

# Global regularity and stability of a hydrodynamic system modeling vesicle and fluid interactions

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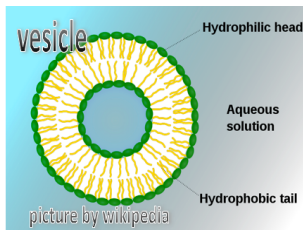
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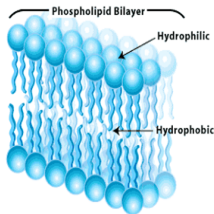
# Outline

- (1) Background
- (2) Phase-field approximation
- (3) Analysis of PDE system
  - ▶ Existence weak/strong solution
  - ▶ Regularity criteria
  - ▶ Stability

# Vesicle Membranes



(a) vesicle



(b) bilayer lipid structure

- Vesicle: a small bubble enclosed by lipid bilayer. It can fuse with the membrane to release its content outside of the cell
- A basic tool used by the cell for organizing cellular substances. Vesicles are involved in metabolism, transport, buoyancy control and enzyme storage.
- A starting point to understand viscoelastic properties, dynamics and rheology of bio-fluids.

# Elastic bending energy

Configurations of vesicle membranes can be characterized by Canham (1970)-Helfrich (1973) models

- Thickness usually small: considered as a closed 2-D surface.
- Helfrich bending elastic energy

$$E_{elastic} = \int_{\Gamma} \left[ \frac{k_0}{2} (H - c_0)^2 + \frac{\bar{k}}{2} K \right] ds.$$

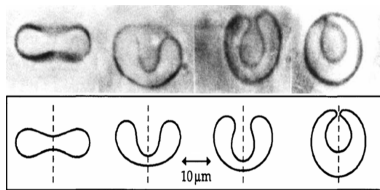
$\Gamma$  – the vesicle membrane,  $k_0, \bar{k}$  – the bending rigidities

$H$  – mean curvature  $K$  – Gaussian curvature

$c_0$  – the spontaneous curvature describing the asymmetry effect of the membrane or its environment.

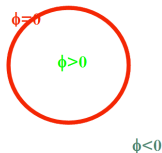
# Equilibrium shape for vesicle

- The equilibrium configuration:  
→ minimizing of the bending energy  $E_{elastic}$  subject to fixed volume/area
- ⇒ Highly nonlinear Euler-Lagrange equation with two Lagrange multipliers:  
complicated free boundary problem, drawbacks for numerical simulations
- Example: changing of surface area / volume may result the change of equilibrium shapes of vesicles



# Phase-field model

- Du, Liu, Wang (JCP, 2004): Phase function  $\phi = \phi(x)$  defined on a computational domain  $\Omega$ , to label the inside ( $\phi > 0$ ) and outside ( $\phi < 0$ ) of the vesicle
- Vesicle membrane  $\Gamma$  : the level set  $\{x : \phi(x) = 0\}$



- Sharp interface  $\Gamma$  is replaced by a diffuse interface, a thin neighborhood of thickness  $\varepsilon$  of the zero level set of  $\phi$ .

# Phase-field approximation

- For simplicity, consider homogeneous membrane with zero  $c_0$ :

$$E_{elastic} = \int_{\Gamma} \frac{k}{2} H^2 ds.$$

- Modified elastic energy (Du, Liu, Wang, JCP, 2004):

$$E_{\varepsilon}(\phi) = \frac{k}{2\varepsilon} \int_{\Omega} \left| \varepsilon \Delta \phi - \frac{1}{\varepsilon} \phi(1 - \phi^2) \right|^2 dx$$

- Two constraints

$$A(\phi) = \text{Volume} = \int_{\Omega} \phi dx = \alpha$$

$$B(\phi) = \text{Surface Area} = \int_{\Omega} \frac{\varepsilon |\nabla \phi|^2}{2} + \frac{(\phi^2 - 1)^2}{4\varepsilon} dx = \beta.$$

# Evolution in fluid under constraints

- The original problem of minimizing the bending energy with the prescribed surface area and volume constraints  
→ finding the function  $\phi$  that minimizes the energy  $E_\varepsilon$  with the constraints of prescribed values for  $A$  and  $B$ .

- Involving fluid interaction:

Incompressible **Navier-Stokes** with extra stress from membrane

+

**Phase-field** transported by fluid

- An energetic variational approach  $\implies$  Coupling system under constraints (Du, Liu, Ryham, Wang, Physica D, 2009):

$$u_t + u \cdot \nabla u + \nabla P = \mu \Delta u + \left( \frac{\delta E_\varepsilon(\phi)}{\delta \phi} + \lambda(t) + \frac{\delta B(\phi)}{\delta \phi} \mu(t) \right) \nabla \phi,$$

$$\nabla \cdot u = 0,$$

$$\phi_t + u \cdot \nabla \phi = -\gamma \left( \frac{\delta E_\varepsilon(\phi)}{\delta \phi} + \lambda(t) + \frac{\delta B(\phi)}{\delta \phi} \mu(t) \right),$$

# Known results without fluid interaction $\varepsilon = 1$

(a) Volume constraint (Colli, Laurencot, IFB, 2011)

$$\phi_t - \Delta \frac{\delta B(\phi)}{\delta \phi} + (3\phi^2 - 1) \frac{\delta B(\phi)}{\delta \phi} - \overline{(3\phi^2 - 1) \frac{\delta B(\phi)}{\delta \phi}} = 0$$

(b) Volume & area constraints (Colli, Laurencot, SIMA, 2012)

$$\phi_t - \Delta \frac{\delta B(\phi)}{\delta \phi} + (3\phi^2 - 1) \frac{\delta B(\phi)}{\delta \phi} = \lambda_1(t) + \lambda_2(t) \frac{\delta B(\phi)}{\delta \phi}$$

- Homogeneous Neumann BC for  $\phi$  and  $\frac{\delta B(\phi)}{\delta \phi}$
- Existence and uniqueness: using gradient flow structure and a time-discrete minimization scheme.
- Remark: analysis for (b) is restricted to the case where critical points of  $B(\phi)$  under a volume constraint cannot be reached during time evolution (e.g., large area  $\beta$  with small volume  $|\alpha|$ )

# Hydrodynamic system for fluid and vesicle interactions

A penalty formulation (Du, Li, Liu, DCDS-B, 2007)

$$E(\phi) = E_\varepsilon(\phi) + \frac{1}{2}M_1(A(\phi) - \alpha)^2 + \frac{1}{2}M_2(B(\phi) - \beta)^2$$

- The PDE system

$$u_t + u \cdot \nabla u + \nabla P = \mu \Delta u + \frac{\delta E(\phi)}{\delta \phi} \nabla \phi, \quad (1)$$

$$\nabla \cdot u = 0, \quad (2)$$

$$\phi_t + u \cdot \nabla \phi = -\gamma \frac{\delta E(\phi)}{\delta \phi}, \quad (3)$$

- BC: periodic in  $Q = [0, 1]^3$

- IC:

$$u|_{t=0} = u_0(x), \text{ with } \nabla \cdot u_0 = 0, \int_Q u_0 dx = 0, \phi|_{t=0} = \phi_0(x).$$

# Known results

System (1)-(3) with no-slip boundary condition for  $u$  and Dirichlet boundary conditions for  $\phi$ :

$$u = 0, \quad \phi = -1, \quad \Delta\phi = 0.$$

(1) Q. Du, M. Li & C. Liu, DCDS-B, 2007:

- ▶ Existence of global weak solutions;
- ▶ Uniqueness under extra regularity  $u \in L^8(0, T; L^4)$

(2) Y. Liu, T. Takahashi & M. Tucsnak, JMFM, 2012:

- ▶ Existence/uniqueness of local strong solution in fractional order Sobolev spaces (via fixed point argument)
- ▶ Almost global solutions under the assumptions of *small*  $(|\Omega| + \alpha)^2$  and initial data (in terms of the existing length  $T$ ).

# Summary of results



*Joint work with X. Xu (CMU), SIMA 2013*

We study the 3D hydrodynamic system (1)-(3) with penalty in periodic setting,

- Existence of global weak solutions
- Existence and uniqueness of local strong solutions
- Regularity criteria for local strong solutions that only involve the velocity field  $u$
- Well-posedness and stability of global strong solutions near 0 (for  $u$ ) and local minimizers of  $E$  (for  $\phi$ )

# Basic energy law

Functional settings for periodic problems:

$$H_p^m(Q) = \{v \in H_{loc}^m(\mathbb{R}^3; \mathbb{R}^3) \mid v(x + e_i) = v(x)\},$$

$$\dot{H}_p^m(Q) = H_p^m(Q) \cap \left\{ v : \int_Q v(x) dx = 0 \right\},$$

$$\mathbf{H} = \{v \in \mathbf{L}_p^2(Q), \nabla \cdot v = 0\}, \text{ where } L_p^2(Q) = H_p^0(Q),$$

$$\mathbf{V} = \{v \in \mathbf{H}_p^1(Q), \nabla \cdot v = 0\}.$$

- Total energy: kinetic + elastic

$$\mathcal{E}(t) = \frac{1}{2} \|u(t)\|^2 + E(\phi(t))$$

The coupling system (1)-(3) has the following dissipative energy law:

$$\frac{d}{dt} \mathcal{E}(t) + \mu \|\nabla u\|^2 + \gamma \left\| \frac{\delta E}{\delta \phi} \right\|^2 = 0, \quad (4)$$

# Existence of global weak solutions

## Theorem

*For any initial datum  $(u_0, \phi_0) \in \dot{\mathbf{H}} \times H_p^2$ ,  $T > 0$ , there exists at least one global weak solution  $(u, \phi)$  to the problem (1)–(3) that satisfies*

$$\begin{aligned} u &\in L^\infty(0, T; \dot{\mathbf{H}}) \cap L^2(0, T; \dot{\mathbf{V}}); \\ \phi &\in L^\infty(0, T; H_p^2) \cap L^2(0, T; H_p^4) \cap H^1(0, T; L_p^2). \end{aligned}$$

*In addition, the weak solution is unique provided that  $u \in L^8(0, T; \mathbf{L}_p^4)$ .*

Sketch of proof (as in Du et al, 2007)

- Galerkin approximation to both  $u$  and  $\phi$
- Derive *a priori* estimates in the approximation system
- Passing to the limit
- Derivation of a Gronwall type inequality for uniqueness under extra regularity for  $u$

# Useful estimates

Basic energy law  $\implies$

$$\|u(t)\| + \|\phi(t)\|_{H^2} \leq C(\mathcal{E}(0)),$$
$$\int_0^\infty (\mu \|\nabla u(t)\|^2 + \gamma \|\frac{\delta E}{\delta \phi}\|^2) dt \leq \mathcal{E}(0).$$

## Lemma

*There exists a positive constant  $C$  depending on  $\|\phi\|_{H^2}$ , such that*

$$\|\nabla \Delta \phi\| \leq C \left\| \frac{\delta E}{\delta \phi} \right\|^{\frac{1}{2}} + C, \quad \|\Delta^2 \phi\| \leq \frac{1}{k\varepsilon} \left\| \frac{\delta E}{\delta \phi} \right\| + C, \quad \forall \phi \in H_p^4.$$

## Further regularity for $\phi$

Due to the “weak” coupling in the phase-field equation (e.g., the convection term  $u \cdot \nabla \phi$ ), we have

### Lemma

*For any smooth solution to the problem (1)–(3), it holds*

$$\frac{d}{dt} \|\nabla \Delta \phi\|^2 + k\gamma\varepsilon \|\nabla \Delta^2 \phi\|^2 \leq C(\|\nabla u\|^2 + 1) \|\nabla \Delta \phi\|^2 + C(1 + \|\nabla u\|^2),$$

*C only depends on  $\mathcal{E}(0)$  and coefficients of the system.*

**Remark:** The uniform Gronwall's inequality and the basic energy law yield

$$\|\nabla \Delta \phi(t+r)\|^2 \leq C \left(1 + \frac{1}{r}\right), \quad \forall t \geq 0, r > 0.$$

If  $\phi_0 \in H_p^3$ , then

$$\|\phi(t)\|_{H^3} \leq C, \quad \|\nabla \phi(t)\|_{L^\infty} \leq C, \quad \forall t \geq 0.$$

# Local strong solution: higher order energy inequality

Define

$$\mathcal{A}(t) = \|\nabla u\|^2(t) + \eta \left\| \frac{\delta E}{\delta \phi} \right\|^2(t), \quad \eta > 0.$$

## Lemma

*For any smooth solution to the problem (1)–(3), if*

$$\|\phi(t)\|_{H^3} + \|\nabla \phi(t)\|_{L^\infty} \leq K, \quad \forall t \geq 0.$$

*then for*

$$\eta = \frac{\mu\gamma}{16k_\varepsilon K^2},$$

*it holds*

$$\frac{d}{dt} \mathcal{A}(t) + \mu \|\Delta u\|^2 + k_\varepsilon \gamma \eta \left\| \Delta \frac{\delta E}{\delta \phi} \right\|^2 \leq C_* (\mathcal{A}^3(t) + \mathcal{A}(t)).$$

*Here  $C_* > 0$  depends on  $\|u_0\|$ ,  $\|\phi_0\|_{H^2}$ ,  $K$  and coefficients of the system.*

# Existence of local strong solution

## Theorem

*For any initial datum  $(u_0, \phi_0) \in \dot{\mathbf{V}} \times H_p^4(Q)$ , there exists  $T_0 \in (0, +\infty)$  such that the problem (1)–(3) admits a unique strong solution  $(u, \phi)$  satisfying*

$$u \in L^\infty(0, T_0; \dot{\mathbf{V}}) \cap L^2(0, T_0; \mathbf{H}_p^2);$$

$$\phi \in L^\infty(0, T_0; H_p^4) \cap L^2(0, T_0; H_p^6) \cap H^1(0, T_0; H_p^2).$$

## Remark:

- Uniqueness results from the fact  $u \in L^\infty(0, T_0; \dot{\mathbf{V}}) \subset L^8(0, T_0; \mathbf{L}_p^4)$ .
- One cannot expect global strong solutions for arbitrary large initial data: (1)–(3) contains the 3D Navier-Stokes equation as a subsystem.

# Global strong solution: large viscosity case

## Lemma

For any  $\mu \geq \mu_0 > 0$ , choosing the parameter  $\eta$  in  $\mathcal{A}(t)$  to be

$$\eta' = \frac{\mu_0 \gamma}{16 k \varepsilon K^2},$$

then the following inequality holds for  $(u, \phi)$ :

$$\frac{d}{dt} \mathcal{A}(t) + \left( \mu - \mu^{\frac{1}{2}} \mathcal{A}(t) \right) \|\Delta u\|^2 + k \varepsilon \gamma \eta' \left\| \Delta \frac{\delta E}{\delta \phi} \right\|^2 \leq C' \mathcal{A}(t),$$

where  $C'$  is a constant depending on  $\|u_0\|$ ,  $\|\phi_0\|_{H^2}$ ,  $K$ ,  $\mu_0$  and coefficients of the system but except  $\mu$ .

## Theorem (Global strong solution under large viscosity)

For any initial data  $(u_0, d_0) \in \dot{\mathbf{V}} \times H_p^4$ , if  $\mu$  is sufficiently large, then there exists a unique global strong solution.

# Regularity criteria: Serrin-type

## Theorem

For  $(u_0, \phi_0) \in \dot{\mathbf{V}} \times H_p^4$ , let  $(u(t), \phi(t))$  be a local smooth solution to the problem (1)-(3) on  $[0, T)$  for some  $0 < T < +\infty$ . Suppose that one of the following conditions holds,

$$(i) \quad \int_0^T \|\nabla u(t)\|_{\mathbf{L}^p}^s dt < +\infty, \text{ for } \frac{3}{p} + \frac{2}{s} \leq 2, \frac{3}{2} < p \leq +\infty,$$

$$(ii) \quad \int_0^T \|u(t)\|_{\mathbf{L}^p}^s dt < +\infty, \text{ for } \frac{3}{p} + \frac{2}{s} \leq 1, 3 < p \leq +\infty.$$

Then  $(u(t), \phi(t))$  can be extended beyond  $T$ .

**Remark:** Despite the nonlinear coupling between the equations for velocity field and the phase function,  $u$  indeed plays a dominant role in regularity for solutions.

# Proof

Just consider Case (i): for  $p > \frac{3}{2}$ ,

$$\begin{aligned}(u \cdot \nabla u, \Delta u) &\leq C \|\nabla u\|_{\mathbf{L}^p} \|\nabla u\|_{\mathbf{L}^{\frac{2p}{p-1}}}^2 \\&\leq C \|\nabla u\|_{\mathbf{L}^p} (\|\nabla u\|^{\frac{2p-3}{p}} \|\Delta u\|^{\frac{3}{p}} + \|\nabla u\|^2) \\&\leq \frac{\mu}{8} \|\Delta u\|^2 + C (\|\nabla u\|_{\mathbf{L}^p} + \|\nabla u\|_{\mathbf{L}^p}^{\frac{2p}{2p-3}}) \|\nabla u\|^2.\end{aligned}$$

Uniform estimates for  $\|\phi\|_{H^3}$  and  $\|\nabla \phi\|_{L^\infty} \implies$

$$\frac{d}{dt} \mathcal{A}(t) \leq C \left(1 + \|\nabla u\|_{\mathbf{L}^p}^{\frac{2p}{2p-3}}\right) \mathcal{A}(t).$$

**Remark:** Beale-Kato-Majda type criterion is also valid (Zhao et al 2012):

$$\int_0^T \|\nabla \times u\|_{\mathbf{L}^\infty} dt < +\infty.$$

# Improved logarithmic regularity criteria

## Theorem

For  $(u_0, \phi_0) \in (\dot{\mathbf{V}} \cap \mathbf{H}_p^2) \times H_p^5$ , let  $(u, \phi)$  be a local smooth solution to the problem (1)–(3) on  $[0, T)$  for some  $0 < T < +\infty$ . If one of the following conditions holds,

(i)

$$\int_0^T \frac{\|\nabla u(t)\|_{\mathbf{L}^p}^s}{1 + \ln(e + \|\nabla u(t)\|_{\mathbf{L}^p})} dt < +\infty, \quad \text{for } \frac{3}{p} + \frac{2}{s} \leq 2, \quad \frac{3}{2} \leq p \leq 6,$$

(ii)

$$\int_0^T \frac{\|u(t)\|_{\mathbf{L}^p}^s}{1 + \ln(e + \|u(t)\|_{\mathbf{L}^\infty})} dt < +\infty, \quad \text{for } \frac{3}{p} + \frac{2}{s} \leq 1, \quad 3 < p \leq +\infty,$$

then  $(u, \phi)$  can be extended beyond  $T$ .

## Sketch of proof for (i)

Recall  $\mathcal{A}(t) = \|\nabla u\|^2(t) + \eta \left\| \frac{\delta E}{\delta \phi} \right\|^2(t)$ . Denote

$$\mathcal{Q}(t) = \frac{1 + \|\nabla u(t)\|_{\mathbf{L}^p}^{\frac{2p}{2p-3}}}{1 + \ln(e + \|\nabla u(t)\|_{\mathbf{L}^p})},$$

then from previous argument

$$\frac{d}{dt} \mathcal{A}(t) \leq C(1 + \|\nabla u\|_{\mathbf{L}^p}^{\frac{2p}{2p-3}}) \mathcal{A}(t) \leq C_* \mathcal{Q}(t) [1 + \ln(e + \|\Delta u(t)\|)] \mathcal{A}(t),$$

Besides, since  $\|u(t)\| + \|\phi(t)\|_{H^3} \leq C$ , we have

$$\frac{d}{dt} \left( \|\Delta u\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2 \right) \leq C(\mathcal{A}^5(t) + \mathcal{A}(t)).$$

## Sketch of proof for (i) (continued)

Assume  $[0, T)$  is the maximal interval of existence but  $\int_0^T \mathcal{Q}(t) dt = M < +\infty$ .  
Fix  $\epsilon \in (0, \frac{1}{5C_*}]$ , for sufficiently small  $t_1$ ,

$$\int_0^{t_1} \mathcal{Q}(t) dt \leq \epsilon.$$

Using Gronwall inequality,  $\forall t \in [0, t_1]$ ,

$$\begin{aligned} \mathcal{A}(t) &\leq \mathcal{A}(0) \exp \left( C_* \left[ 1 + \ln \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right) \right] \int_0^t \mathcal{Q}(s) ds \right) \\ &\leq \mathcal{A}(0) e^{C_* \epsilon} \left( e + \sup_{[0,t]} \|\Delta u(\cdot)\| \right)^{C_* \epsilon}. \end{aligned}$$

which implies for  $t \in [0, t_1]$ ,

$$\frac{d}{dt} \left( \|\Delta u\|^2(t) + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2(t) \right) \leq C \left( e + \sup_{[0,t]} \|\Delta u\|(\cdot) \right).$$

## Sketch of proof for (i) (continued)

After integration we get

$$\begin{aligned} & \sup_{[0, t_1]} \left( \|\Delta u(\cdot)\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi}(\cdot) \right\|^2 \right) \\ & \leq \|\Delta u(0)\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2(0) + CT \left( e + \sup_{[0, t_1]} \|\Delta u(\cdot)\| \right) \\ & \leq \|\Delta u(0)\|^2 + \eta_1 \left\| \nabla \frac{\delta E}{\delta \phi} \right\|^2(0) + \frac{1}{2} \sup_{[0, t_1]} \|\Delta u(\cdot)\|^2 + C_T. \end{aligned}$$

By iteration, we can prove

$$\|u\|_{L^\infty(0, T; \mathbf{H}_p^2)} \leq C, \quad \|\phi\|_{L^\infty(0, T; H_p^5)} \leq C,$$

which indicates  $[0, T)$  is not the maximal interval of existence.

# Stability: An alternative result on existence of global strong solution

## Theorem

*For any initial data  $(u_0, \phi_0) \in \dot{\mathbf{V}} \times H_p^4$ , there exists  $\varepsilon_0 \in (0, 1)$ , either*  
(1) *The problem (1)–(3) has a unique global strong solution  $(u, \phi)$  with uniform-in-time estimate*

$$\|u(t)\|_{\mathbf{V}} + \|\phi(t)\|_{H^4} \leq C, \quad \forall t \geq 0,$$

*or (2) there is a  $T_* \in (0, +\infty)$  such that  $\mathcal{E}(T_*) \leq \mathcal{E}(0) - \varepsilon_0$ .*

**Remark:** If the total energy  $\mathcal{E}(t)$  cannot “drop” too much for all time, then there exists a unique bounded global strong solution.

## Corollary

*For  $(u_0, \phi_0) \in \dot{\mathbf{V}} \times H_p^4$ , if  $\|u_0\|$  and  $\|\phi - \phi^*\|_{H^2}$  are small,  $\phi^*$  is the absolute minimizer of  $E(\phi)$ , then problem (1)–(3) has a unique global strong solution.*

# Eventual regularity of weak solutions

## Corollary (Eventual regularity of weak solutions)

*Suppose  $(u, \phi)$  is a global weak solution of the problem (1)–(3). Then there exists a time  $T_0 \in (0, +\infty)$  such that  $(u, \phi)$  becomes a strong solution on  $[T_0, +\infty)$ .*

Key to the proof:

- the basic energy law (4)  $\implies$

$$\int_0^\infty \mathcal{A}(t) dt < +\infty.$$

- higher energy inequality for  $\mathcal{A}(t)$ :

$$\frac{d}{dt} \mathcal{A}(t) \leq C_*(\mathcal{A}^3(t) + \mathcal{A}(t)).$$

# Local energy minimizer of elastic energy

## Definition

$\phi^* \in H_p^2$  is called a local minimizer of  $E(\phi)$ , if there exists a  $\delta > 0$ ,  $E(\phi^*) \leq E(\phi)$  for all  $\phi \in H_p^2$  satisfying  $\|\phi - \phi^*\|_{H^2} < \delta$ . If for all  $\phi \in H_p^2$ ,  $E(\phi^*) \leq E(\phi)$ , then  $\phi^*$  is an absolute minimizer.

## Lemma

*Let  $\mathcal{B}$  be a bounded closed convex subset of  $H_p^2$ . The approximate elastic energy  $E(\phi)$  admits at least one minimizer  $\phi^* \in \mathcal{B}$  such that  $E(\phi^*) = \inf_{\phi \in \mathcal{B}} E(\phi)$ .*

**Remark:** If  $\phi$  is a minimizer of  $E(\phi)$ , then it is a critical point of  $E(\phi)$ . Meanwhile, any critical point of  $E(\phi)$  in  $H_p^2$  is equivalent to a weak solution to the forth-order nonlocal elliptic problem

$$\frac{\delta E}{\delta \phi} = 0, \quad \text{with} \quad \phi(x + \mathbf{e}_i) = \phi(x).$$

## Theorem

Let  $\phi^* \in H_p^4(Q)$  be a local minimizer of  $E(\phi)$ . For any  $R > 0$ , consider the initial data

$$(u_0, \phi_0) \in \mathbf{B} = \{(u, \phi) \in \dot{\mathbf{V}} \times H_p^4(Q) : \|u\|_{\mathbf{H}^1} \leq R, \|\phi_0 - \phi^*\|_{H^4} \leq R\}.$$

For any  $\epsilon > 0$ , there exists  $\sigma \in (0, \delta)$ , such that if the initial data  $(u_0, \phi_0) \in \mathbf{B}$  satisfies the condition

$$\|u_0\| + \|\phi_0 - \phi^*\|_{H^2} \leq \sigma,$$

then there exists a unique global strong solution satisfying

$$\|\phi(t) - \phi^*\|_{H^2} \leq \epsilon, \quad \forall t \geq 0,$$

and

$$\lim_{t \rightarrow +\infty} (\|u(t)\|_{\mathbf{H}^1} + \|\phi(t) - \phi_\infty\|_{H^4}) = 0.$$

# Key to the proof

- Basic energy law (4)  $\implies \int_0^\infty \|\phi_t\|^2 dt < +\infty$
- Łojasiewicz–Simon inequality

## Lemma

*There exist constants  $\beta > 0$ ,  $\theta \in (0, \frac{1}{2})$  such that for any  $\phi \in H_p^4(Q)$  with  $\|\phi - \phi^*\|_{H^2} < \beta$ , it holds*

$$\left\| \frac{\delta E}{\delta \phi} \right\| \geq |E(\phi) - E(\psi)|^{1-\theta}.$$

- Łojasiewicz–Simon approach  $\implies$  control on  $\int_s^t \|\phi_t\| d\tau$ .

## Remark:

- (i) The asymptotic limit  $\phi_\infty$  is also a local minimizer of  $E(\phi)$  at the same energy level as  $\phi^*$ . If  $\phi^*$  is an isolated local minimizer, then  $\phi_\infty = \phi^*$  and  $\phi^*$  is asymptotically stable.
- (ii) Estimate on convergence can be obtained

# Further questions

- Asymptotic behavior as  $\varepsilon \rightarrow 0$ 
  - ▶ Formal argument under constraints: Du, Liu, Ryham, Wang (2005), Wang (2008).
  - ▶ Rigorous approach without or with constraints: Bellettini, Mugnai (2010)
- Asymptotic behavior as  $M_1, M_2 \rightarrow +\infty$
- Fluid interaction with constraints
- Fluid interaction model with variable density and viscosity
- Non-homogeneous membranes and generalizations: Wang, Du (2008); Lowengrub, Rätz, Voigt (2009); Elliott, Stinner (2010, 2013), etc

The End

Thank You !