



Weierstrass Institute for
Applied Analysis and Stochastics



DFG Research Center MATHEON



Modeling and analysis of a nonlinear PDE-system for phase separation and damage

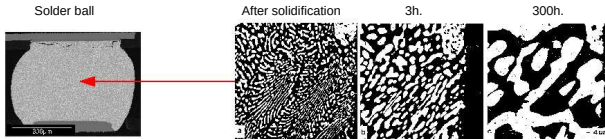
Christiane Kraus

(joint work with Christian Heinemann, WIAS,
Elena Bonetti, Antonio Segatti, Pavia)

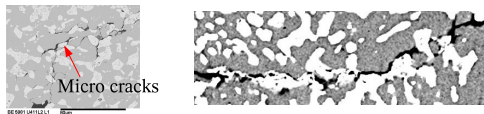
1. Introduction
2. Modeling of a system for phase separation and damage processes
3. Numerical simulations
4. Existence results for the introduced system
 - ▷ different chemical energy densities
 - ▷ different elastic energy densities

Morphology in solder joints

▷ Phase separation and coarsening



▷ Crack initiation and propagation along the phase boundary

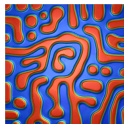


Tin-lead solders in electronic devices (mobile phones, PCs, micro-chips,...)

Phase separation and coarsening

- ▷ Phase field models of Cahn-Hilliard type with elasticity

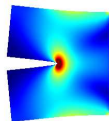
Analysis: Carrive/Miranville/Pietrus 00, Garcke 00, Miranville 00/01,
Bonetti/Colli/Dreyer/Gilardi/Schimperna/Sprekels 02,
Pawłow/Zajączkowski 08/09/10



Damage

- ▷ Elliptic system/differential inclusion

Engineering: Marigo 93, Frémond/Nedjar 96,
Bourdin 00, Miehe 07, Ubachs/Schreurs/Geers 06, Hakim/Karma 09
Analysis: Mielke/Roubíček 03/10, Bonetti/Schimperna/Segatti 05,
Thomas 10, Knees/Rossi/Zanini 11, Rocca/Rossi 12



Phase separation and coarsening

- ▷ Phase field models of Cahn-Hilliard type with elasticity

Analysis: Carrive/Miranville/Pietrus 00, Garcke 00, Miranville 00/01,
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Aim

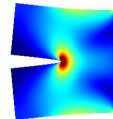
- ▷ Developing of a **phase field model** for **phase separation** and **damage processes**
- ▷ Analytical properties, numerical simulations



Damage

- ▷ Elliptic system/differential inclusion

Engineering: Marigo 93, Frémond/Nedjar 96,
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Two components

$$c_+, c_-, \quad c_+ + c_- = 1 \quad \implies c = c_+ - c_-$$

Variables

c : concentration, u : displacement field

z : damage, $0 \leq z \leq 1$

Free energy

$$\hat{\mathcal{E}}(c, u, z) = \int_{\Omega} \left(\frac{1}{2} |\nabla c|^2 + \psi(c) + \frac{1}{2} |\nabla z|^2 + h(z) + W(c, e(u), z) \right) dx$$

Chemical energy density

ψ : polynomial or logarithmic type

Elastic energy density

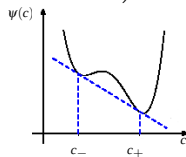
$$e(u) = \frac{1}{2} (\nabla u + (\nabla u)^T)$$

$$W_1(c, e(u)) = \frac{1}{2} (e(u) - e^*(c)) : \mathbb{C}(c) (e(u) - e^*(c))$$

$e^*(c)$: eigenstrain, $\mathbb{C}(c)$: stiffness tensor (sym., pos. def.),

$$W(c, e(u), z) = (g(z) + \delta) W_1(c, e(u)), \quad \delta > 0,$$

g : monotonically increasing with $g(0) = 0$, $g \in C^1([0, 1])$.



Free energy

$$\hat{\mathcal{E}}(c, u, z) = \int_{\Omega} \left(\frac{1}{2} |\nabla c|^2 + \psi(c) + \frac{1}{2} |\nabla z|^2 + h(z) + W(\mathbf{c}, \mathbf{e}(\mathbf{u}), \mathbf{z}) \right) dx$$

Dissipation potential

$$\hat{\mathcal{R}}(\partial_t z) = \int_{\Omega} \left(\frac{\beta}{2} |\partial_t z|^2 - \alpha \partial_t z \right) dx$$

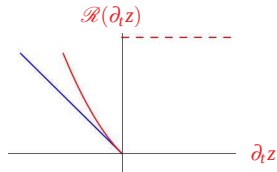
Constraints for the damage variable

$0 \leq z \leq 1$, ($z = 0$ completely damaged, $z = 1$ undamaged)

$\partial_t z \leq 0$ unidirectional process

$$\mathcal{E}(c, u, z) := \hat{\mathcal{E}}(c, u, z) + \int_{\Omega} I_{[0,1]}(z) dx$$

$$\mathcal{R}(\partial_t z) := \hat{\mathcal{R}}(\partial_t z) + \int_{\Omega} I_{(-\infty, 0]}(\partial_t z) dx$$



Free energy $\mathcal{E}(c, u, z) = \int_{\Omega} \left(\frac{1}{2} |\nabla c|^2 + \psi(c) + \frac{1}{2} |\nabla z|^2 + h(z) + W(c, e(u), z) + I_{[0,1]}(z) \right) dx$

Dissipation potential $\mathcal{R}(\partial_t z) = \int_{\Omega} \left(\frac{\beta}{2} |\partial_t z|^2 - \alpha \partial_t z + I_{(-\infty, 0]}(\partial_t z) \right) dx$

Evolution system (ES) in classical formulation

Evolution law for the mass concentration

$$\partial_t c = -\operatorname{div} J, \quad J = -M \nabla w$$

$$w = D_c \mathcal{E}(c, u, z) = -\Delta c + \psi'(c) + W_{,c}(c, e(u), z)$$

Quasistatic balance of forces

$$0 = D_u \mathcal{E}(c, u, z) = \operatorname{div} (W_{,e}(c, e(u), z))$$

Evolution law for the damage variable

$$0 \in D_z \mathcal{E}(c, u, z) + \partial \mathcal{R}(\partial_t z)$$

$$\iff 0 = -\Delta z + h'(z) + W_{,z}(c, e(u), z) + r + \beta \partial_t z - \alpha + s \text{ with}$$

$$r \in \partial I_{[0,1]}(z), \quad s \in \partial I_{(-\infty, 0]}(\partial_t z).$$

Assumptions

$\Omega \subset \mathbb{R}^N$ bounded Lipschitz domain,
 $D \subset \partial\Omega$ with $\mathcal{H}^{n-1}(D) > 0$.

Initial-boundary conditions (IBC)

Dirichlet conditions $u(t) = b(t)$ on $D \times (0, T)$,

Neumann conditions $\begin{aligned} \mathbb{M} \nabla w \cdot \nu &= 0 && \text{on } \partial\Omega \times (0, T), \\ \nabla c \cdot \nu &= 0 && \text{on } \partial\Omega \times (0, T), \\ \nabla z \cdot \nu &= 0 && \text{on } \partial\Omega \times (0, T), \\ W_{,e} \cdot \nu &= 0 && \text{on } \partial(\Omega \setminus D) \times (0, T), \end{aligned}$

Initial conditions $\begin{aligned} c(0) &= c^0, \\ z(0) &= z^0 \quad \text{with } 0 \leq z^0 \leq 1. \end{aligned}$

Assumptions:

- ▷ $\psi \in C^1(\mathbb{R})$, $h \in C^1([0, 1])$, $W_1 \in C^1(\mathbb{R} \times \mathbb{R}^{N \times N})$,
 $W_1(c, e) = W_1(c, (e)^t)$.

Growth assumptions for W_1 :

- ▷ $|W_1(c, e)| \leq C(|c|^2 + |e|^2 + 1)$,
- ▷ $\eta |e_1 - e_2|^2 \leq (W_{1,e}(c, e_1) - W_{1,e}(c, e_2)) : (e_1 - e_2)$,
- ▷ $|W_{1,e}(c, e_1 + e_2)| \leq C(W_1(c, e_1, z) + |e_2| + 1)$,
- ▷ $|W_{1,c}(c, e)| \leq C(|c|^2 + |e|^2 + 1)$.

Growth assumption for ψ :

- ▷ $|\psi'(c)| \leq C(|c|^{2^*/2} + 1)$

Weak formulation

$q = (c, w, u, z)$ is a weak solution of (ES) with (IBC) if

- ▶ $c \in L^\infty(0, T; H^1(\Omega))$, $w \in L^2(0, T; H^1(\Omega))$,
 $u \in L^\infty(0, T; H^1(\Omega; \mathbb{R}^N))$, $z \in L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))$,
 $0 \leq z \leq 1$ a.e. and $\partial_t z \leq 0$ a.e.

- ▶ for all $\xi \in L^2(0, T; H^1(\Omega))$, $\partial_t \xi \in L^2(\Omega_T)$ and $\xi(T) = 0$:

$$\int_{\Omega_T} (c - c^0) \partial_t \xi \, dx \, ds = - \int_{\Omega_T} \mathbb{M} \nabla w \cdot \nabla \xi \, dx \, ds$$

- ▶ for all $\xi \in H^1(\Omega) \cap L^\infty(\Omega)$ and a.e. $t \in (0, T)$:

$$\int_{\Omega} w \xi \, dx = \int_{\Omega} \left(\nabla c \cdot \nabla \xi + \psi'(c) \xi + W_{,c}(c, e(u), z) \right) \xi \, dx$$

- ▶ for all $\xi \in H^1(\Omega; \mathbb{R}^N)$ with $\xi = 0$ on D and a.e. $t \in (0, T)$:

$$\int_{\Omega} W_{,e}(c, e(u), z) : e(\xi) \, dx = 0$$

Weak formulation (continued)

▷ Variational inequality

for all $\xi \in H_-^1(\Omega) \cap L^\infty(\Omega)$ and a.e. $t \in (0, T)$:

$$0 \leq \int_{\Omega} \left(\nabla z \cdot \nabla \xi + h'(z) \xi + W_{,z}(c, e(u), z) \xi - \alpha \xi + \beta \partial_t z \xi \right) dx + \langle r, \xi \rangle, \quad (1)$$

where $r \in \partial I_{H_+^1(\Omega) \cap L^\infty(\Omega)}(z)$.

▷ Energy estimate

for a.e. $t \in (0, T)$:

$$\begin{aligned} \mathcal{E}(c(t), u(t), z(t)) + \int_0^t \int_{\Omega} \left(-\alpha \partial_t z + \beta |\partial_t z|^2 + \mathbb{M} \nabla w \cdot \nabla w \right) dx ds \\ \leq \mathcal{E}(c(0), u(0), z(0)) + \int_0^t \int_{\Omega} W_{,e}(c, e(u), z) : e(\partial_t b) dx ds \end{aligned} \quad (2)$$

Weak formulation (continued)

▷ Variational inequality

for all $\xi \in H^1_-(\Omega) \cap L^\infty(\Omega)$ and a.e. $t \in (0, T)$:

$$0 \leq \int_{\Omega} \left(\nabla z \cdot \nabla \xi + h'(z) \xi + W_{,z}(c, e(u), z) \xi - \alpha \xi + \beta \partial_t z \xi \right) dx + \langle r, \xi \rangle, \quad (1)$$

where $r \in \partial I_{H^1_+(\Omega) \cap L^\infty(\Omega)}(z)$.

▷ Energy estimate

for a.e. $t \in (0, T)$:

$$\begin{aligned} \mathcal{E}(c(t), u(t), z(t)) + \int_0^t \int_{\Omega} \left(-\alpha \partial_t z + \beta |\partial_t z|^2 + \mathbb{M} \nabla w \cdot \nabla w \right) dx ds \\ \leq \mathcal{E}(c(0), u(0), z(0)) + \int_0^t \int_{\Omega} W_{,e}(c, e(u), z) : e(\partial_t b) dx ds \end{aligned} \quad (2)$$

Theorem (Heinemann & K.)

For smooth solutions, (1) and (2) are equivalent to

$$0 \in \partial \mathcal{R}(\partial_t z) + \partial_z \mathcal{E}(c, u, z).$$

Notion of weak solutions

- ▷ Cahn-Hilliard system with elasticity
- ▷ variational inequality for z
- ▷ energy estimate

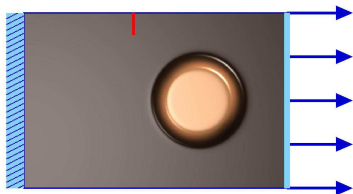
Assumptions

- ▷ Initial and boundary conditions
- ▷ Conditions for ψ , h and W

Theorem (Heinemann & K.) Existence of weak solutions

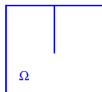
Let $b \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^N))$, $c^0 \in H^1(\Omega)$ and $z^0 \in H^1(\Omega)$.

Then there exists a weak solution (c, u, w, z) of (ES) with the previous initial and boundary conditions.

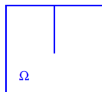


(Müller, WIAS)

- ▷ loading proportional to the time t
- ▷ initial damage seed
- ▷ different thermal expansions of the phases



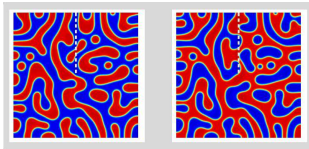
Damage field is constant in time



Damage field is constant in time

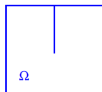
Homogeneous elasticity: $\mathbb{C}(c) = \mathbb{C}$

$$c(0, \cdot) = c_-$$



$$c(0, \cdot) = c_+$$

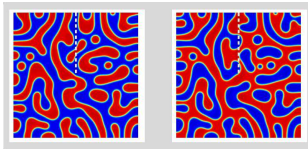
(Müller, WIAS)



Damage field is constant in time

Homogeneous elasticity: $\mathbb{C}(c) = \mathbb{C}$

$$c(0, \cdot) = c_-$$

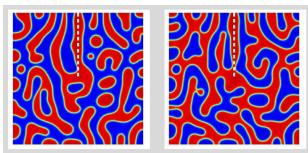


$$c(0, \cdot) = c_+$$

(Müller, WIAS)

Inhomogeneous elasticity: $\mathbb{C}(c) = m(c)\mathbb{C}_1 + (1 - m(c))\mathbb{C}_2$

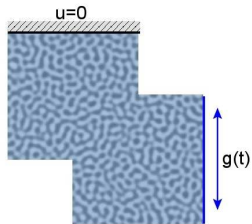
$$c(0, \cdot) = c_-$$

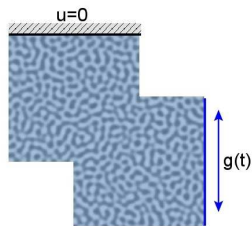


$$c(0, \cdot) = c_+$$

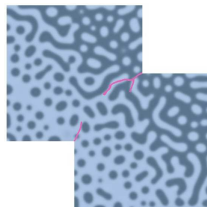
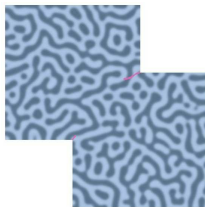
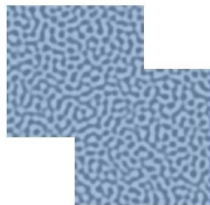
(Müller, WIAS)

- ▷ Softer material where stresses are large
- ▷ Phase boundary aligned with damage

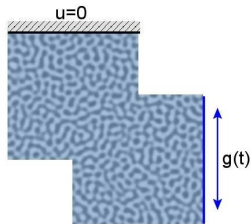




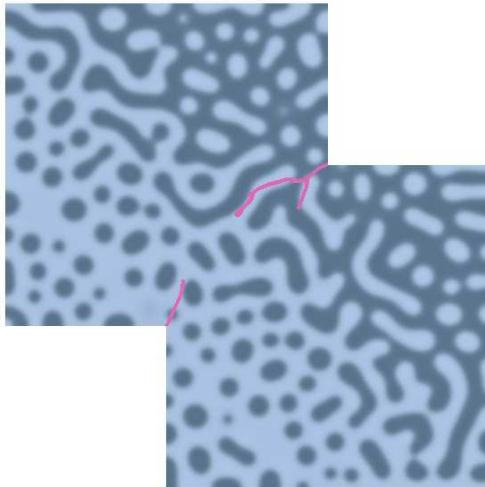
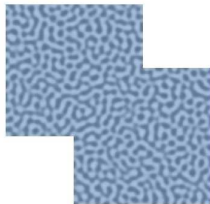
- ▷ damage propagation along the phase boundary
- ▷ crack branching



(Müller, WIAS)



- ▷ damage propagation along t
- ▷ crack branching



Notion of weak solutions

- ▷ Cahn-Hilliard system with elasticity
- ▷ variational inequality for z
- ▷ energy estimate

Assumptions

- ▷ Initial and boundary conditions
- ▷ Conditions for ψ and W

Theorem (Heinemann & K.) Existence of weak solutions

Let $b \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^N))$, $c^0 \in H^1(\Omega)$ and $z^0 \in H^1(\Omega)$.

Then there exists a weak solution (c, u, w, z) of (ES) with the previous initial and boundary conditions.

Regularized energy

$$\mathcal{E}_{\varepsilon}(c, u, z) = \mathcal{E}(c, u, z) + \int_{\Omega} \left(\frac{\varepsilon}{q} |\nabla z|^q + \frac{\varepsilon}{4} |\nabla u|^4 \right) dx, \quad q > N.$$

Regularized evolution system (ES_ε) in classical formulation

Evolution law for the mass concentration

$$\begin{aligned} \partial_t c &= \operatorname{div}(\mathbb{M} \nabla w), \\ w &= D_c \mathcal{E}_{\varepsilon}(c, u, z) = -\Delta c + \psi'(c) + W_{,c}(c, e(u), z) + \varepsilon \partial_t c \end{aligned}$$

Quasistatic balance of forces

$$0 = D_u \mathcal{E}_{\varepsilon}(c, u, z) = \operatorname{div}(W_{,e}(c, e(u), z)) + \varepsilon \operatorname{div}(|\nabla u|^2 \nabla u)$$

Evolution law for the damage variable

$$0 \in \partial \mathcal{R}(\partial_t z) + \partial_z \mathcal{E}_{\varepsilon}(c, u, z)$$

Boundary conditions: $u(t) = b(t)$ on D , $(W_{,e} + \varepsilon |\nabla u|^2 \nabla u) \cdot \nu = 0$ on $\partial\Omega \setminus D$,
 $\mathbb{M} \nabla w \cdot \nu = 0$, $\nabla c \cdot \nu$, $\nabla z \cdot \nu = 0$ on $\partial\Omega$.

(initial conditions as before)

Existence of weak solutions of (ES_ε) for fixed $\varepsilon > 0$

► Implicit Euler scheme in time

Discretization fineness $\tau = \frac{T}{M}$

u^0 : minimizer of $u \rightarrow \mathcal{E}_\varepsilon(c^0, u, z^0)$

Let $q^{m-1} = (c^{m-1}, u^{m-1}, z^{m-1})$ be given.

Recursive construction of q^m

Minimize

$$\mathcal{F}_\varepsilon^m(q) = \mathcal{E}_\varepsilon(q) + \frac{\tau}{2} \left\| \frac{c - c^{m-1}}{\tau} \right\|_{\hat{H}^1}^2 + \tau \mathcal{R} \left(\frac{z - z^{m-1}}{\tau} \right) + \frac{\varepsilon \tau}{2} \left\| \frac{c - c^{m-1}}{\tau} \right\|_{L^2}^2$$

with the constraints

$$0 \leq z \leq z^{m-1}, \quad \int_{\Omega} (c - c^0) dx = 0, \quad u|_D = b(m\tau)|_D.$$

Weighted scalar product:

$$(c_1, c_2)_{\hat{H}^1} := (\mathbb{M} \nabla (-\operatorname{div}(\mathbb{M} \nabla))^{-1} c_1, \nabla (-\operatorname{div}(\mathbb{M} \nabla))^{-1} c_2)_{L^2} \quad (\text{Garcke 00})$$

Minimizer of $\mathcal{F}_\varepsilon^m$: $q^m = (c^m, u^m, z^m)$

$$\begin{aligned} q_M(t) &:= q^m \text{ for } t \in ((m-1)\tau, m\tau], \\ q_M^-(t) &:= q^m \text{ for } t \in [m\tau, (m+1)\tau), \end{aligned} \quad m = 1, 2, \dots, M.$$

▷ Boundedness of the energy

Theorem (Heinemann & K.)

For all $M \in \mathbb{N}$ and all $t \in [0, T]$:

$$\mathcal{E}_\varepsilon(q_M(t)) \leq C, \quad q_M(t) = (c_M(t), u_M(t), z_M(t)).$$

- ▷ A-priori estimates, weak convergence properties
- ▷ Strong convergence properties

Theorem (Heinemann & K.)

There exists a subsequence $\{M_k\}$ such that

$$\begin{aligned} c_{M_k}, c_{M_k}^- &\rightarrow c \quad \text{in } L^{2^*}(0, T; H^1(\Omega)), \\ u_{M_k}, u_{M_k}^- &\rightarrow u \quad \text{in } L^4(0, T; W^{1,4}(\Omega; \mathbb{R}^N)), \\ z_{M_k}, z_{M_k}^- &\rightarrow z \quad \text{in } L^q(0, T; W^{1,q}(\Omega)) \end{aligned}$$

as $k \rightarrow \infty$.

$\implies (c, u, z)$ is a solution of the extended Cahn-Hilliard-system

Variational inequality:

- ▷ The triple $(c_m(t), u_m(t), z_m(t))$ satisfies for all $\xi \in W^{1,q}(\Omega)$ with $0 \leq \xi + z_m(t) \leq z_m^-(t)$ and all $t \in [0, T]$ the estimate:

$$0 \leq \int_{\Omega} \left((\varepsilon |\nabla z_m(t)|^{q-2} + 1) \nabla z_m(t) \cdot \nabla \xi + (h'(z_m(t)) + W_z(c_m(t), e(u_m(t)), z_m(t)) - \alpha + \beta \partial_t \hat{z}_m(t)) \xi \right) dx$$

$$\downarrow \quad m \rightarrow \infty?$$

- ▷ The limiting triple $(c(t), u(t), z(t))$ should satisfy for all $\xi \in W_-^{1,q}(\Omega)$ and for a.e. $t \in [0, T]$ the following estimate:

$$0 \leq \int_{\Omega} \left((\varepsilon |\nabla z(t)|^{q-2} + 1) \nabla z(t) \cdot \nabla \xi + (h'(z(t)) + W_z(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t)) \xi \right) dx + \langle r(t), \xi \rangle,$$

where $r(t) \in \partial I_{W_+^{1,q}(\Omega)}(z(t))$.

Variational inequality:

- ▷ The triple $(c_m(t), u_m(t), z_m(t))$ satisfies for all $\xi \in W^{1,q}(\Omega)$ with $0 \leq \xi + z_m(t) \leq z_m^-(t)$ and all $t \in [0, T]$ the estimate:

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Approximation properties $\downarrow m \rightarrow \infty$

- ▷ For all $\xi \in W_-^{1,q}(\Omega)$ with $\{\xi = 0\} \supseteq \{z = 0\}$ and a.e. $t \in [0, T]$:

$$0 \leq \int_{\Omega} \left((\varepsilon |\nabla z(t)|^{q-2} + 1) \nabla z(t) \cdot \nabla \xi(t) + (h'(z(t)) + W_z(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t)) \xi(t) \right) dx$$

▷ For all $\xi \in W_-^{1,q}(\Omega)$ with $\{\xi = 0\} \supseteq \{z = 0\}$ and a.e. $t \in [0, T]$:

$$0 \leq \int_{\Omega} \left((\varepsilon |\nabla z(t)|^{q-2} + 1) \nabla z(t) \cdot \nabla \xi + (h'(z(t)) + W_z(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t)) \xi \right) dx$$

▷ **Aim**

For all $\xi \in W_-^{1,q}(\Omega)$ and a.e. $t \in [0, T]$ with some $r(t) \in \partial I_{W_+^{1,q}(\Omega)}(z(t))$:

$$0 \leq \int_{\Omega} \left((\varepsilon |\nabla z(t)|^{p-2} + 1) \nabla z(t) \cdot \nabla \xi + (h'(z(t)) + W_z(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t)) \xi \right) dx + \langle r(t), \xi \rangle$$

Lemma (Heinemann & K.)

Let $f \in L^q(\Omega; \mathbb{R}^N)$, $g \in L^q(\Omega)$ and $l \in W^{1,q}(\Omega)$ ($q > N$) with $l \geq 0$, $f \cdot \nabla l \geq 0$ and $\{f = 0\} \supseteq \{l = 0\}$ a.e. . If

$$\int_{\Omega} (f \cdot \nabla \xi + g \xi) dx \geq 0 \quad \text{for all } \xi \in W_-^{1,q}(\Omega) \text{ with } \{\xi = 0\} \supseteq \{l = 0\}$$

then

$$\int_{\Omega} (f \cdot \nabla \xi + g \xi) dx \geq \int_{\{l=0\}} g^+ \xi dx \quad \text{for all } \xi \in W_-^{1,q}(\Omega).$$

▷ For all $\xi \in W_-^{1,q}(\Omega)$ and a.e. $t \in [0, T]$

$$\begin{aligned} & \int_{\Omega} \left((\varepsilon |\nabla z(t)|^{q-2} + 1) \nabla z(t) \cdot \nabla \xi + (h'(z(t)) + W_{,z}(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t)) \xi \right) dx \\ & \stackrel{\text{Lemma}}{\geq} \int_{\{z(t)=0\}} \left(h'(z(t)) + W_{,z}(c(t), e(u(t)), z(t)) - \alpha + \beta \partial_t z(t) \right)^+ \xi \, dx \\ & \geq \int_{\{z(t)=0\}} \left(h'(z(t)) + W_{,z}(e(u(t)), c(t), z(t)) \right)^+ \xi \, dx \end{aligned}$$

▷ Variational inequality

For all $\xi \in W_-^{1,q}(\Omega)$ and a.e. $t \in [0, T]$

$$\begin{aligned} 0 \leq \int_{\Omega} & \left((\varepsilon |\nabla z(t)|^{q-2} + 1) \nabla z(t) \cdot \nabla \xi + (h'(z(t)) + W_{,z}(c(t), e(u(t)), z(t)) \right. \\ & \left. - \alpha + \beta \partial_t z(t)) \xi \right) dx + \langle r(t), \xi \rangle \end{aligned}$$

with

$$r(t) = -\chi_{\{z(t)=0\}} [h'(z(t)) + W_{,z}(c(t), e(u(t)), z(t))]^+ \in \partial I_{W_+^{1,q}(\Omega)}(z(t)).$$

▷ A-priori energy estimate

$$\begin{aligned} \mathcal{E}_\varepsilon(q_m(t)) + \int_0^{t_m} \int_\Omega \left(-\alpha \partial_t \hat{z}_m + \frac{\beta}{2} |\partial_t \hat{z}_m|^2 + \frac{\varepsilon}{2} |\partial_t \hat{c}_m|^2 + \frac{1}{2} \mathbb{M} \nabla w_m \cdot \nabla w_m \right) dx ds \\ \leq \mathcal{E}_\varepsilon(q^0) + \int_0^{t_m} \int_\Omega W_{,e}(c_m^-, e(u_m^- + b - b_m^-), z_m^-) : e(\partial_t b) dx ds \\ + \varepsilon \int_0^{t_m} \int_\Omega |\nabla(u_m^- + b - b_m^-)|^2 \nabla(u_m^- + b - b_m^-) : \nabla \partial_t b dx ds \end{aligned}$$

with $t_m := \min\{\tau m : \tau l \geq t, l \in \mathbb{N}_0\}$, τ discretization fineness.

▷ Aim

Energy estimate

$$\begin{aligned} \mathcal{E}_\varepsilon(q(t)) + \int_0^t \int_\Omega \left(-\alpha \partial_t z + \beta |\partial_t z|^2 + \varepsilon |\partial_t c|^2 + \mathbb{M} \nabla w \cdot \nabla w \right) dx ds \\ \leq \mathcal{E}_\varepsilon(q^0) + \int_0^t \int_\Omega W_{,e}(c, e(u), z) : e(\partial_t b) dx ds \\ + \varepsilon \int_0^t \int_\Omega |\nabla u|^2 \nabla u : \nabla \partial_t b dx ds \end{aligned}$$

Theorem (Heinemann & K.) Asymptotic energy estimate

For every $t \in [0, T]$:

$$\begin{aligned} \mathcal{E}_\varepsilon(q_m(t)) &+ \int_0^t \int_\Omega \left(-\alpha \partial_t \hat{z}_m + \beta |\partial_t \hat{z}_m|^2 + \varepsilon |\partial_t \hat{c}_m|^2 + \mathbb{M} \nabla \mathbf{w}_m \cdot \nabla \mathbf{w}_m \right) dx ds \\ &\leq \mathcal{E}_\varepsilon(q^0) + \int_0^t \int_\Omega W_{,e}(c_m^-, e(u_m^- + b - b_m^-), z_m^-) : e(\partial_t b) dx ds \\ &\quad + \varepsilon \int_0^t \int_\Omega |\nabla(u_m^- + b - b_m^-)|^2 \nabla(u_m^- + b - b_m^-) : \nabla \partial_t b dx ds + \kappa_m \end{aligned}$$

with $\kappa_m \rightarrow 0$ for $m \rightarrow \infty$.

Proof:

- ▷ viscous term: $\varepsilon \partial_t c$
- ▷ regularization term: $\varepsilon \int_\Omega |\nabla u|^4 dx$

1. Existence of weak solutions of (ES_ε) for fixed $\varepsilon > 0$

- ▷ implicit Euler scheme
- ▷ convergence properties of the time-discrete solutions
- ▷ Cahn-Hilliard system with elasticity
- ▷ variational inequality
- ▷ energy estimate

2. Existence of weak solutions of the original system (ES)

- ▷ a-priori estimates for solutions $(c_\varepsilon, w_\varepsilon, u_\varepsilon, z_\varepsilon)$ of (ES_ε) w.r.t. $0 < \varepsilon \leq 1$
- ▷ convergence properties: $(c_\varepsilon, w_\varepsilon, u_\varepsilon, z_\varepsilon) \rightarrow (c, w, u, z)$ as $\varepsilon \searrow 0$
- ▷ (c, w, u, z) is a weak solution of (ES).

- ▷ Chemical free energy density of logarithmic type

$$\psi(c) = \theta \sum_{k=1}^n c_k \log c_k + \frac{1}{2} c \cdot \mathbb{A} c$$

Theorem (Heinemann & K.) Existence of weak solutions

Let $D = \partial\Omega$ and

- $b \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^N)), z^0 \in H^1(\Omega),$
- $c^0 \in H^1(\Omega; \mathbb{R}^n)$ with $\sum_{k=1}^n c_k = 1$ and $c_k^0 > 0$ for $k = 1, \dots, n.$

Then there exists a weak solution (c, u, w, z) of (ES) with

$$c_k > 0, \quad k = 1, \dots, n,$$

and the previous initial and boundary conditions b, c^0 and z^0 .

Proof:

- ▷ based on [EL91] and [Gar00]
- ▷ Higher integrability of ∇u for time dependent Dirichlet conditions:
 $\{u_\delta\}$ is bounded in $W^{1,2+\nu}(\Omega; \mathbb{R}^N), \nu > 0$ small.
(inequality of Sobolev-Poincaré type, reverse Hölder inequality, perturbation arguments [Geh73], [GM79])

Free energy

$$\mathcal{E}(c, u, z) = \int_{\Omega} \left(\frac{1}{2} |\nabla c|^2 + \psi(c) + \frac{1}{2} |\nabla z|^2 + h(z) + W(c, e(u), z) + I_{[0,1]}(z) \right) dx$$

▷ Chemical energy density ψ

- (i) ψ of polynomial type
- (ii) ψ of logarithmic type

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▷ Elastic energy density W

$$W(c, e(u), z) = (g(z) + \delta) W_1(c, e(u)), \quad \delta > 0.$$

Damage literature:

$$W_1(e(u)) = (g(z) + \delta) W_1(e(u)), \quad W_1(e(u)) = \frac{1}{2} e(u) : \mathbb{C} e(u).$$

(Mielke & Roubířek 03/10, Thomas 10, Knees/Rossi/Zanini 11,...)

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Aim: Modeling of different elastic behavior for undamaged and damaged states

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$$W(c, e(u), z) = g_1(z) W_1(c, e(u)) + (1 - g_1(z)) W_2(c, e(u))$$

W_1 (as discussed)

Example:

$$W_2(c, e(u)) = \frac{1}{2} ((e(u) - \hat{e}(c)) : \hat{\mathbb{C}}(c)(e(u) - \hat{e}(c)))^{p/2}, \quad 1 < p < 2.$$

$g_1 \in C^1([0, 1])$: monotonically increasing with $g_1(0) = 0$ and $g_1(1) = 1$

- ▷ $z = 1$: W_1 (standard elastic energy density)
- ▷ $z = 0$: W_2 (weaker elastic energy density)
- ▷ $z \in (0, 1)$: W_1 and W_2 contribute

$$W_1(c, e(u)) \geq W_2(c, e(u))$$

Assumptions:

$$\triangleright W_2 \in C^1(\mathbb{R} \times \mathbb{R}^{N \times N}), \quad W_2(c, e) = W_2(c, (e)^t).$$

Growth assumptions for W_2 :

$$\triangleright |W_2(c, e)| \leq C(|c|^s + |e|^p + 1), \quad p \in (1, 2), \quad s < \frac{n(p-1)}{n-p}$$

$$\triangleright \eta |e_1 - e_2|^p \leq (W_{2,e}(c, e_1) - W_{2,e}(c, e_2)) : (e_1 - e_2)$$

$$\triangleright |W_{2,e}(c, e_1 + e_2)| \leq C(W_2(c, e_1) + |e_2|^{p-1} + 1)$$

$$\triangleright |W_{2,c}(c, e)| \leq C(|c|^s + |e|^p + 1)$$

Additional assumption for W_1 :

$$\triangleright l_c(\cdot) := W_{1,e}(\cdot, c) - W_{1,e}(0, c) \text{ is 1-homogeneous.}$$

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Theorem (Bonetti & Heinemann & K. & Segatti)

Existence of weak solutions

Let $b \in W^{1,1}([0, T]; W^{1,\infty}(\Omega; \mathbb{R}^N))$, $c^0 \in H^1(\Omega)$ and $z^0 \in H^1(\Omega)$.

Then there exists a weak solution (c, u, w, z) of (ES) with the refined elastic energy and the previous initial and boundary conditions.

Limit of approximate system

▷ Balance of forces

Test functions $\xi \in L^1(0, T; W_D^{1,p}(\Omega; \mathbb{R}^n))$

$$-\operatorname{div}\left(\sqrt{g_1(z)} \theta_{u;c;z} + (1 - g_1(z)) \eta_{u;c}\right) = 0$$

with $\sqrt{g_1(z)} \theta_{u;c;z}, \eta_{u;c} \in L^\infty(0, T; W^{1,p'}(\Omega))$

▷ Variational inequality

$W_{,z}(c_\varepsilon, e(u_\varepsilon), z_\varepsilon) \rightharpoonup \zeta \in L^2(0, T; (H^1(\Omega))')$

Aim: Identifying limits

$$\theta_{c;u;z} = \sqrt{g_1(z)} W_{1,e}(c, e(u))$$

$$\eta_{c;u} = W_{2,e}(c, (e(u)))$$

$$\zeta = g_1(z) W_{1,z}(c, e(u)) + (1 - g_1(z)) W_{2,z}(c, e(u))$$

Some ingredients: Lower semicontinuous properties for expressions with W_1 and W_2 ,
Properties of subdifferentials, 1-homogeneous property of $W_{1,e}$.

Free energy

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▷ Elastic energy density W

$$W(c, e(u), z) = (g(z) + \delta) W_1(c, e(u)), \quad \delta > 0,$$

$$W(c, e(u), z) = g_2(z) (g_1(z) W_1(c, e(u)) + (1 - g_1(z)) W_2(c, e(u))),$$
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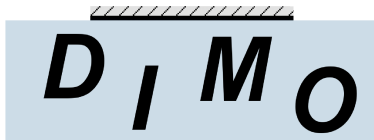
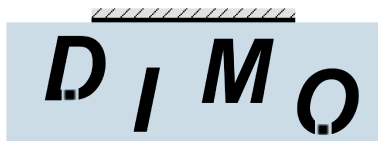
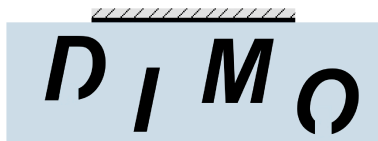
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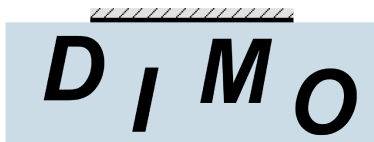
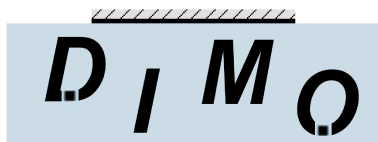
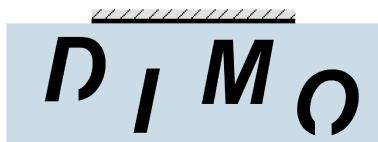
▷ Mobility

Mobility $\mathbb{M}$ degenerates:

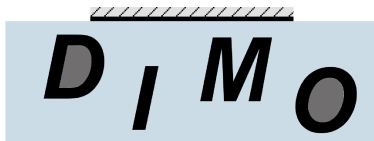
$$\mathbb{M}(z) \geq 0 \quad \text{if and only if} \quad \mathbb{M}(z) = 0 \iff z = 0$$







Idea: Time-dependent domain, free boundary problem



Challenges

- ▷ Jumps in the energy
 - ↪ Accounting in the energy inequalitySeveral energy estimates are established by Γ -convergence
- ▷ Time-dependent domain
 - ↪ Bad smoothness propertiesCovering result for such sets with smoothly compactly embedded domains

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Then there exists a weak solution (c, u, w, z) of the doubly degenerating system in an SBV-framework.

Interested in that problem?



Afternoon session

Poster by Christian Heinemann

Thank you for your attention!

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- ▶ C. Heinemann and C. Kraus, *Existence of weak solutions for Cahn–Hilliard systems coupled with elasticity and damage*, Adv. Math. Sci. Appl., **21**, 2011, 321-359.
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