

On some nonlinear parabolic equation involving variable exponents

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Doubly nonlinear parabolic equation

Nonlinear parabolic equations of the form

$$\beta(\partial_t u) - \Delta u = f \quad \text{in } \Omega \times (0, T)$$

with a maximal monotone graph $\beta : \mathbb{R} \rightarrow \mathbb{R}$, a domain Ω of $\mathbb{R}^N$, and a given function $f = f(x, t) : \Omega \times (0, T) \rightarrow \mathbb{R}$ have been studied in various contexts.

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The linear Laplacian Δ is often replaced with nonlinear variants such as the so-called m -Laplacian Δ_m given by

$$\Delta_m u = \operatorname{div} (|\nabla u|^{m-2} \nabla u), \quad 1 < m < \infty,$$

and then, the equation above is called a doubly nonlinear parabolic equation.

Doubly nonlinear evolution equation

By setting $u(t) := u(\cdot, t)$, such a nonlinear parabolic equation is interpreted as an abstract evolution equation,

$$A(u'(t)) + B(u(t)) = f(t) \text{ in } X, \quad 0 < t < T, \quad (\star)$$

with unknown function $u : (0, T) \rightarrow X$, two (possibly nonlinear) operators A, B in X and $f : (0, T) \rightarrow X$.

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It is quite natural to build the existence and regularity theory for $(\star)$ in some class of **vector-valued functions, like the Lebesgue-Bochner space $L^p(0, T; X)$.**

Abstract theories

$$A(u'(t)) + B(u(t)) = f(t) \text{ in } X, \quad 0 < t < T.$$

Barbu ('75), Arai ('79), Senba ('86)

- established in $L^2(0, T; H)$;
- time differentiation of the equation, which transforms it into another (more tractable) type of doubly nonlinear equation;
- some peculiar monotonicity condition,

$$(Bu - Bv, A(u - v))_H \geq 0.$$

Abstract theories (contd.)

$$A(u'(t)) + B(u(t)) = f(t) \text{ in } X, \quad 0 < t < T.$$

Colli-Visintin ('90), Colli ('94)

- established in $L^2(0, T; H)$ or $L^p(0, T; V)$, $1 < p < \infty$;
- a power growth for $A : V \rightarrow V^*$,

$$c_0 \|u\|_V^p \leq \langle Au, u \rangle_V + C, \quad \|Au\|_{V^*}^{p'} \leq C(\|u\|_V^p + 1).$$

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These approaches have been extensively studied by many authors.

Recently, it was extended by Mielke-Rossi-Savaré ('13) to a quite broad class of doubly nonlinear evolution.

Doubly nonlinear parabolic eq. with power nonlinearity

Let us consider

$$|\partial_t u|^{p-2} \partial_t u - \Delta_m u = f(x, t) \quad \text{in } Q := \Omega \times (0, T),$$

$$u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

where $1 < p, m < \infty$.

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$$\begin{aligned} |\partial_t u|^{p-2} \partial_t u - \Delta_m u &= f(x, t) && \text{in } Q := \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega, \end{aligned}$$

where $1 < p, m < \infty$.

- It is a simple case of the **generalized Ginzburg-Landau eq. (Gurtin '96)**,

$$\beta(u, \nabla u, \partial_t u) \partial_t u = \operatorname{div} [\partial_{\nabla u} \psi(u, \nabla u)] - \partial_u \psi(u, \nabla u) + \gamma.$$

- This problem falls within **the framework of Colli ('94) in $L^p(0, T; V)$** by setting $V = L^p(\Omega)$, $Au = |u|^{p-2}u$, $Bu = -\Delta_m u$.

Indeed, **the p -power growth condition holds:**

$$\|u\|_V^p = \langle Au, u \rangle_V, \quad \|Au\|_{V^*}^{p'} = \|u\|_V^p.$$

Target problem of this talk

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain. In this talk, we discuss

$$(1) \quad |\partial_t u|^{p(x)-2} \partial_t u - \Delta_{m(x)} u = f(x, t) \quad \text{in } Q := \Omega \times (0, T),$$

$$(2) \quad u = 0 \quad \text{on } \partial\Omega \times (0, T),$$

$$(3) \quad u(\cdot, 0) = u_0 \quad \text{in } \Omega,$$

where $1 < p(x), m(x) < \infty$ are **variable exponents** and $\Delta_{m(x)}$ stands for the **$m(x)$ -Laplacian** given by

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$$\Delta_{m(x)} u = \operatorname{div} (|\nabla u|^{m(x)-2} \nabla u) .$$

Eq. (1) can describe **mixed settings** of several types of (generalized) Ginzburg-Landau models, e.g.,

$$\underbrace{p(x) \equiv 2 \text{ in } \Omega_1}_{\text{usual GL model}}, \quad \underbrace{p(x) \not\equiv 2 \text{ in } \Omega_2}_{\text{generalized GL model}}, \quad \Omega = \Omega_1 \oplus \Omega_2.$$

Lebesgue and Sobolev spaces with variable exponents

$$p^- := \operatorname{ess\,inf}_{x \in \Omega} p(x) \quad \text{and} \quad p^+ := \operatorname{ess\,sup}_{x \in \Omega} p(x),$$

$$\mathcal{P}(\Omega) := \{p \in \mathcal{M}(\Omega) : 1 \leq p^-\},$$

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Variable exponent Lebesgue and Sobolev spaces are defined as follows:

$$L^{p(x)}(\Omega) := \left\{ u \in \mathcal{M}(\Omega) : \int_{\Omega} |u(x)|^{p(x)} dx < \infty \right\},$$

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$$W^{1,p(x)}(\Omega) := \{u \in L^{p(x)}(\Omega) : \partial_{x_i} u \in L^{p(x)}(\Omega) \text{ for } i = 1, \dots, N\},$$

$$\|u\|_{W^{1,p(x)}(\Omega)} := \left(\|u\|_{L^{p(x)}(\Omega)}^2 + \|\nabla u\|_{L^{p(x)}(\Omega)}^2 \right)^{1/2}.$$

Difficulties arising from variable exponents

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and note that

$$\left(|u|^{p(x)-2} u \right) u = |u|^{p(x)}, \quad \left| |u|^{p(x)-2} u \right|^{p'(x)} = |u|^{p(x)}.$$

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However, it only implies, e.g.,

$$\langle |u|^{p(x)-2} u, u \rangle_V \geq c \|u\|_V^{p^-}, \quad \left\| |u|^{p(x)-2} u \right\|_{V^*}^{(p^+)' } \leq C \|u\|_V^{p^+}$$

with $p^+ := \text{ess sup } p(x)$, $p^- := \text{ess inf } p(x)$ and $V^* = L^{p'(x)}(\Omega)$.

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with $p^+ := \text{ess sup } p(x)$, $p^- := \text{ess inf } p(x)$ and $V^* = L^{p'(x)}(\Omega)$.

Since $p^+ > p^-$, it does not fall within the frame of $L^p(0, T; V)$.

Difficulties arising from variable exponents (contd.)

More precisely, testing the following equation by $\partial_t u$,

$$|\partial_t u|^{p(x)-2} \partial_t u - \Delta_{m(x)} u = 0,$$

we have

$$\int_{\Omega} |\partial_t u|^{p(x)} dx + \int_{\Omega} \underbrace{|\nabla u|^{m(x)-2} \nabla u \cdot \nabla (\partial_t u)}_{\partial_t \left(\frac{1}{m(x)} |\nabla u|^{m(x)} \right)} dx = 0.$$

The integration of both sides over $(0, t)$ implies

$$\begin{aligned} \int_0^t \int_{\Omega} |\partial_{\tau} u|^{p(x)} dx d\tau + \int_{\Omega} \frac{1}{m(x)} |\nabla u(x, t)|^{m(x)} dx \\ \leq \int_{\Omega} \frac{1}{m(x)} |\nabla u(x, 0)|^{m(x)} dx. \end{aligned}$$

Difficulties arising from variable exponents (contd.)

To estimate $|\partial_t u|^{p(x)-2} \partial_t u$ in $L^{p'(x)}(\Omega) = (L^{p(x)}(\Omega))^*$, we use

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The Lebesgue-Bochner space $L^q(0, T; V)$ approach:

One needs to estimate $\partial_t u$ and $|\partial_t u|^{p(x)-2} \partial_t u$ in some L-B spaces.

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One needs to estimate $\partial_t u$ and $|\partial_t u|^{p(x)-2} \partial_t u$ in some L-B spaces.

However, in the variable exponent setting, we should emphasize that

$$\int_{\Omega} |w(x)|^{p(x)} dx \neq \|w\|_{L^{p(x)}(\Omega)}^{\textcolor{red}{p(x)}} \quad \text{for } w \in L^{p(x)}(\Omega).$$

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$$\int_{\Omega} |w(x)|^{p(x)} dx \neq \|w\|_{L^{p(x)}(\Omega)}^{p(x)} \quad \text{for } w \in L^{p(x)}(\Omega).$$

Instead, we use

$$\sigma_{p(x)}^- (\|w\|_{L^{p(x)}}) \leq \int_{\Omega} |w(x)|^{p(x)} dx \leq \sigma_{p(x)}^+ (\|w\|_{L^{p(x)}}) \quad \forall w \in L^{p(x)}(\Omega)$$

with $\sigma_{p(x)}^-(s) := \min\{s^{p^-}, s^{p^+}\}$ and $\sigma_{p(x)}^+(s) := \max\{s^{p^-}, s^{p^+}\}$.

Difficulties arising from variable exponents (contd.)

Then for $V = L^{p(x)}(\Omega)$, one may obtain estimates, e.g.,

$$\int_0^T \|\partial_t u\|_V^{p^-} dt \leq \int_0^T \left(\int_{\Omega} |\partial_t u|^{p(x)} dx \right) dt,$$

$$\int_0^T \left\| |\partial_t u|^{p(x)-2} \partial_t u \right\|_{V^*}^{(p')^-} dt \leq \int_0^T \left(\int_{\Omega} |\partial_t u|^{p(x)} dx \right) dt.$$

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However, since $(p')^- = (p^+)' < (p^-)'$ by $p^+ > p^-$, we note that

$$L^{(p')^-}(0, T; V^*) \not\approx \left(L^{p^-}(0, T; V) \right)^*.$$

This fact will prevent us to identify the limit of approximate solutions to prove the existence of solution.

Lebesgue space approach

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$$||\partial_t u|^{p(x)-2} \partial_t u||_{L^{p'(x)}(Q)} \leq \left(\|\partial_t u\|_{L^{p(x)}(Q)} \right)^{p^+/p^-}.$$

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Here we note that

$$L^{p'(x)}(Q) \sim \left(L^{p(x)}(Q) \right)^*.$$

Lebesgue space approach (contd.)

Namely, setting

$$\mathcal{V} := L^{p(x)}(Q) \quad \left(\mathcal{V}^* = L^{p'(x)}(Q) \right), \quad Q = \Omega \times (0, T)$$

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In order to treat the equation in $L^{p(x)}(Q) \times L^{p'(x)}(Q)$, we note that:

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♠ $L^{p(x)}(Q) \not\approx L^{p(x)}(0, T; L^{p(x)}(\Omega))$, which is meaningless for $p(x)$;

♠ lack of devices: e.g., chain-rules for subdifferential are usually formulated in Lebesgue-Bochner space settings.

Aim of this talk

We shall discuss the existence and regularity of solutions for

$$\begin{aligned} |\partial_t u|^{p(x)-2} \partial_t u - \Delta_{m(x)} u &= f(x, t) && \text{in } Q := \Omega \times (0, T), \\ u &= 0 && \text{on } \partial\Omega \times (0, T), \\ u(\cdot, 0) &= u_0 && \text{in } \Omega. \end{aligned}$$

To this end, we shall

- work in a mixed frame of Lebesgue space $L^{p(x)}(Q)$ and usual Lebesgue-Bochner spaces;
- develop devices of subdifferential calculus,
- in particular, a chain-rule for subdifferentials in the mixed frame.

Basic assumptions (H)

$$(H1) \quad m \in \mathcal{P}_{\log}(\Omega), \quad p \in \mathcal{P}(\Omega), \quad 1 < p^-, m^-, p^+, m^+ < \infty,$$

$$(H2) \quad \operatorname{ess\,inf}_{x \in \Omega} (m^*(x) - p(x)) > 0, \quad m^*(x) := \frac{Nm(x)}{(N - m(x))_+},$$

$$(H3) \quad f \in L^{p'(x)}(Q), \quad u_0 \in W_0^{1,m(x)}(\Omega).$$

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Remark.

- By (H1), $L^{p(x)}(\Omega)$ and $W^{1,m(x)}(\Omega)$ are uniformly convex and separable Banach spaces.
- Since $m(\cdot) \in \mathcal{P}_{\log}(\Omega)$, one can define

$$W_0^{1,m(x)}(\Omega) := \overline{C_0^\infty(\Omega)}^{W^{1,m(x)}(\Omega)}, \quad \|u\|_{W_0^{1,m(x)}(\Omega)} := \|\nabla u\|_{L^{m(x)}(\Omega)}.$$

- Moreover, (H2) ensures that $W_0^{1,m(x)}(\Omega) \xrightarrow{\text{compact}} L^{p(x)}(\Omega)$.

Main results

Definition 1 (Strong solutions)

We call $u \in L^{p(x)}(Q)$ a **strong solution of (1)–(3)** in Q whenever the following conditions hold true:

- (i) $t \mapsto u(\cdot, t)$ is continuous with values in $L^{p(x)}(\Omega)$ on $[0, T]$,
and it is weakly continuous with values in $W_0^{1,m(x)}(\Omega)$ on $[0, T]$,
- (ii) $\partial_t u \in L^{p(x)}(Q)$, $\Delta_{m(x)} u \in L^{p'(x)}(Q)$,
- (iii) the equation (1) holds for a.e. $(x, t) \in Q$,
- (iv) the initial condition (3) is satisfied for a.e. $x \in \Omega$.

Theorem 2 (Existence)

Assume (H). Then the Cauchy-Dirichlet problem (1)–(3) admits (at least) one strong solution u .

Main results (contd.)

Theorem 3 (Time-Regularization of strong solutions)

In addition to (H), suppose that

$$t\partial_t f \in L^{p'(x)}(Q).$$

Then, the Cauchy-Dirichlet problem (1)–(3) admits a strong solution u , which additionally satisfies

$$\operatorname{ess\,sup}_{t \in (\delta, T)} \|\partial_t u(\cdot, t)\|_{L^{p(x)}(\Omega)} < \infty,$$

$$\operatorname{ess\,sup}_{t \in (\delta, T)} \|\Delta_{m(x)} u(\cdot, t)\|_{L^{p'(x)}(\Omega)} < \infty$$

for any $\delta \in (0, T)$.

Lebesgue-Bochner space setting

Set $V = L^{p(x)}(\Omega)$ and $X = W_0^{1,m(x)}(\Omega)$ with $\|u\|_V := \|u\|_{L^{p(x)}(\Omega)}$,
 $\|u\|_X := \|\nabla u\|_{L^{m(x)}(\Omega)}$,

$$\langle v, u \rangle_V = \int_{\Omega} u(x)v(x) \, dx \quad \text{for all } u \in V, \, v \in V^* = L^{p'(x)}(\Omega).$$

Lebesgue-Bochner space setting

Set $V = L^{p(x)}(\Omega)$ and $X = W_0^{1,m(x)}(\Omega)$ with $\|u\|_V := \|u\|_{L^{p(x)}(\Omega)}$, $\|u\|_X := \|\nabla u\|_{L^{m(x)}(\Omega)}$,

$$\langle v, u \rangle_V = \int_{\Omega} u(x)v(x) \, dx \quad \text{for all } u \in V, v \in V^* = L^{p'(x)}(\Omega).$$

By (H2), it follows that $X \xhookrightarrow{\text{compact}} V$ and $V^* \xhookrightarrow{\text{compact}} X^*$.

Define functionals ψ and ϕ on V by

$$\psi(u) := \int_{\Omega} \frac{1}{p(x)} |u(x)|^{p(x)} \, dx \quad \text{for } u \in V$$
$$\phi(u) := \begin{cases} \int_{\Omega} \frac{1}{m(x)} |\nabla u(x)|^{m(x)} \, dx & \text{if } u \in X, \\ \infty & \text{if } u \in V \setminus X. \end{cases}$$

Lebesgue-Bochner space setting (contd.)

Denote by ∂_Ω (∂_Q) the subdifferential in $V = L^{p(x)}(\Omega)$ ($\mathcal{V} = L^{p(x)}(Q)$).

Then (1)–(3) can be reduced to

$$\partial_\Omega \psi(u'(t)) + \partial_\Omega \phi(u(t)) = Pf(t) \text{ in } V^*, \quad 0 < t < T, \quad u(0) = u_0,$$

where $Pf(t) := f(\cdot, t)$.

Identification between two settings

For each $u \in \mathcal{M}(Q)$, write

$$Pu(t) := u(\cdot, t) \quad \text{for } t \in (0, T).$$

Proposition 4 (Identification between L-B and L spaces)

Let $1 \leq p < \infty$ and let $p(x)$ be such that $1 \leq p^- \leq p^+ < \infty$.

- (i) P is a linear, bijective, isometric mapping from $L^p(Q)$ to $L^p(0, T; L^p(\Omega))$. Furthermore, if $u \in L^{p(x)}(Q)$, then $Pu \in L^{p^-}(0, T; L^{p(x)}(\Omega))$.
- (ii) The inverse $P^{-1} : L^p(0, T; L^p(\Omega)) \rightarrow L^p(Q)$ is well-defined, and for $u = u(t) \in L^p(0, T; L^p(\Omega))$, $u(t) = P^{-1}u(\cdot, t)$ for a.e. $t \in (0, T)$.

Remark. $L^\infty(0, T; L^\infty(\Omega)) \neq L^\infty(Q)$.

Identification between two settings (contd.)

For each $u \in \mathcal{M}(Q)$, write

$$Pu(t) := u(\cdot, t) \quad \text{for } t \in (0, T).$$

Proposition 4 (Identification between L-B and L spaces)

Let $1 \leq p < \infty$ and let $p(x)$ be such that $1 \leq p^- \leq p^+ < \infty$.

- (iii) If $u \in L^{p(x)}(Q)$ with $\partial_t u \in L^{p(x)}(Q)$, then Pu belongs to the space $W^{1,p^-}(0, T; L^{p(x)}(\Omega))$ and $(Pu)' = P(\partial_t u)$.
- (iv) If $u \in W^{1,p}(0, T; L^p(\Omega))$, then $\partial_t(P^{-1}u) \in L^p(Q)$ and $\partial_t(P^{-1}u) = P^{-1}(u')$.

Identification of subdifferentials in two settings

Let $\varphi : V = L^{p(x)}(\Omega) \rightarrow (-\infty, \infty]$ be proper, l.s.c., convex and define $\Phi : \mathcal{V} = L^{p(x)}(Q) \rightarrow (-\infty, \infty]$ by

$$\Phi(u) := \begin{cases} \int_0^T \varphi(Pu(t)) \, dt & \text{if } \varphi(Pu(\cdot)) \in L^1(0, T), \\ \infty & \text{otherwise.} \end{cases}$$

Proposition 5 (Identification of subdifferentials)

For $u \in \mathcal{V}$, $\xi \in \mathcal{V}^*$ with $1 < p^- \leq p^+ < \infty$,

$\xi \in \partial_Q \Phi(u)$ iff $P\xi(t) \in \partial_\Omega \varphi(Pu(t))$ for a.e. $t \in (0, T)$.

Lebesgue space setting

Define Ψ and Φ on $\mathcal{V} = L^{p(x)}(Q)$ as, for $u \in \mathcal{V}$,

$$\Psi(u) := \iint_Q \frac{1}{p(x)} |u(x, t)|^{p(x)} dx dt = \int_0^T \psi(Pu(t)) dt,$$

$$\Phi(u) := \begin{cases} \int_0^T \phi(Pu(t)) dt & \text{if } Pu(t) \in X \text{ for a.e. } t \in (0, T), \\ & t \mapsto \phi(Pu(t)) \in L^1(0, T), \\ \infty & \text{otherwise.} \end{cases}$$

Then by the latter proposition, the evolution eqn ($\Leftrightarrow$ (1)–(3)) is rewritten by

$$\partial_Q \Psi(\partial_t(P^{-1}u)) + \partial_Q \Phi(P^{-1}u) = f \text{ in } \mathcal{V}^*, \quad u(0) = u_0.$$

Construction of solutions

Step 1 (time-discretization). For $n = 0, \dots, N - 1$,

$$\partial_{\Omega}\psi\left(\frac{u_{n+1} - u_n}{h}\right) + \partial_{\Omega}\phi(u_{n+1}) = f_{n+1} \text{ in } V^*,$$

where $h := T/N$, $t_n := nh$ and $f_n := \frac{1}{h} \int_{t_{n-1}}^{t_n} P f(\theta) \, d\theta$.

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Moreover, define

a piecewise constant interpolant $\bar{u}_N : (0, T) \rightarrow X = W_0^{1,m(x)}(\Omega)$;

a piecewise linear interpolant $u_N : (0, T) \rightarrow X$.

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Moreover, define

a piecewise constant interpolant $\bar{u}_N : (0, T) \rightarrow X = W_0^{1,m(x)}(\Omega)$;

a piecewise linear interpolant $u_N : (0, T) \rightarrow X$. Then we get

$$\partial_{\Omega}\psi(u'_N(t)) + \partial_{\Omega}\phi(\bar{u}_N(t)) = \bar{f}_N(t) \text{ in } V^*, \quad \text{for a.e. } t \in (0, T)$$

with $u_N(0) = u_0$ in the Lebesgue-Bochner space setting, and equivalently,

$$\partial_Q\Psi(\partial_t(P^{-1}u_N)) + \partial_Q\Phi(P^{-1}\bar{u}_N) = P^{-1}\bar{f}_N \text{ in } \mathcal{V}^*, \quad u_N(0) = u_0$$

in the Lebesgue space setting.

Construction of solutions (contd.)

Step 2 (energy estimates). Test discretized eqn. by $(u_{n+1} - u_n)/h$. Then

$$\iint_Q |\partial_t(P^{-1}u_N)|^{p(x)} dx dt + \sup_{t \in (0,T]} \phi(\bar{u}_N(t)) \leq C,$$

which also gives

$$\|\partial_t(P^{-1}u_N)\|_{\mathcal{V}} \leq C, \quad \sup_{t \in (0,T]} |\bar{u}_N(t)|_X + \sup_{t \in [0,T]} |u_N(t)|_X \leq C.$$

Recall that

$$\mathcal{A} = \partial_Q \Psi : v \mapsto |v|^{p(x)-2}v \text{ is bounded from } \mathcal{V} \text{ to } \mathcal{V}^*.$$

We conclude that

$$\|\partial_Q \Psi(\partial_t(P^{-1}u_N))\|_{\mathcal{V}^*} \leq C,$$

which also implies the boundedness of $\partial_Q \Phi(P^{-1}\bar{u}_N)$ in $\mathcal{V}^*$ by comparison.

Construction of solutions (contd.)

Step 3 (Convergence).

$$u_N \rightarrow u \quad \begin{array}{l} \text{strongly in } C([0, T]; V), \\ \text{weakly star in } L^\infty(0, T; X), \end{array}$$

$$\bar{u}_N \rightarrow u \quad \begin{array}{l} \text{strongly in } L^\infty(0, T; V), \\ \text{weakly star in } L^\infty(0, T; X), \end{array}$$

$$P^{-1}\bar{u}_N \rightarrow \hat{u} = P^{-1}u \quad \text{strongly in } \mathcal{V},$$

$$\partial_t(P^{-1}u_N) \rightarrow \partial_t\hat{u} \quad \text{weakly in } \mathcal{V},$$

$$\partial_Q\Phi(P^{-1}\bar{u}_N) \rightarrow \xi \quad \text{weakly in } \mathcal{V}^*,$$

$$\partial_Q\Psi(\partial_t(P^{-1}u_N)) \rightarrow \eta \quad \text{weakly in } \mathcal{V}^*.$$

Thus $\eta + \xi = f$ in $\mathcal{V}^*$.

Construction of solutions (contd.)

Step 3 (Convergence).

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$$P^{-1}\bar{u}_N \rightarrow \hat{u} = P^{-1}u \quad \text{strongly in } \mathcal{V},$$

$$\partial_t(P^{-1}u_N) \rightarrow \partial_t\hat{u} \quad \text{weakly in } \mathcal{V},$$

$$\partial_Q\Phi(P^{-1}\bar{u}_N) \rightarrow \xi \quad \text{weakly in } \mathcal{V}^*,$$

$$\partial_Q\Psi(\partial_t(P^{-1}u_N)) \rightarrow \eta \quad \text{weakly in } \mathcal{V}^*.$$

Thus $\eta + \xi = f$ in $\mathcal{V}^*$.

From the maximal monotonicity of $\partial_Q\Phi$ in $\mathcal{V} \times \mathcal{V}^*$, one can immediately obtain $\xi \in \partial_Q\Phi(\hat{u})$. It remains to show $\eta \in \partial_Q\Psi(\partial_t\hat{u})$.

Construction of solutions (contd.)

To this end, by a standard technique, one has

$$\begin{aligned}
 & \iint_Q \partial_Q \Psi \left(\partial_t(P^{-1}u_N) \right) \partial_t(P^{-1}u_N) \, dx \, dt \\
 &= \iint_Q \left(P^{-1}\bar{f}_N - \partial_Q \Phi(P^{-1}\bar{u}_N) \right) \partial_t(P^{-1}u_N) \, dx \, dt \\
 &\leq \iint_Q (P^{-1}\bar{f}_N) \partial_t(P^{-1}u_N) \, dx \, dt - \phi(u_N(T)) + \phi(u_0).
 \end{aligned}$$

Passing to the limit as $N \rightarrow \infty$, we have

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \iint_Q \partial_Q \Psi \left(\partial_t(P^{-1}u_N) \right) \partial_t(P^{-1}u_N) \, dx \, dt \\
 &\leq \iint_Q f \partial_t \hat{u} \, dx \, dt - \phi(u(T)) + \phi(u_0)
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Construction of solutions (contd.)

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 &\leq \iint_Q \left(P^{-1}\bar{f}_N \right) \partial_t(P^{-1}u_N) \, dx \, dt - \phi(u_N(T)) + \phi(u_0).
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 & \limsup_{n \rightarrow \infty} \iint_Q \partial_Q \Psi \left(\partial_t(P^{-1}u_N) \right) \partial_t(P^{-1}u_N) \, dx \, dt \\
 &\leq \iint_Q f \partial_t \hat{u} \, dx \, dt - \phi(u(T)) + \phi(u_0) \stackrel{?}{=} \iint_Q \eta \partial_t \hat{u} \, dx \, dt.
 \end{aligned}$$

Construction of solutions (contd.)

Here we need:

Proposition 6 (Chain rule for subdifferentials in a mixed frame)

Let $p(\cdot) \in \mathcal{P}(\Omega)$ satisfy $1 < p^- \leq p^+ < \infty$. Let $u \in \mathcal{V}$ be such that $\partial_t u \in \mathcal{V}$. Suppose that there exists $\xi \in \mathcal{V}^*$ such that $\xi \in \partial_Q \Phi(u)$. Then, the function $t \mapsto \varphi(Pu(t))$ is absolutely continuous over $[0, T]$. Moreover, for each $t \in (0, T)$, we have

$$\frac{d}{dt} \varphi(Pu(t)) = \langle \eta, (Pu)'(t) \rangle_V \quad \text{for all } \eta \in \partial_\Omega \varphi(Pu(t)),$$

whenever Pu and $\varphi(Pu(\cdot))$ are differentiable at t . In particular, for $0 \leq s < t \leq T$, we have

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \iint_{\Omega \times (s,t)} \xi \partial_\tau u \, dx \, d\tau.$$

Construction of solutions (contd.)

Applying the latter chain-rule, we conclude that

$$\begin{aligned}
 & \limsup_{n \rightarrow \infty} \iint_Q \partial_Q \Psi (\partial_t(P^{-1}u_N)) \partial_t(P^{-1}u_N) \, dx \, dt \\
 & \leq \iint_Q f \partial_t \hat{u} \, dx \, dt - \phi(u(T)) + \phi(u_0) \\
 & \stackrel{u=P\hat{u}}{=} \iint_Q f \partial_t \hat{u} \, dx \, dt - \phi(P\hat{u}(T)) + \phi(P\hat{u}(0)) \\
 & \stackrel{\text{Prop 6}}{=} \iint_Q f \partial_t \hat{u} \, dx \, dt - \iint_Q \xi \partial_t \hat{u} \, dx \, dt \stackrel{\xi+\eta=f}{=} \iint_Q \eta \partial_t \hat{u} \, dx \, dt,
 \end{aligned}$$

whence follows

$$\eta \in \partial_Q \Psi(\partial_t \hat{u}).$$

Consequently, $\hat{u}$ solves (1)–(3). □

Outline of proof for the chain-rule

Step 1 (Approximation). We start with approximating the proper, l.s.c., convex functional φ defined on $V = L^{p(x)}(\Omega)$ as follows:

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$$\varphi_\lambda(u) := \min_{v \in V} \left(\int_{\Omega} \frac{\lambda}{p(x)} \left| \frac{v(x) - u(x)}{\lambda} \right|^{p(x)} dx + \varphi(v) \right) \quad \text{for } u \in V,$$

which is a variant of Moreau-Yosida regularization of φ .

Outline of proof for the chain-rule

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Then φ_λ enjoys similar properties to the usual M-Y regularization with the modified resolvent J_λ and modified Yosida approximation A_λ of $A = \partial_\Omega \varphi$.

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which is a variant of Moreau-Yosida regularization of φ .

Then φ_λ enjoys similar properties to the usual M-Y regularization with the **modified resolvent** J_λ and **modified Yosida approximation** A_λ of $A = \partial_\Omega \varphi$.

One can also define the modified Moreau-Yosida approximation Φ_λ of Φ defined on $\mathcal{V}$.

Outline of proof for the chain-rule

Let $A : V \rightarrow V^*$ be a maximal monotone operator.

- The **modified resolvent** $J_\lambda : V \rightarrow V$ of A is given by,
for each $u \in V$, $J_\lambda u := u_\lambda$, which is a unique solution of

$$Z_\Omega \left(\frac{u_\lambda - u}{\lambda} \right) + A(u_\lambda) \ni 0 \quad \text{in } V^*,$$

where $Z_\Omega(u) := |u|^{p(x)-2}u$ for $u \in V$.

- The **modified Yosida approximation** $A_\lambda : V \rightarrow V^*$ of A is given by

$$A_\lambda(u) := Z_\Omega \left(\frac{u - J_\lambda u}{\lambda} \right) \in A(J_\lambda u) \quad \text{for each } u \in V.$$

Outline of proof for the chain-rule

Step 2 (Correspondence of φ_λ and Φ_λ).

Lemma 7 (Correspondence of φ_λ and Φ_λ)

It follows that

$$\Phi_\lambda(u) = \int_0^T \varphi_\lambda(Pu(t)) \, dt \quad \text{for all } u \in \mathcal{V}.$$

In particular, for $u \in \mathcal{V}$ and $\xi \in \mathcal{V}^*$,

$$\xi_\lambda = \partial_Q \Phi_\lambda(u) \quad \text{iff} \quad P\xi_\lambda(t) = \partial_\Omega \varphi_\lambda(Pu(t)) \quad \text{for a.a. } t \in (0, T).$$

Outline of proof for the chain-rule

Step 3 (Chain-rule for φ_λ). Let $u \in \mathcal{V}$ be such that $\partial_t u \in \mathcal{V}$. Then

$$u, \partial_t u \in L^{p(x)}(Q) \quad \Rightarrow \quad Pu \in W^{1, \textcolor{red}{p}^-}(0, T; L^{p(x)}(\Omega)).$$

Moreover, **since $\partial\varphi_\lambda$ is bounded**, we see that

$$\partial_\Omega \varphi_\lambda(Pu(\cdot)) \in L^\infty(0, T; V).$$

Let $\xi_\lambda := \partial_Q \Phi_\lambda(u)$ and use a standard chain-rule in the L-B space setting.

$$\begin{aligned} \varphi_\lambda(Pu(t)) - \varphi_\lambda(Pu(s)) &\stackrel{\textcolor{blue}{\text{chain}}}{=} \int_s^t \langle \partial_\Omega \varphi_\lambda(Pu(\tau)), (Pu)'(\tau) \rangle_V \, d\tau \\ &\stackrel{\textcolor{blue}{\text{Lem 7}}}{=} \int_s^t \langle P\xi_\lambda(\tau), (Pu)'(\tau) \rangle_V \, d\tau \\ &= \iint_{\Omega \times (s, t)} \textcolor{red}{\xi_\lambda \partial_t u} \, dx d\tau, \quad 0 \leq s \leq t \leq T. \end{aligned}$$

Outline of proof for the chain-rule

Step 4 (Boundedness and convergence).

Lemma 8 (Boundedness of modified Yosida approx.)

Let $u \in V$, $\eta \in Au$ and let A_λ be the modified Yosida approximation.

$$\int_{\Omega} \frac{1}{p'(x)} |A_\lambda u(x)|^{p'(x)} dx \leq \int_{\Omega} \frac{1}{p'(x)} |\eta(x)|^{p'(x)} dx.$$

An analogous statement also holds for maximal monotone $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$.

Outline of proof for the chain-rule

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An analogous statement also holds for maximal monotone $\mathcal{A} : \mathcal{V} \rightarrow \mathcal{V}^*$.

Thus since $\xi_\lambda = \partial_Q \Phi_\lambda(Pu)$, we have, for any $\eta \in \partial_Q \Phi(Pu)$,

$$\iint_Q \frac{1}{p'(x)} |\xi_\lambda|^{p'(x)} dx dt \leq \iint_Q \frac{1}{p'(x)} |\eta|^{p'(x)} dx dt < \infty.$$

Hence

$$\xi_\lambda \rightarrow \xi \quad \text{weakly in } \mathcal{V}^* \quad \text{and} \quad \xi \in \partial_Q \Phi(Pu).$$

Outline of proof for the chain-rule

We have obtained

$$\varphi_{\lambda}(Pu(t)) - \varphi_{\lambda}(Pu(s)) = \iint_{\Omega \times (s,t)} \xi_{\lambda} \partial_t u \, dx d\tau.$$

Outline of proof for the chain-rule

We have obtained

$$\varphi_{\lambda}(Pu(t)) - \varphi_{\lambda}(Pu(s)) = \iint_{\Omega \times (s,t)} \xi_{\lambda} \partial_t u \, dx d\tau.$$

Using the fact

$$\varphi_{\lambda}(u) \rightarrow \varphi(u) \quad \forall u \in V,$$

we have obtained the formula,

$$\varphi(Pu(t)) - \varphi(Pu(s)) = \iint_{\Omega \times (s,t)} \xi \partial_t u \, dx d\tau,$$

which also implies the absolute continuity of $t \mapsto \varphi(Pu(t))$. □

Thank you for your attention !

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