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Phase-field Approach to Models of Regional Economic Trend

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1. Motivation

Disaster: (mega)earthquake+tsunami in Japan

Government+Non-disaster regions

↓ supports

Disaster region

What kinds of supports ?

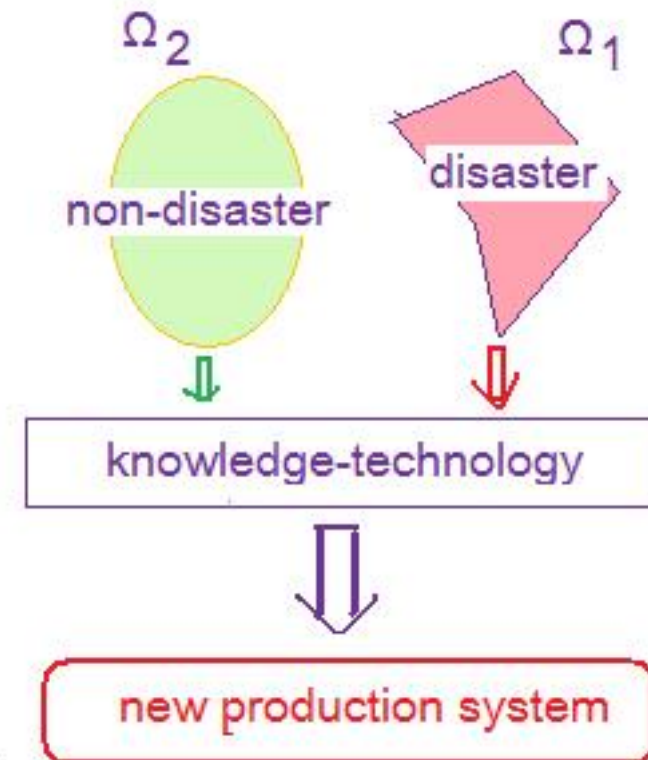
How much and how long ?

Self-reliant recovery ?

Importance of

- mutual help system
for life conditions,
- economic and technological
cooperation

among several regions.



It is also pointed out that

- it takes a long time to rebuild or reform the (local) society which once collapsed.
- it is not easy to expect a constant labor force for industrial production.
- in such a situation, the knowledge-technology progress is extremely important in place of labor force.

In this talk, we focus on the influence of the knowledge-technology progress for the economic growth.

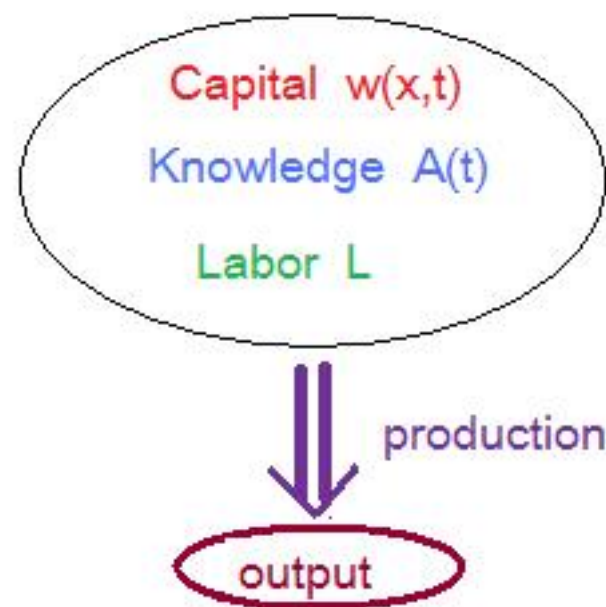
In the economics,

capital $w = w(x, t)$

knowledge(technology)

$A := (A_1(t), A_2(t), \dots, A_N(t))$

labor force $L = L(x, t)$, constant



2. Economic energy and dynamical systems (prototype)

$$E(w, A) := \int_{\Omega} \left\{ \frac{\kappa}{2} |\nabla w(x)|^2 + \frac{b}{2} |w(x)|^2 - \frac{\sigma P(L \cdot A)}{1 + \alpha} |w(x)|^{1+\alpha} \right\} dx \\ + I_K(A) + G(A),$$

where Ω is bounded and smooth domain in $\mathbf{R}^2$ or $\mathbf{R}^3$,

- w : capital, $A = (A_1, A_2, \dots, A_N)$: knowledge-technology level;
- $\kappa > 0$, $b > 0$ (**depreciation rate**), $0 < \alpha < 1$, $\sigma > 0$ (**saving rate**),
 $L = (L_1, L_2, \dots, L_N)$ (**labor force**), $L_i > 0$, constants;
- $G(\cdot) : \mathbf{R}_+ \rightarrow \mathbf{R}$, C^2 -class;
- $P(\cdot) : \mathbf{R}_+ \rightarrow \mathbf{R}_+$, C^2 -class and increasing,
(Ex.) $P(\xi) := \xi^{1-\alpha}$, $\xi \geq 0$,
 $L \cdot A = \sum_{i=1}^N L_i A_i$ is called the "effective labor".
 $P(L \cdot A)w^\alpha$: **production function** of Cobb-Douglas type;
- (**New concept**) knowledge-technology field K ,
 K : compact convex in $\mathbf{R}_+^N$, $0 \notin K$, prescribed in $\mathbf{R}_+^N$;
 $I_K(\cdot)$: indicator function of K .

Equilibrium state

$$-\kappa \Delta w + bw = \sigma P(L \cdot A) w^\alpha \quad \text{in } \Omega,$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Gamma := \partial\Omega,$$

$$\partial I_K(A) + g(A) \ni \left(\frac{\sigma P'(L \cdot A)}{1 + \alpha} \int_{\Omega} |w(x)|^{1+\alpha} dx \right) L \quad \text{in } \mathbf{R}^N,$$

where $g = \nabla G$ and $P'(r) = \frac{dP(r)}{dr}$; note that $\frac{\partial P(L \cdot A)}{\partial A_i} = P'(L \cdot A) L_i$.

Dynamics of w and A :

$$(1) \quad w_t - \kappa \Delta w + bw = \sigma P(L \cdot A) w^\alpha \quad \text{in } Q := \Omega \times (0, T),$$

$$\frac{\partial w}{\partial n} = 0 \quad \text{on } \Sigma := \Gamma \times (0, T),$$

$$(2) \quad A' + \partial I_K(A) + g(A) \ni \left(\frac{\sigma P'(L \cdot A)}{1 + \alpha} \int_{\Omega} |w|^{1+\alpha} dx \right) L \quad \text{in } (0, T),$$

$$(3) \quad w(\cdot, 0) = w_0 > 0, \quad A(0) = A_0 > 0.$$

Note: Most of economic models are derived by various choices of pseudo-potential of dissipation (in the theory due to M. Frémond).

Theorem 1. Assume that $w_0 \in H^1(\Omega)$ and w_0 is close to 0 with $w_0 \geq c_0$ a.e. in Ω for a constant $c_0 > 0$ such that $bw_0 \leq \sigma P(LA_0)w_0^\alpha$, and $A_0 \in K$. Then, (1)-(3) has one and only one solution $\{w, A\}$ such that

$$w \in W^{1,2}(0, T; H^2(\Omega)),$$

$$A \in W^{1,2}(0, T; \mathbb{R}^N), \quad A(t) \in K \quad (\forall t \in [0, T]).$$

Proof. The proof is standard.

Remark.(i) The basic idea is found in a paper of

A. Visintin (1985 Stefan problem with phase relaxation)

(ii) For the large-time behaviour of solutions ($N = 1$) some interesting results are found in

M. Ôtani (1981-1984 Non-monotone perturbations).

The structure of energy functional is the difference of two convex functions.

(iii) A perturbation result and asymptotic stability results are also useful,

K.Shirakawa, A.Ito, N. Yamazaki, K. (1998)

3. Formation of knowlegde-technology field

In real applications, the field K of knowlegde-technology is not fixed, but is **cultivated by spending much time and making an investment**: K is a **moving compact convex set** in $\mathbf{R}_+^N$ depending on w in a non-local way; for instance,

$$K := K(\bar{w}(t); t), \quad \bar{w}(t) := \int_{t-\tau_0}^t \int_{\Omega} \rho(t-s) w(x, s) dx ds, \quad \forall t \geq 0,$$

where

- $w(x, s) = w_0(x), \quad \forall x \in \Omega, \quad \forall s \leq 0;$
- $\rho(\cdot) \geq 0$, smooth, $\text{supp}(\rho) \subset [-\tau_0, \tau_0], \quad \tau_0 > 0, \quad \int_{\mathbf{R}} \rho(r) dr = 1;$
- $K(r; t) : \mathbf{R}^2 \rightarrow \mathcal{K}$, Lip. cont. in Hausdorff distance,
 $\mathcal{K} := \{K \subset \mathbf{R}_+^N \mid K \text{ is compact, convex, } \text{int.} K \neq \emptyset, 0 \notin K\}$

In this case the dynamics of A is governed by:

$$(2') A'(t) + \partial I_{K(\bar{w}(t); t)}(A(t)) + g(A(t)) \ni \left(\frac{\sigma P'(L \cdot A(t))}{1 + \alpha} \int_{\Omega} |w(x, t)|^{1+\alpha} dx \right) L.$$

The problem $\{(1), (2'), (3)\}$ is a sort of **quasi-variational inequalities**, since the constraint $K(\bar{w}(t); t)$ depends on the unknown w .

4. Mathematical results

$$(1) \quad w_t - \kappa \Delta w + bw = \sigma P(L \cdot A)w^\alpha \quad \text{in } Q, \quad \frac{\partial w}{\partial n} = 0 \text{ on } \Sigma,$$

$$(2') \quad A'(t) + \partial I_{K(\bar{w}(t);t)}(A(t)) + g(A(t)) \ni \left(\frac{\sigma P'(L \cdot A(t))}{1 + \alpha} \int_{\Omega} |w(x, t)|^{1+\alpha} dx \right) L.$$

$$(3) \quad w(\cdot, 0) = w_0 > 0, \quad A(0) = A_0 > 0.$$

Theorem 2. Assume that the same condition as in Theorem 1 is satisfied for the initial data, and $\overline{K(r, t)}$ is included in a compact convex set $\hat{K}$ in $\mathbf{R}^N$ for all (r, t) . Then, problem $\{(1), (2'), (3)\}$ has at least one solution $\{w, A\}$ such that

$$w \in W^{1,2}(0, T; H^2(\Omega)),$$

$$A \in W^{1,2}(0, T; \mathbf{R}^N), \quad A(t) \in K(\bar{w}(t); t), \quad \forall t \in [0, T].$$

Proof. Apply the theory of parabolic quasi-variational inequalities (R.Kano-K-Y.Murase 2009).

Outline of the proof:

$\forall A \in X := \{A \in W^{1,2}(0,T) \mid A(t) \in \hat{K}, \forall t \in [0,T]\},$

solve (1) to get the solution w :

$$|w|_{W^{1,2}(0,T;L^2(\Omega))} + |w|_{C([0,T];H^1(\Omega))} + |w|_{L^2(0,T;H^2(\Omega))} \leq M_1 \quad (*)$$

where M_1 is a positive constant independent of $A \in X$. Next, $\forall w$ satisfying (*), solve (2') to get the solution $\tilde{A}$. Define a mapping

$$S : X \rightarrow X \text{ by } S(A) = \tilde{A}.$$

By the energy estimate, $\exists M_2 > 0$, depending only on M_1 , such that

$$|\tilde{A}|_{W^{1,2}(0,T)} \leq M_2. \quad (**)$$

Now, putting $X_0 := \{A \in X \mid |A|_{W^{1,2}(0,T)} \leq M_2\}$, we see that

- X_0 is compact and convex in $C([0,T])$ and $S(X_0) \subset X_0$.
- S is continuous in X_0 with respect to the topology on $C([0,T])$.

Therefore, S has at least one fixed point $A \in X_0$, $S(A) = A$. The couple of $\{w, A\}$, w being the solution of (1), gives a solution of $\{(1), (2'), (3)\}$.

5. Model with knowledge-technology cooperation

Two regions Ω_1, Ω_2 with common knowledge-technology field.
Economic elements in $\Omega_k, k = 1, 2$: $A := (A_1, A_2, \dots, A_N)$, w_k ,
 $L_k := (L_{k1}, L_{k2}, \dots, L_{kN})$, P_k , α_k , σ_k , b_k , κ_k .

$$w_{k,t} - \kappa_k \Delta w_k + b_k w_k = \sigma_k P_k(L_k \cdot A) w_k^{\alpha_k} \quad \text{in } Q_k := \Omega_k \times (0, T),$$

$$\frac{\partial w_k}{\partial n} = 0, \quad \text{on } \Sigma_k := \partial\Omega_k \times (0, T), \quad k = 1, 2,$$

$$\begin{aligned} A'(t) + \partial I_{K(\bar{w}(t); t)}(A(t)) + g(A(t)) \ni & \left(\frac{\sigma_1 P'_1(L_1 \cdot A(t))}{1 + \alpha_1} \int_{\Omega_1} |w_1(x, t)|^{1+\alpha_1} dx \right) L_1 \\ & + \left(\frac{\sigma_2 P'_2(L_2 \cdot A(t))}{1 + \alpha_2} \int_{\Omega_2} |w_2(x, t)|^{1+\alpha_2} dx \right) L_2, \end{aligned}$$

$$w_k(\cdot, 0) = w_{k0} \quad \text{in } \Omega_k, \quad k = 1, 2, \quad A(0) = A_0,$$

where $\bar{w}(t) := (\bar{w}_1(t), \bar{w}_2(t))$, $\bar{w}_k(t) := \int_{t-\tau_0}^t \int_{\Omega} \rho(t-s) w_k(x, s) dx ds$.

Our quasi-variational approach is possible to the above system.

6. Further questions

General economic energy (diffusion effect of knowledge):

$$E(w, A; t) := \int_{\Omega} \left\{ \frac{\kappa}{2} |\nabla w(x)|^2 + \frac{b}{2} |w(x)|^2 - \frac{\sigma P(L \cdot A(x))}{1 + \alpha} |w(x)|^{1+\alpha} \right\} dx \\ + \int_{\Omega} \left\{ \frac{\kappa_a}{2} \sum_{i=1}^N |\nabla A_i(x)|^2 + I_{K(\hat{w}(x,t);t)}(A(x)) + G(A(x)) \right\} dx,$$

where $\hat{w}(x, t) = w(x, t)$ or $\int_{t-\tau_0}^t \rho(t-s)w(x, s)ds$.

In this case, the dynamics of $\{w, A\}$ is governed by:

$$w_t - \kappa \Delta w + bw = \sigma P(L \cdot A)w^\alpha \quad \text{in } Q,$$

$$A_t - \kappa_a \Delta A + \partial I_{K(\hat{w}(\cdot, t);t)}(A) + g(A) \ni \left(\frac{\sigma P'(L \cdot A)}{1 + \alpha} |w|^{1+\alpha} \right) L \quad \text{in } Q$$

with initial and boundary conditions.

This system can be treated in the same way to get a solution.

Large-time behaviour

(1) In the case of time-independent K and the inclusion

$$L \in \partial I_K(A_L) + g(A_L)$$

has a unique solution A_L in $\mathbf{R}^N$, it could be shown that

$$(w(t), A(t)) \rightarrow (w_L, A_L) \text{ in } L^2(\Omega) \times \mathbf{R}^N \text{ as } t \rightarrow \infty,$$

where w_L is a constant satisfying

$$bw_L = \sigma P(L \cdot A_L)w_L^\alpha.$$

(2) In general case, it is not sure that the solution converges to a steady state as $t \rightarrow \infty$. But, of course, it is possible to discuss the large time behaviour in the sense of **global attractors**.

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