $\exists c > 0, \omega \in (0, 1), \alpha \in (0, 1)$ , depending only on  $\mu$ , s.t.

- (i)  $\|\theta\|_{\mathcal{V}} + \|\chi\|_{\mathcal{W}} \leq c$
- (ii)  $\alpha \leq \theta \leq \alpha^{-1} \, dm\text{-a.e. in } \overline{\Omega}$
- (iii)  $-1 + \omega \leq \chi \leq 1 - \omega \, dm\text{-a.e. in } \overline{\Omega}$

**Remark :** The assumption implies that  $\mathcal{X}_\mu$  is not empty. It can be replaced by

$$\liminf_{|r| \rightarrow 1} \lambda'(r) \operatorname{sgn}(r) > 0, \, \lambda = \lambda_b \text{ and } \lambda_\Gamma$$

We prove :

- The existence of a coercive Lyapunov functional (the energy  $\mathcal{E}$ )
- That trajectories are precompact
- That the set of stationary states is bounded independently of the size of the initial data