

Existence of the free boundary in a diffusive flow in porous media

Gabriela Marinoschi

*Institute of Mathematical Statistics and Applied
Mathematics of the Romanian Academy, Bucharest*

1

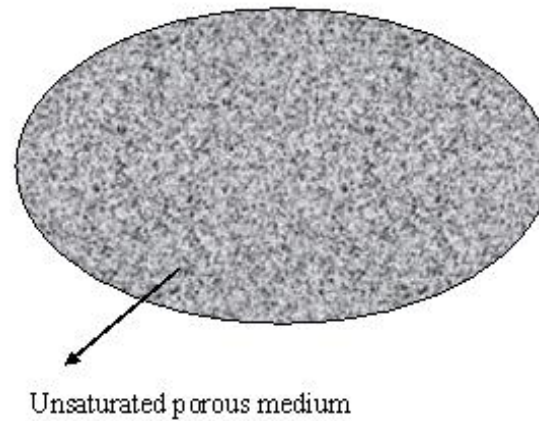
Problem presentation

*Prove the existence of the solution to a two phase flow
in a porous medium*

Compute the solution and the diffusive interface evolution

G.M., J. Optimiz. Theory Appl., 2012

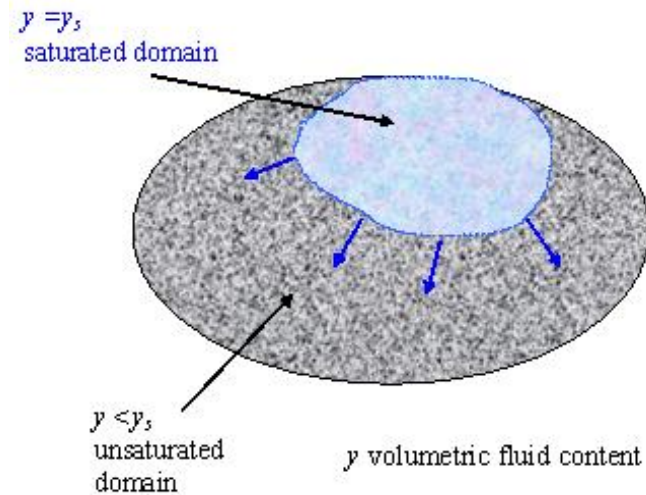
Problem presentation



$\Omega \subset \mathbb{R}^3$, open bounded,

$\Gamma = \partial\Omega$ sufficiently smooth

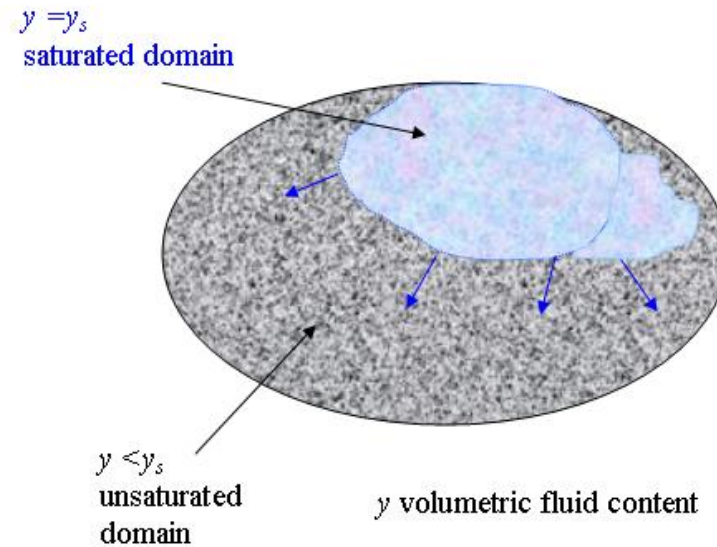
Problem presentation



$$Q = (0, T) \times \Omega, \quad T < \infty, \quad \Sigma = (0, T) \times \Gamma$$

$$Q_s = \{(t, x) \in Q; y(t, x) = y_s\} \quad Q_u = \{(t, x) \in Q; y(t, x) < y_s\}$$

Problem presentation



$$Q = (0, T) \times \Omega, \quad T < \infty, \quad \Sigma = (0, T) \times \Gamma$$

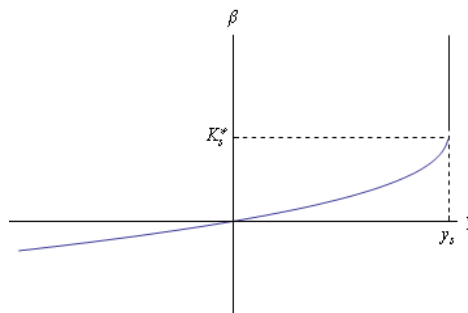
$$Q_s = \{(t, x) \in Q; y(t, x) = y_s\} \quad Q_u = \{(t, x) \in Q; y(t, x) < y_s\}$$

Problem presentation

$$\begin{aligned} \frac{\partial y}{\partial t} - \Delta \beta(t, x, y) &\ni f && \text{in } Q \\ \frac{\partial \beta(t, x, y)}{\partial \nu} + \alpha \beta(t, x, y) &\ni 0 && \text{on } \Sigma, \quad \alpha > 0 \\ y(0, x) &= y_0 && \text{in } \Omega. \end{aligned} \quad (NE)$$

Problem presentation

$$\begin{aligned}
 \frac{\partial y}{\partial t} - \Delta \beta(t, x, y) &\ni f && \text{in } Q \\
 \frac{\partial \beta(t, x, y)}{\partial \nu} + \alpha \beta(t, x, y) &\ni 0 && \text{on } \Sigma, \quad \alpha > 0, \\
 y(0, x) &= y_0 && \text{in } \Omega.
 \end{aligned}
 \tag{NE}$$



$$\beta : Q \times (-\infty, y_s] \rightarrow \mathbb{R}, \quad \beta(t, x, \cdot) \in C^1(-\infty, y_s), \quad \lim_{r \nearrow y_s} \beta(t, x, r) = K_s^*, \quad \text{a.e. } (t, x) \in Q.$$

Fast diffusion in porous media (G.M. Springer, 2006; A. Favini, G.M. Springer 2012)

Problem presentation

$$j : Q \times \mathbb{R} \rightarrow (-\infty, \infty], \quad j(t, x, r) = \begin{cases} \int_0^r \overset{\circ}{\beta}(t, x, s) ds, & r \leq y_s, \\ +\infty, & \text{otherwise} \end{cases}$$

$(t, x) \rightarrow j(t, x, r)$ is measurable on Q for all $r \in (-\infty, y_s]$

$j(t, x, \cdot)$ proper convex l.s.c. a.e. $(t, x) \in Q$

$\partial j(t, x, \cdot) = \beta(t, x, \cdot)$ a.e. $(t, x) \in Q$

$j(t, x, r) \leq K_s^* |r|$, for any $r \leq y_s$, a.e. $(t, x) \in Q$

Problem presentation

Conjugate of j

$$j^*(t, x, \omega) = \sup_{r \in \mathbb{R}} (\omega r - j(t, x, r)) \text{ a.e. } (t, x) \in Q$$

Legendre-Fenchel relations

$$j(t, x, r) + j^*(t, x, \omega) \geq r\omega \text{ for all } r \in \mathbb{R}, \omega \in \mathbb{R}, \text{ a.e. } (t, x) \in Q,$$

$$j(t, x, r) + j^*(t, x, \omega) = r\omega \text{ if and only if } \omega \in \partial j(t, x, r), \text{ a.e. } (t, x) \in Q.$$

Problem presentation

Conjugate of j

$$j^*(t, x, \omega) = \sup_{r \in \mathbb{R}} (\omega r - j(t, x, r)) \text{ a.e. } (t, x) \in Q$$

Legendre-Fenchel relations

$$j(t, x, r) + j^*(t, x, \omega) \geq r\omega \text{ for all } r \in \mathbb{R}, \omega \in \mathbb{R}, \text{ a.e. } (t, x) \in Q,$$

$$j(t, x, r) + j^*(t, x, \omega) = r\omega \text{ if and only if } \omega \in \partial j(t, x, r), \text{ a.e. } (t, x) \in Q.$$

$$C_3 |\omega| + C_3^0 \leq j^*(t, x, \omega) \text{ for any } \omega \in \mathbb{R}, \text{ a.e. } (t, x) \in Q, \quad C_3 > 0.$$

Problem presentation

$$\begin{aligned}\frac{\partial y}{\partial t} - \Delta \beta(t, x, y) &\ni f && \text{in } Q, \\ \frac{\partial \beta(t, x, y)}{\partial \nu} + \alpha \beta(t, x, y) &\ni 0 && \text{on } \Sigma, \\ y(0, x) &= y_0 && \text{in } \Omega.\end{aligned}\tag{NE}$$

j singular potential, j^* minimal growth conditions, no time and space regularity

Problem presentation

$$\begin{aligned}
 \frac{\partial y}{\partial t} - \Delta \beta(t, x, y) &\ni f && \text{in } Q \\
 \frac{\partial \beta(t, x, y)}{\partial \nu} + \alpha \beta(t, x, y) &\ni 0 && \text{on } \Sigma \\
 y(0, x) &= y_0 && \text{in } \Omega.
 \end{aligned}
 \tag{NE}$$

j singular potential, j^* minimal growth conditions, no time and space regularity



Minimize $J(y, w)$ for all $(y, w) \in U$ (P)

connected with (NE) by j and j^*

¹ H. Brezis, I. Ekeland, C.R. Acad. Sci. Paris, 282, 971–974, 1197–1198, 1976

Problem presentation

- ▶ H. Brezis, I. Ekeland (C.R. Acad. Sci. Paris, 1976)
- ▶ G. Auchmuty (NA, 1988)
- ▶ N. Ghoussoub, L. Tzou (Math. Ann. 2004)
- ▶ N. Ghoussoub (Springer, 2009)
- ▶ A. Visintin (Adv. Math. Sci. Appl. 2008)
- ▶ U. Stefanelli (SICON 2008, J. Convex Analysis 2009)
- ▶ V. Barbu (JMAA 2011)
- ▶ G. M. (JOTA 2012, 2013)

Problem presentation

Definition. Let $f \in L^1(Q)$, $y_0 \in L^\infty(\Omega)$, $y_0 \leq y_s$ a.e. on Ω .

A **weak solution** to (NE) is a pair (y, w)

$$y \in L^\infty(Q), \quad w \in (L^\infty(Q))', \quad (w = w_a + w_s, \quad w_a \in L^1(Q))$$

$$w_a(t, x) \in \partial j(t, x, y) = \beta(t, x, y) \text{ a.e. on } Q, \quad w_s \in N_{D(\varphi)}(y)$$

$$-\int_Q y \frac{d\psi}{dt} dx dt + \int_\Omega y_0 \psi(0) + \langle w, A_{0,\infty} \psi \rangle_{(L^\infty(Q))', L^\infty(Q)} = \int_Q f \psi dx dt \quad (*)$$

for any $\psi \in W^{1,1}([0, T]; L^1(\Omega)) \cap L^\infty(0, T; D(A_{0,\infty}))$, $\psi(T) = 0$.

$$A_{0,\infty} \psi = -\Delta \psi, \quad A_{0,\infty} : D(A_{0,\infty}) \subset L^\infty(\Omega) \rightarrow L^\infty(\Omega),$$

$$D(A_{0,\infty}) = \{\psi \in W^{2,\infty}(\Omega); \frac{\partial \psi}{\partial \nu} + \alpha \psi = 0 \text{ on } \Gamma\}.$$

2 **A duality approach**

A duality approach

$$J : L^\infty(Q) \times L^1(Q)$$

$$\text{Min}_{(y,w) \in U} \left\{ \int_Q (j(t, x, y(t, x)) + j^*(t, x, w(t, x))) dx dt - y(t, x)w(t, x) dx dt \right\}$$

$$U = \left\{ (y, w); y \in L^\infty(Q), y(t, x) \leq y_s \text{ a.e.}, w \in L^1(Q), \frac{dy}{dt} - \Delta w = f, y(0) = y_0 \right\}$$

A duality approach

$$J : L^\infty(Q) \times (L^\infty(Q))'$$

$$J(y, w) = \int_Q j(t, x, y(t, x)) dx dt + \varphi^*(t, x, w) - w(y)$$

A duality approach

$$w \in (L^\infty(Q))', \quad w = w_a + w_s$$

$L^1(Q) \ni w_a =$ "absolutely continuous component", $w_s =$ singular component

A duality approach

$$\varphi : L^\infty(Q) \rightarrow (-\infty, \infty] \quad \varphi(y) = \int_Q j(t, x, y(t, x)) dx dt \text{ proper, convex, lsc}$$

$$\varphi^* : (L^\infty(Q))' \rightarrow (-\infty, \infty] \quad \varphi^*(w) = \int_Q j^*(t, x, w_a(t, x)) dx dt + \delta_{D(\varphi)}^*(w_s)$$

² R.T. Rockafeller, Pacific J. Math. J, 39, 2, 1971

A duality approach

$$\varphi : L^\infty(Q) \rightarrow (-\infty, \infty] \quad \varphi(y) = \int_Q j(t, x, y(t, x)) dx dt \text{ proper, convex, lsc}$$

$$\varphi^* : (L^\infty(Q))' \rightarrow (-\infty, \infty] \quad \varphi^*(w) = \int_Q j^*(t, x, w_a(t, x)) dx dt + \delta_{D(\varphi)}^*(w_s)$$

$$\delta_{D(\varphi)}^*(v) = \sup\{v(\psi); \psi \in D(\varphi)\} = \text{conjugate of the indicator function of } D(\varphi)$$

$$D(\varphi) = \{y \in L^\infty(Q); \varphi(y) < +\infty\} = \{y \in L^\infty(Q); y(t, x) \leq y_s \text{ a.e.}\}.$$

³ R.T. Rockafeller, Pacific J. Math. J, 39, 2, 1971

A duality approach

$$J : L^\infty(Q) \times (L^\infty(Q))'$$

$$J(y, w) = \int_Q j(t, x, y(t, x)) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) - w(y)$$

A duality approach

$$J : L^\infty(Q) \times (L^\infty(Q))'$$

$$J(y, w) = \int_Q j(t, x, y(t, x)) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) - w(y)$$

In some good cases we have a Lemma of integration by parts:

$$-w(y) = \frac{1}{2} \|y(T)\|_{(H^1(\Omega))'}^2 - \frac{1}{2} \|y_0\|_{(H^1(\Omega))'}^2 - \int_Q f A_{0,\infty}^{-1} y dx dt \quad (**)$$

A duality approach

$$J : L^\infty(Q) \times (L^\infty(Q))'$$

$$J(y, w) = \int_Q j(t, x, y(t, x)) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) - w(y)$$

In some good cases we have a Lemma of integration by parts:

$$\left. \frac{dy}{dt} - \Delta w = f \right| \cdot \Delta^{-1} y \quad (**)$$

A duality approach

Minimize (J) for all $(y, w) \in U$ (P)

$$J(y, w) = \begin{cases} \int_Q j(t, x, y(t, x)) dx dt + \int_Q j^*(t, x, w_a(t, x)) dx dt + \delta_{D(\varphi)}^*(w_s) \\ + \frac{1}{2} \|y(T)\|_{(H^1(\Omega))'}^2 - \frac{1}{2} \|y_0\|_{(H^1(\Omega))'}^2 - \int_Q f A_{0,\infty}^{-1} y dx dt & \text{if } (y, w) \in U, \end{cases}$$

A duality approach

Minimize (J) for all $(y, w) \in U$ (P)

$$J(y, w) = \begin{cases} \int_Q j(t, x, y(t, x)) dx dt + \int_Q j^*(t, x, w_a(t, x)) dx dt + \delta_{D(\varphi)}^*(w_s) \\ + \frac{1}{2} \|y(T)\|_{(H^1(\Omega))'}^2 - \frac{1}{2} \|y_0\|_{(H^1(\Omega))'}^2 - \int_Q f A_{0,\infty}^{-1} y dx dt & \text{if } (y, w) \in U, \end{cases}$$

$$U = \left\{ (y, w); y \in L^\infty(Q), y(t, x) \in [0, y_s] \text{ a.e., } y(T) \in (H^1(\Omega))', w \in (L^\infty(Q))', \right. \\ \left. (y, w) \text{ verifies } (*) \right\}$$

$$- \int_Q y \frac{d\psi}{dt} dx dt + \int_\Omega y_0 \psi(0) + \langle w, A_{0,\infty} \psi \rangle_{(L^\infty(Q))', L^\infty(Q)} = \int_Q f \psi dx dt \quad (*)$$

for any $\psi \in W^{1,1}([0, T]; L^1(\Omega)) \cap L^\infty(0, T; D(A_{0,\infty}))$, $\psi(T) = 0$.

A duality approach

Theorem

J is proper, convex, l.s.c. on $L^\infty(Q) \times (L^\infty(Q))'$.

Problem (P) has at least a solution (y, w) .

⁴ Alaoglu theorem

A duality approach

Theorem

J is proper, convex, l.s.c. on $L^\infty(Q) \times (L^\infty(Q))'$.

Problem (P) has at least a solution (y, w) .

*A solution to (P) is called a *generalized solution* to (NE)*

⁵ Alaoglu theorem

A duality approach

Theorem

Let (y, w) be the null minimizer in (P) , i.e., $J(y, w) = 0$.

Assume that

$$\frac{1}{2} \|y(T)\|_{(H^1(\Omega))'}^2 - \frac{1}{2} \|y_0\|_{(H^1(\Omega))'}^2 - \int_Q f A_{0,\infty}^{-1} y dx dt = -w(y). \quad (**)$$

Then, (y, w) is the weak solution to (NE) , i.e.,

$$w_a(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q$$

and

$$w_s \in N_{D(\varphi)}(y).$$

A duality approach

Proof sketch

$$\begin{aligned} J(y, w) = & \int_Q j(t, x, y) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) \\ & + \frac{1}{2} \|y(T)\|_{(H^1(\Omega))'}^2 - \frac{1}{2} \|y_0\|_{(H^1(\Omega))'}^2 - \int_Q f A_{0,\infty}^{-1} y dx dt = 0 \end{aligned}$$

A duality approach

Proof sketch

$$J(y, w) = \int_Q j(t, x, y) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) - \int_Q w_a y dx dt - w_s(y) = 0$$

$$\delta_{D(\varphi)}^*(w_s) = \sup\{w_s(z); z \in D(\varphi)\} \geq w_s(y)$$

$$\int_Q (j(t, x, y) + j^*(t, x, w_a) - w_a y) dx dt = 0 \implies w_a(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q$$

$$-w_s(y) + \delta_{D(\varphi)}^*(w_s) = 0.$$

$$w_s(y) \geq w_s(z) \text{ for any } z \in D(\varphi) \implies w_s \in N_{D(\varphi)}(y).$$

A duality approach

Proof sketch

$$J(y, w) = \int_Q j(t, x, y) dx dt + \int_Q j^*(t, x, w_a) dx dt + \delta_{D(\varphi)}^*(w_s) - \int_Q w_a y dx dt - w_s(y) = 0$$

$$\delta_{D(\varphi)}^*(w_s) = \sup\{w_s(z); z \in D(\varphi)\} \geq w_s(y)$$

$$\int_Q (j(t, x, y) + j^*(t, x, w_a) - w_a y) dx dt = 0 \implies w_a(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. } (t, x) \in Q$$

$$-w_s(y) + \delta_{D(\varphi)}^*(w_s) = 0.$$

$$w_s(y) \geq w_s(z) \text{ for any } z \in D(\varphi) \implies w_s \in N_{D(\varphi)}(y).$$

A duality approach

$$\frac{dy}{dt}(\psi) + w_a(A_{0,\infty}\psi) + w_s(A_{0,\infty}\psi) = f(\psi),$$

for any $\psi \in W^{1,\infty}([0, T]; L^1(\Omega)) \cap L^\infty(0, T; D(A_{0,\infty}))$, $\psi(T) = 0$.

If $\frac{dy}{dt} \in (L^\infty(Q))'$

$$\left(\frac{dy}{dt}\right)_a(\psi) + \left(\frac{dy}{dt}\right)_s(\psi) + w_a(A_{0,\infty}\psi) + w_s(A_{0,\infty}\psi) = f(\psi).$$

A duality approach

If

$$y \in \text{int}D(\varphi) = \{y \in L^\infty(Q); \text{ess sup } y(t, x) < y_s\}$$

$\implies$

$$N_{D(\varphi)}(y) = \{0\} \implies w_s = 0$$

$$\frac{dy}{dt} - \Delta\beta(t, x, y) = f, \text{ a.e. on } Q$$

with initial condition and b.c.

$$Q = Q_u = \{(t, x); y(t, x) < y_s\} \quad \text{unsaturated flow}$$

A duality approach

If

$$y \in \partial D(\varphi) = \{y \in L^\infty(Q); \text{ess sup } y(t, x) = y_s\}$$

$\implies$

$$Q_s = \{(t, x); y(t, x) = y_s\} \quad Q_u = \{(t, x); y(t, x) < y_s\}$$

$$\left(\frac{dy}{dt}\right)_s - \Delta w_s \ni 0, \text{ in } D'(\overset{\circ}{Q}_s) \quad \left(\frac{dy}{dt}\right)_a - \Delta \beta(t, x, y) = f, \text{ a.e. on } Q_u$$

saturated

unsaturated

A duality approach

If

$$y \in \partial D(\varphi) = \{y \in L^\infty(Q); \text{ess sup } y(t, x) = y_s\}$$

$\Rightarrow$

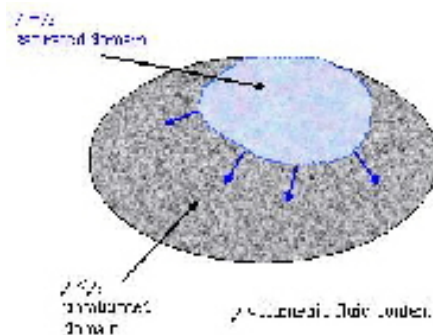
$$Q_s = \{(t, x); y(t, x) = y_s\}$$

$$Q_u = \{(t, x); y(t, x) < y_s\}$$

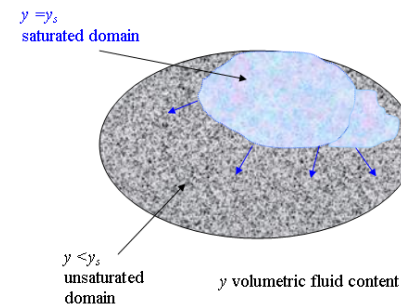
$$\left(\frac{dy}{dt}\right)_s - \Delta w_s \ni 0, \text{ in } D'(\overset{\circ}{Q}_s)$$

$$\left(\frac{dy}{dt}\right)_a - \Delta \beta(t, x, y) = f, \text{ a.e. on } Q_u$$

saturated



unsaturated



3 **A direct approach**

A direct approach: regularity for $\beta(t, x, r)$

$$\begin{aligned}
\frac{\partial y}{\partial t} - \Delta \beta(t, x, y) + \nabla \cdot K_0(t, x, y) &\ni f && \text{in } Q \\
\frac{\partial \beta(t, x, y)}{\partial \nu} + \alpha \beta(t, x, y) &\ni 0 && \text{on } \Sigma \\
y(0, x) &= y_0 && \text{in } \Omega.
\end{aligned} \tag{NE}$$

$$\beta(\cdot, \cdot, r) \in W^{2,\infty}(Q), \text{ for any } r \in (-\infty, y_s],$$

$$(\beta(t, x, r) - \beta(t, x, \bar{r}))(r - \bar{r}) \geq \rho(r - \bar{r})^2, \text{ for any } r, \bar{r} \in (-\infty, y_s], \rho > 0$$

$$K_0(t, x, y) = a(t, x)K(y), \quad a_j \in W^{1,\infty}(Q),$$

K Lipschitz and bounded.

A direct approach: regularity for $\beta(t, x, r)$

Definition. Let $f \in L^2(0, T; V')$, $y_0 \in L^\infty(\Omega)$, $y_0 \leq y_s$ a.e.

We call a *solution* to (NE) a pair (y, w) such that

$$\begin{aligned} y &\in C([0, T]; L^2(\Omega)) \cap L^2(0, T; V) \cap W^{1,2}([0, T]; V') \\ w &\in L^2(0, T; V), \quad w(t, x) \in \beta(t, x, y(t, x)) \text{ a.e. on } Q, \\ y &\leq y_s, \text{ a.e. } (t, x) \in Q, \end{aligned}$$

which satisfies the equation

$$\int_0^T \left\langle \frac{dy}{dt}(t), \phi(t) \right\rangle_{V', V} dt + \int_Q (\nabla w - K_0(t, x, y)) \cdot \nabla \phi dx dt = \int_0^T \langle f(t), \phi(t) \rangle_{V', V} dt$$

for any $\phi \in L^2(0, T; V)$, and the condition $y(0) = y_0$.

$$V = H^1(\Omega), \quad V' = (H^1(\Omega))'$$

A direct approach: regularity for $\beta(t, x, r)$

For each t

$$A(t) : D(A(t)) \subset (H^1(\Omega))' \rightarrow (H^1(\Omega))',$$

$$D(A(t)) = \{z \in L^2(\Omega); \exists w \in V, w(x) \in \beta(t, x, z) \text{ a.e. } x \in \Omega\},$$

by the relation

$$\langle A(t)z, \psi \rangle_{V', V} := \int_{\Omega} (\nabla w - K_0(t, x, y)) \cdot \nabla \psi dx, \text{ for any } \psi \in V.$$

$$\begin{aligned} \frac{dy}{dt}(t) + A(t)y(t) &\ni f(t), \text{ a.e. } t \in (0, T), \\ y(0) &= y_0. \end{aligned}$$

A direct approach: regularity for $\beta(t, x, r)$

Step. 1 Prove the existence to the regularized Cauchy problem⁶

$$\begin{aligned}\frac{dy}{dt}(t) + A_\varepsilon(t)y(t) &= f(t), \text{ a.e. } t \in (0, T), \\ y(0) &= y_0.\end{aligned}$$

- (a) *The domain of $A_\varepsilon(t)$ is independent of t and $D(A_\varepsilon(t)) = D(A_\varepsilon(0)) = V$.*
- (b) *For each $\varepsilon > 0$ and $t \in [0, T]$ fixed, the operator $A_\varepsilon(t)$ is quasi m -accretive on V' .*
- (c) *For $\theta \in V$ and $0 \leq s, t \leq T$ we have*

$$\|A_\varepsilon(t)\theta - A_\varepsilon(s)\theta\|_{V'} \leq |t - s| g(\|\theta\|_{V'}) (\|A(t)\theta\|_{V'} + 1),$$

where $g : [0, \infty) \rightarrow [0, \infty)$ is a nondecreasing function.

Step 2. passing to the limit.

⁶ M.G. Crandall, A. Pazy, Proceedings of The American Mathematical Society, 1979

C. Dafermos, M. Slemrod, J. Functional Anal., 1973

4 Numerical results

Numerical results

$$\begin{aligned}\frac{dy}{dt}(t) + Ay(t) &= f(t), \text{ a.e. } t \in (0, T), \\ y(0) &= y_0.\end{aligned}$$

$$\begin{aligned}\frac{y_{i+1}^h - y_i^h}{h} + Ay_{i+1}^h &= f_{i+1}^h, \quad i = 1, \dots, n-1 \\ y_0^h &= y_0\end{aligned}$$

- Stability and convergence
- Algorithm based on the proof of the m -accretivity of $A^h = A + \frac{1}{h}I$

$$\left(\frac{1}{h}I + A^h\right) \theta = g$$

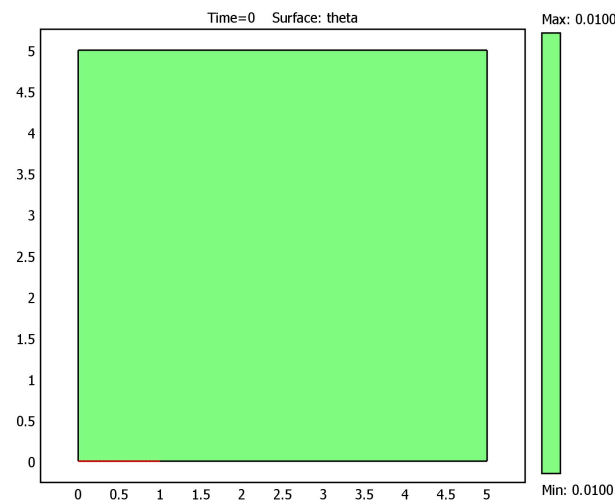
for any $g \in V'$, has a solution $\theta \in D(A)$.

Numerical results

$$\Omega = \{(x, y); x \in (0, 5), y \in (0, 5)\}, \quad T = 20$$

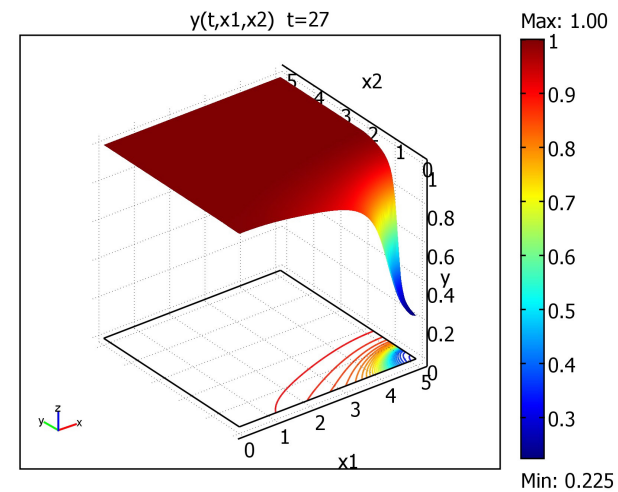
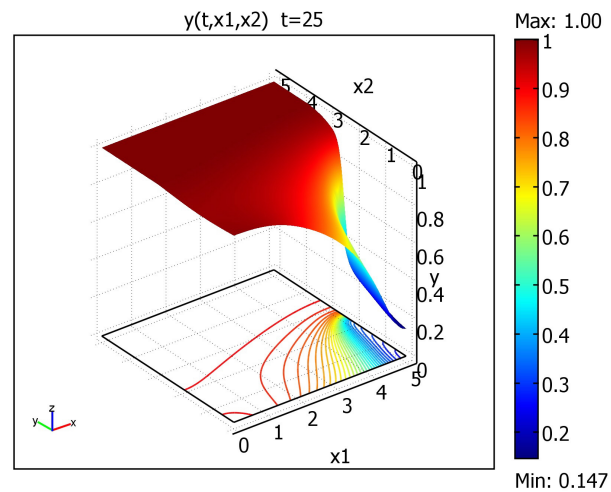
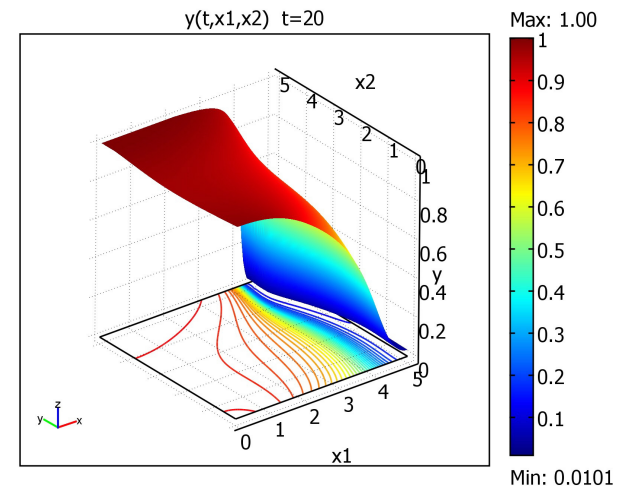
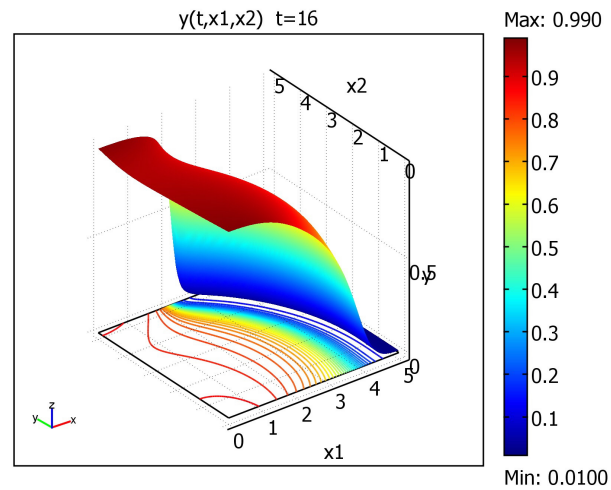
$$\beta(r) = \begin{cases} \frac{(c-1)r}{(c-r)}, & r \in [0, 1), \\ [1, \infty), & r = 1. \end{cases}, \quad K(r) = \frac{(c-1)r^2}{c-r}, \quad r \in [0, 1]$$

$$\theta_0(x, y) = 0.01, \quad f = 0, \quad \alpha = 10^{-8}, \quad rain = \exp(30 - x^2) / \exp(30), \quad c = 1.02$$



↑ ↑ ↑ ↑ *rain*

Numerical results



- [1] G. Auchmuty, Variational principles for operator equations and initial value problems, *Nonlinear Anal.* 12, 531–564, 1988
- [2] F. Bagagiolo, A. Visintin, Porous media filtration with hysteresis. *Adv. Math. Sci. Appl.*, 14, 379-403, 2004
- [3] V. Barbu, *Nonlinear Differential Equations of Monotone Type in Banach Spaces*, Springer, New York, 2010
- [4] V. Barbu, A variational approach to stochastic nonlinear parabolic problems. *J. Math. Anal. Appl.* 384, 2–15, 2011
- [5] V. Barbu, G. Da Prato and M. Röckner, Existence of strong solutions for stochastic porous media equation under general monotonicity conditions. *Ann. Probab.* 37, 428–452, 2009
- [6] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas independant du temps, *C.R. Acad. Sci. Paris Sér. A* 282 (1976) 971–974
- [7] H. Brezis, I. Ekeland, Un principe variationnel associé à certaines equations paraboliques. Le cas dependant du temps, *C.R. Acad. Sci. Paris Sér. A* 282 (1976) 1197–1198
- [8] M.G. Crandall, A. Pazy, An approximation of integrable functions by step functions with an application, *Proceedings of The American Mathematical Society*, 76, 1, 74-80, 1979

- [9] C.M Dafermos, M. Slemrod, Asymptotic behavior of nonlinear contraction semigroups, J. Functional Anal., 13, 97-106, 1973
- [10] A. Favini, G. Marinoschi, Degenerate nonlinear diffusion equations, Lecture Notes in Mathematics, 2049, Springer Verlag, Berlin, 2012
- [11] W. Fenchel, Convex Cones, Sets and Functions., Princeton Univ., 1953
- [12] N. Ghoussoub, L. Tzou, A variational principle for gradient flows, Math. Ann. 330 (2004) 519–549.
- [13] N. Ghoussoub, Self-dual Partial Differential Systems and Their Variational Principles, Springer, NewYork (2009)
- [14] J.L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, Paris, 1969
- [15] G. Marinoschi, Functional Approach to Nonlinear Models of Water Flow in Soils, Mathematical Modelling: Theory and Applications, volume 21, Springer, Dordrecht, 2006
- [16] G. Marinoschi, Existence to time-dependent nonlinear diffusion equations via convex optimization. J. Optim. Theory Appl. 154, 792-817, 2012
- [17] G. Marinoschi, A variational approach to nonlinear diffusion equations with time periodic coefficients, Annals of the University of Bucharest (mathematical series), 3 (LXI),

173-185, 2012

- [18] B. Nayroles, Deux théorèmes de minimum pour certains systèmes dissipatifs. C. R. Acad. Sci. Paris Sér. A-B 282, A1035–A1038, 1976
- [19] R.T. Rockafeller, Pacific J. Math. J, 39, 2, 1971
- [20] U. Stefanelli, The Brezis-Ekeland principle for doubly nonlinear equations, SIAM J. Control Optim. 47, 1615–1642, 2008
- [21] U. Stefanelli, The discrete Brezis-Ekeland principle, J. Convex Anal. 16, 71–87, 2009
- [22] J.J. Telega, Extremum principles for nonpotential and initial-value problems, Arch. Mech. 54 (2002) 663–690
- [23] A. Visintin, Existence results for some free boundary filtration problems. Ann. Mat. Pura Appl., 124 (1980), 293-320
- [24] A. Visintin, Extension of the Brezis-Ekeland-Nayroles principle to monotone operators, Adv. Math. Sci. Appl. 18 (2008), 633-650

Thank you for your attention