

Global solvability of some double-diffusive convection systems

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(BF)

Double-diffusive convection flow based upon Brinkman-Forchheimer equations

$$\text{(BF)}_{(\pi)}^{D(N)} \left\{ \begin{array}{l}
 \mathbf{u}_t = \nu \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - a \mathbf{u} - \nabla p + \mathbf{g}T + \mathbf{h}C + \mathbf{f}_1 \quad \text{in } \Omega \times \{t > 0\}, \\
 T_t + \mathbf{u} \cdot \nabla T = \Delta T + f_2 \quad \text{in } \Omega \times \{t > 0\}, \\
 C_t + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T + f_3 \quad \text{in } \Omega \times \{t > 0\}, \\
 \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times \{t > 0\}, \\
 \mathbf{u} = 0 ; T = 0 \left(\frac{\partial T}{\partial n} = 0 \right); C = 0 \left(\frac{\partial C}{\partial n} = 0 \right) \text{ on } \partial\Omega \times \{t > 0\} \\
 \mathbf{u}|_{t=0} = \mathbf{u}_0(x) ; \quad T|_{t=0} = T_0(x) ; \quad C|_{t=0} = C_0(x), \\
 (\mathbf{u}(0) = \mathbf{u}(S) ; \quad T(0) = T(S) ; \quad C(0) = C(S),)
 \end{array} \right.$$

- $\mathbf{u}(x, t)$: solenoidal velocity of the fluid, $\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}$, $T_t = \frac{\partial T}{\partial t}$, $C_t = \frac{\partial C}{\partial t}$,
- $T(x, t)$: temperature,
- $C(x, t)$: concentration of solute (salt for oceanography),
- $p(x, t)$: pressure,
- $\mathbf{g}, \mathbf{h}, \rho, a$: constant vector term derived from gravity, Soret coefficient, and Darcy coefficient
- $\Omega \subset \mathbb{R}^N$: bounded domain, $\mathbf{f}_1, f_2, f_3$: external forces

Navier-Stokes Equations

$$(\text{NS})_{(\pi)} \left\{ \begin{array}{l} \mathbf{u}_t = \nu \Delta \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} - \nabla p + \mathbf{f}(t) \quad \text{in } \Omega \times \{t > 0\}, \\ \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times \{t > 0\}, \\ \mathbf{u}|_{\partial\Omega} = 0 \\ \mathbf{u}|_{t=0} = \mathbf{u}_0(x), \quad (\mathbf{u}(0) = \mathbf{u}(S)) \end{array} \right.$$

- $\mathbf{u}(x, t)$: solenoidal velocity of the fluid, $\mathbf{u}_t = \frac{\partial \mathbf{u}}{\partial t}$,
- $p(x, t)$: pressure.

Known Results

- (NS) $N = 2$: $\exists$ unique global solution
- (NS) $N = 3$: $\exists$ unique local solution, $\exists$ unique global **small** solution
- (NS) $_{\pi}$ $N = 2$: $\exists S$ -periodic solution for any $\mathbf{f} \in L^2(0, S; \mathbb{L}^2(\Omega))$
- (NS) $_{\pi}$ $N = 3$: $\exists S$ -periodic solution for **small** $\mathbf{f} \in L^2(0, S; \mathbb{L}^2(\Omega))$

Known Results: Dirichlet BC

Theorem 1 (Terasawa-Ô (2010))

For all $N \leq 3$ and for any $\mathbf{u}_0 \in \mathbb{H}_\sigma^1(\Omega)$, $T_0, C_0 \in H_0^1(\Omega)$, $\mathbf{f}_1 \in L_{loc}^2([0, \infty); \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L_{loc}^2([0, \infty); L^2(\Omega))$, $(\text{BF})^D$ has a unique (global) solution $U = (\mathbf{u}, T, C)^t$ satisfying

$$\left\{ \begin{array}{l} \mathbf{u}_t, \mathcal{A}\mathbf{u} \in L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \text{ where } \mathcal{A} : \text{Stokes Operator} \\ T_t, C_t, \Delta T, \Delta C \in L^2(0, S; L^2(\Omega)), \\ \mathbf{u} \in C([0, S]; \mathbb{H}_\sigma^1(\Omega)), \quad T, C \in C([0, S]; H_0^1(\Omega)) \quad \forall S \in (0, \infty). \end{array} \right.$$

Theorem 2 (Uchida-Ô (2013))

For all $N \leq 3$ and for any $\mathbf{f}_1 \in L^2(0, S; \mathbb{L}^2(\Omega))$, $f_2, f_3 \in L^2(0, S; L^2(\Omega))$, $(\text{BF})_\pi^D$ has a S -periodic solution $U = (\mathbf{u}, T, C)^t$ satisfying

$$\left\{ \begin{array}{l} \mathbf{u}_t, \mathcal{A}\mathbf{u} \in L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \text{ where } \mathcal{A} : \text{Stokes Operator} \\ T_t, C_t, \Delta T, \Delta C \in L^2(0, S; L^2(\Omega)), \\ \mathbf{u} \in C([0, S]; \mathbb{H}_\sigma^1(\Omega)), \quad T, C \in C([0, S]; H_0^1(\Omega)). \end{array} \right.$$

Neumann BC

Double-difusive convection flow with Neumann BC

$$(\text{BF})_{(\pi)}^N \left\{ \begin{array}{l}
 \mathbf{u}_t = \nu \Delta \mathbf{u} - a \mathbf{u} - \nabla p + \mathbf{g}T + \mathbf{h}C + \mathbf{f}_1 \quad \text{in } \Omega \times \{t > 0\}, \\
 T_t + \mathbf{u} \cdot \nabla T = \Delta T + f_2 \quad \text{in } \Omega \times \{t > 0\}, \\
 C_t + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T + f_3 \quad \text{in } \Omega \times \{t > 0\}, \\
 \nabla \cdot \mathbf{u} = 0 \quad \text{in } \Omega \times \{t > 0\}, \\
 \mathbf{u} = 0 ; \quad \frac{\partial T}{\partial n} = 0 ; \quad \frac{\partial C}{\partial n} = 0 \quad \text{on } \partial\Omega \times \{t > 0\} \\
 \mathbf{u}|_{t=0} = \mathbf{u}_0(x) ; \quad T|_{t=0} = T_0(x) ; \quad C|_{t=0} = C_0(x), \\
 (\mathbf{u}(0) = \mathbf{u}(S) ; \quad T(0) = T(S) ; \quad C(0) = C(S),)
 \end{array} \right.$$

Function Spaces

- Ω : bounded domain in $\mathbb{R}^N$, $Q = \Omega \times (0, S)$,
- $\mathbb{C}_\sigma^\infty(\Omega) = \{\mathbf{u} = (u^1, u^2, \dots, u^N)^t; u^j \in C_0^\infty(\Omega) \ \forall j = 1, 2, \dots, N, \nabla \cdot \mathbf{u} = 0\}$,
- $\mathbb{L}^2(\Omega) = (L^2(\Omega))^N$, $\mathbb{H}^1(\Omega) = (H^1(\Omega))^N = (W^{1,2}(\Omega))^N$,
- $\mathbb{L}_\sigma^2(\Omega) =$ The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{L}^2(\Omega)$ -norm,
- $\mathbb{H}_\sigma^1(\Omega) =$ The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{H}^1(\Omega)$ -norm,
- $H = \mathbb{L}_\sigma^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, $H^0 = \mathbb{L}^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$,
- $C_\pi([0, S]; H) = \{U \in C([0, S]; H); U(0) = U(S)\}$,
- $\mathcal{P}_\Omega =$ The orthogonal projection from $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}_\sigma^2(\Omega)$,
- $\mathcal{A} = -\mathcal{P}_\Omega \Delta$: The Stokes operator with domain $D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)$,
- $A_N = -\Delta$ with domain $D(A_N) = \{u \in H^2(\Omega); \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\}$,
- $\mathcal{A}^\alpha$, A_D^α and A_N^α denote the fractional powers of $\mathcal{A}$, A_D and A_N of order α .

Main Results: Initial Boundary Value Problem

Theorem 3 (Uchida-Ô (2013))

Let $N \leq 3$ and $(f_1, f_2, f_3)^t \in L^2(0, S; H^0)$. Then for each initial data $U_0 = (\mathbf{u}_0, T_0, C_0)^t \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha)$ with $\alpha \in [1/4, 1/2]$, $(\text{BF})^N$ admits a unique solution $U = (\mathbf{u}, T, C)^t \in C([0, S]; H)$ satisfying $U(0) = U_0$ and

$$(\#)_\alpha \quad \begin{cases} t^{1/2-\alpha} \partial_t \mathbf{u}, t^{1/2-\alpha} \mathcal{A} \mathbf{u} \in L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \\ t^{1/2-\alpha} \|\nabla \mathbf{u}\|_{\mathbb{L}^2(\Omega)} \in L_*^p(0, S) \quad \text{for all } p \in [2, \infty], \\ t^{1/2-\alpha} \partial_t T, t^{1/2-\alpha} \partial_t C, t^{1/2-\alpha} \Delta T, t^{1/2-\alpha} \Delta C \in L^2(0, S; L^2(\Omega)), \\ t^{1/2-\alpha} \|\nabla T\|_{L^2(\Omega)}, t^{1/2-\alpha} \|\nabla C\|_{L^2(\Omega)} \in L_*^p(0, S) \quad \text{for all } p \in [2, \infty], \end{cases}$$

where $L_*^p = L^p(dt/t)$, i.e., $\|f\|_{L_*^p(0,S)} = (\int_0^S |f(t)|^p t^{-1} dt)^{1/p}$ for $1 \leq p < \infty$ and $L_*^\infty = L^\infty$.

Main Results: Periodic Problem

Theorem 4 (Uchida-Ô (2013))

Let $N \leq 3$ and $(f_1, f_2, f_3)^t \in L^2(0, S; H^0)$ such that

$$f_2, f_3 \in \left\{ f; \int_Q f(x, t) dx dt = 0 \right\}. \quad (1)$$

Then $(\text{BF})_\pi^N$ admits a solution $U = (\mathbf{u}, T, C)^t \in C_\pi([0, S]; H)$ satisfying

$$(\#)_{1/2} \quad \begin{cases} \partial_t \mathbf{u}, \mathcal{A} \mathbf{u} \in L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \\ \mathbf{u} \in C([0, S]; \mathbb{H}_\sigma^1(\Omega)), \\ \partial_t T, \partial_t C, \Delta T, \Delta C \in L^2(0, S; L^2(\Omega)), \\ T, C \in C([0, S]; H^1(\Omega)). \end{cases}$$

Remark 1

Condition (1) is the necessary condition for the existence of the periodic solution of $(\text{BF})^N$.

Reduction to an Abstract Problem

Let φ be a proper lower semi-continuous convex function from H into $(-\infty, +\infty]$. Define the effective domain of φ by $D(\varphi) = \{U \in H; \varphi(U) < +\infty\}$ and the subdifferential of φ by

$$\partial\varphi(U) = \{f \in H; \varphi(V) - \varphi(U) \geq (f, V - U)_H \text{ for all } V \in H\}$$

with domain $D(\partial\varphi) = \{U \in H; \partial\varphi(U) \neq \emptyset\}$. Then $A = \partial\varphi$ becomes a maximal monotone operator.

It is well known that $J_\lambda U = (I + \lambda A)^{-1}U \rightarrow U$, as $\lambda \rightarrow +0$ for all $U \in \overline{D(A)}$. Then for $\alpha \in (0, 1)$, $p \in [1, \infty]$, by measuring how fast $J_\lambda U$ converges to U , we can define a nonlinear interpolation class $\mathcal{B}_{\alpha,p}(A)$ associated with A by

$$\mathcal{B}_{\alpha,p}(A) = \{U \in \overline{D(A)}; t^{-\alpha}|U - J_t U|_H \in L_*^p(0, 1)\}.$$

We often use the notation

$$|U|_{\mathcal{B}_{\alpha,p}(A)} = \left| t^{-\alpha}|U - J_t U|_H \right|_{L_*^p(0,1)}.$$

If A is non-negative self-adjoint operator, then $D(A^\alpha) = \mathcal{B}_{\alpha,2}(A)$.

In the later arguments, it will be shown that the leading terms $(\mathcal{A}, A_N, A_N)^t$ can be given as the subdifferential of a suitable lower semi-continuous convex function on H .

Operate $\mathcal{P}_\Omega$ to the first equation of (BF) to erase ∇p . Then we have

$$\begin{cases} \partial_t \mathbf{u} + \nu \mathcal{A} \mathbf{u} = -a \mathbf{u} + \mathcal{P}_\Omega \mathbf{g} T + \mathcal{P}_\Omega \mathbf{h} C + \mathcal{P}_\Omega \mathbf{f}_1, \\ \partial_t T + A_N T + \mathbf{u} \cdot \nabla T = f_2, \\ \partial_t C + A_N C + \mathbf{u} \cdot \nabla C = -\rho A_N T + f_3. \end{cases} \quad (2)$$

Here, for each parameter $\eta \in (0, 1]$, H_η designates the Hilbert space H endowed with the following inner product:

$$(U_1, U_2)_H = (\mathbf{u}_1, \mathbf{u}_2)_{\mathbb{L}_\sigma^2} + (T_1, T_2)_{L^2} + \frac{\eta^2}{9\rho^2} (C_1, C_2)_{L^2} \quad U_i = (\mathbf{u}_i, T_i, C_i)^t, \quad (i = 1, 2). \quad (3)$$

Next define φ by

$$\varphi(U) = \begin{cases} \frac{\nu}{2} \|\nabla \mathbf{u}\|_{L^2}^2 + \frac{1}{2} \|\nabla T\|_{L^2}^2 + \frac{\eta^2}{18\rho^2} \|\nabla C\|_{L^2}^2 & \text{if } U \in D(\varphi) = \mathbb{H}_\sigma^1 \times H^1 \times H^1, \\ +\infty & \text{if } U \in H \setminus D(\varphi). \end{cases}$$

Then the subdifferential $\partial\varphi$ is given by

$$\partial\varphi(U) = \begin{pmatrix} -\nu \mathcal{P}_\Omega \Delta \mathbf{u} \\ -\Delta T \\ -\Delta C \end{pmatrix} \quad \text{with domain } D(\partial\varphi) = (\mathbb{H}^2 \cap \mathbb{H}_\sigma^1) \times D(A_N) \times D(A_N). \quad (4)$$

Furthermore, we put

$$U(t) = \begin{pmatrix} \mathbf{u}(t) \\ T(t) \\ C(t) \end{pmatrix}, \quad \frac{dU}{dt}(t) = \begin{pmatrix} \partial_t \mathbf{u}(t) \\ \partial_t T(t) \\ \partial_t C(t) \end{pmatrix},$$

$$B(U(t)) = \begin{pmatrix} a\mathbf{u}(t) - \mathcal{P}_\Omega \mathbf{g} T(t) - \mathcal{P}_\Omega \mathbf{h} C(t) \\ \mathbf{u} \cdot \nabla T(t) \\ \mathbf{u} \cdot \nabla C(t) - \rho \Delta T(t) \end{pmatrix}, \quad F(t) = \begin{pmatrix} \mathcal{P}_\Omega \mathbf{f}_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}.$$

Then the initial boundary value problem and periodic problem for (2) are reduced to the following abstract problems in H :

$$(\text{CP}) \begin{cases} \frac{dU}{dt}(t) + \partial\varphi(U(t)) + B(U(t)) = F(t) & t \in [0, S], \\ U(0) = U_0, \end{cases} \quad (5)$$

$$(\text{AP}) \begin{cases} \frac{dU}{dt}(t) + \partial\varphi(U(t)) + B(U(t)) = F(t) & t \in [0, S], \\ U(0) = U(S). \end{cases} \quad (6)$$

Abstract Results

In order to prove Theorems 2.1 and 2.2, we rely on abstract results.

(A1) For any $L \in (0, +\infty)$, the set $\{U \in H; \varphi(U) + \|U\|_H^2 \leq L\}$ is compact in H .

(A2) $B(\cdot)$ is φ -demiclosed in the following sense:

$U_n \rightarrow U$ strongly in $C([0, S]; H)$, $\partial\varphi(U_n) \rightharpoonup \partial\varphi(U)$ weakly in $L^2(0, S; H)$,
 $B(U_n) \rightharpoonup b$ weakly in $L^2(0, S; H)$, then $b(t) = B(U(t))$ holds for a.e. $t \in [0, S]$.

(A3) $^0_\alpha$ For a given exponent $\alpha \in (0, 1/2)$, there exists a monotone increasing function $\ell(\cdot)$ such that

$$\|B(U)\|_H \leq \ell(\|U\|_H) \{ \varepsilon \|\partial\varphi(U)\|_H + \frac{1}{\varepsilon} |\varphi(U)|^{\frac{1-\alpha}{1-2\alpha}} + 1 \} \quad \forall U \in D(\partial\varphi),$$

where ε is a positive constant determined by the initial data U_0 and the external force $F(t)$, more precisely, ε is a monotone decreasing function of $|U_0|_H + |U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)} + |F|_{L^2(0,S;H)}$.

(A4) There exists a monotone increasing function $\ell(\cdot)$ and $k \in (0, 1)$ such that

$$\|B(U)\|_H^2 \leq k \|\partial\varphi(U)\|_H^2 + \ell(\varphi(U) + \|U\|_H^2) \quad \forall U \in D(\partial\varphi).$$

Then the following results hold.

Theorem A-1 [$\hat{O}$, JDE, 1982]

Let $U_0 \in \mathcal{B}_{\alpha,p}(\partial\varphi)$ with $p \in [1, 2]$ and $F \in L^2(0, S; H)$, and let (A1), (A2) and $(A3)_\alpha^0$ be satisfied. Then there exists $S_0 \in (0, S]$ depending on $|U_0|_H$ and $|U_0|_{\mathcal{B}_{\alpha,p}(\partial\varphi)}$ such that (CP) has a solution $U(t)$ in $[0, S_0]$ satisfying

$$t^{1/2-\alpha} dU/dt, t^{1/2-\alpha} \partial\varphi(U(t)), t^{1/2-\alpha} B(U(t)) \in L^2(0, S_0; H),$$

$$t^{-\alpha} \|U(t) - U_0\|_H, t^{1/2-\alpha} |\varphi(U(t))|^{1/2} \in L_*^q(0, S_0) \quad \forall q \in [2, \infty].$$

Theorem A-2 [$\hat{O}$, JDE, 1982]

Let (A1), (A2) and (A4) be satisfied and let $U_0 \in D(\varphi)$ and $F \in L^2(0, S; H)$. Then there exists $S_0 \in (0, S]$ depending on $|U_0|_H$ and $\varphi(U_0)$ such that (CP) has a solution $U(t)$ in $[0, S_0]$ satisfying

$$dU/dt, \partial\varphi(U(t)), B(U(t)) \in L^2(0, S_0; H), \quad \varphi(U(t)) \in W^{1,1}(0, S_0; \mathbb{R}^1).$$

Remark 2

In [$\hat{O}$, JDE, 1982], Theorem A-1 is actually proved under a different assumption $(A3)_\alpha$ which is slightly stronger than $(A3)_\alpha^0$. However it is easy to see that the proof of Theorem A-1 holds true with $(A3)_\alpha$ replaced by $(A3)_\alpha^0$.

(A5) There exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0, 1)$ such that

$$\|B(U)\|_H^2 \leq k \|\partial\varphi(U)\|_H^2 + \ell(\|U\|_H)(\varphi(U) + 1)^2 \quad \forall U \in D(\partial\varphi).$$

(A6) There exist positive constants α, K such that

$$(-\partial\varphi(U) - B(U), U)_H + \alpha\varphi(U) \leq K \quad \forall U \in D(\partial\varphi).$$

Theorem A-3 [Ô, JDE, 1984]

Let (A1), (A2), (A5) and (A6) be satisfied. Then for every $F \in L^2(0, S; H)$, (AP) has a strong solution $U \in C_\pi([0, S]; H)$ such that

$$dU/dt, \partial\varphi(U), B(U) \in L^2(0, S; H),$$

$$\varphi(U) \in W^{1,1}(0, S_0; \mathbb{R}^1) \text{ and } \varphi(U(0)) = \varphi(U(S)).$$

Proof of Theorem 3

- Local Existence
 - Reduce our problem to an abstract Cauchy Problem.
 - Apply Theorem A-1 to the problem.
- Existence of Global Solution in time
 - Establish some a priori estimates.
- Uniqueness of the Solution

Check of (A.1)

$$\{U \in H; \varphi(U) + |U|_H^2 \leq L\}$$

$$= \{u \in H; \frac{\nu}{2} |\nabla u|_{L^2}^2 + \frac{1}{2} \|\nabla T\|_{L^2}^2 + \frac{\eta^2}{18\rho^2} \|\nabla C\|_{L^2}^2 + \|u\|_{L^2}^2 + \|T\|_{L^2}^2 + \frac{\eta^2}{9\rho^2} \|C\|_{L^2}^2 \leq L\}$$

From Rellich-Kondrachev's theorem, the level set is compact in

$$H(\Omega) = \mathbb{L}_\sigma^2(\Omega) \times L^2(\Omega) \times L^2(\Omega).$$

Check of (A.2)

Assume

$$\left\{ \begin{array}{l} \mathbf{u}_k \rightarrow \mathbf{u} \text{ in } C([0, S]; \mathbb{L}_\sigma^2(\Omega)), \\ T_k \rightarrow T \text{ in } C([0, S]; L^2(\Omega)), \\ C_k \rightarrow C \text{ in } C([0, S]; L^2(\Omega)), \\ -\nu \mathcal{P} \Delta \mathbf{u}_k \rightharpoonup -\nu \mathcal{P} \Delta \mathbf{u} \text{ in } L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \\ -\Delta T_k \rightharpoonup -\Delta T \text{ in } L^2(0, S; L^2(\Omega)), \\ -\Delta C_k \rightharpoonup -\Delta C \text{ in } L^2(0, S; L^2(\Omega)). \end{array} \right.$$

$$\left\{ \begin{array}{l} a\mathbf{u}_k - \mathcal{P} \mathbf{g} T_k - \mathcal{P} \mathbf{h} C_k \rightharpoonup h_1 \text{ in } L^2(0, S; \mathbb{L}_\sigma^2(\Omega)), \\ \mathbf{u}_k \cdot \nabla T_k \rightharpoonup h_2 \text{ in } L^2(0, S; L^2(\Omega)), \\ \mathbf{u}_k \cdot \nabla C_k - \rho \Delta T_k \rightharpoonup h_3 \text{ in } L^2(0, S; L^2(\Omega)). \end{array} \right.$$

then we have to show

$$\left\{ \begin{array}{l} h_1 = a\mathbf{u} - \mathcal{P} \mathbf{g} T - \mathcal{P} \mathbf{h} C \text{ for a.e. } t \in [0, S], \\ h_2 = \mathbf{u} \cdot \nabla T \text{ for a.e. } t \in [0, S], \\ h_3 = \mathbf{u} \cdot \nabla C - \rho \Delta T \text{ for a.e. } t \in [0, S]. \end{array} \right.$$

Check of (A.2)

Take $\forall \phi \in C_0^\infty(\Omega \times (0, S))$

$$\begin{aligned}\langle \mathbf{u}_k \cdot \nabla T_k, \phi \rangle &= -\langle \nabla \cdot \mathbf{u}_k T_k, \phi \rangle - \langle \mathbf{u}_k T_k, \nabla \phi \rangle = -\langle \mathbf{u}_k T_k, \nabla \phi \rangle \\ &\rightarrow -\langle \mathbf{u} T, \nabla \phi \rangle = \langle \mathbf{u} \cdot \nabla T, \phi \rangle\end{aligned}$$

Let us recall the assumption we imposed:

$$\mathbf{u}_k \cdot \nabla T_k \rightharpoonup h_2 \quad \text{in} \quad L^2(0, S; L^2(\Omega)).$$

So we obtain

$$h_2 = \mathbf{u} \cdot \nabla T \quad \text{for} \quad a.e. \quad t \in [0, S]$$

Check of (A.3) $^0_\alpha$: Choose $\eta = \varepsilon$ and denote H_η simply by H .

By the definition of $B(U)$ and the inner product of H , we get

$$\begin{aligned} \|B(U)\|_H &\leq a\|\mathbf{u}\|_{\mathbb{L}^2_\sigma} + |\mathbf{g}|\|T\|_{L^2} + |\mathbf{h}|\|C\|_{L^2} \\ &\quad + \|\mathbf{u}\|\nabla T\|_{L^2} + \frac{\varepsilon}{3\rho} (\|\mathbf{u}\|\nabla C\|_{L^2} + \rho\|\Delta T\|_{L^2}). \end{aligned} \quad (7)$$

Here using Hölder's inequality and the fact that $|w|_{L^3}^2 \leq |w|_{L^2}|w|_{L^6}$, we get

$$\begin{aligned} \|\mathbf{u}\|\nabla T\|_{L^2} &\leq \|\mathbf{u}\|_{L^6}\|\nabla T\|_{L^3} \leq \|\mathbf{u}\|_{L^6}\|\nabla T\|_{L^2}^{1/2}\|\nabla T\|_{L^6}^{1/2} \\ &\leq \frac{\varepsilon}{4}\|\Delta T\|_{L^2} + \frac{\gamma_0}{\varepsilon}(\|\nabla \mathbf{u}\|_{L^2} + 1)^2\|\nabla T\|_{L^2}. \end{aligned} \quad (8)$$

Hence, by virtue of (7) and (8), we obtain

$$\begin{aligned} \|B(U)\|_H &\leq \left(\frac{\varepsilon}{4} + \frac{\varepsilon}{3}\right)\|\Delta T\|_{L^2} + \frac{\varepsilon}{4}\frac{\varepsilon}{3\rho}\|\Delta C\|_{L^2} \\ &\quad + \frac{\gamma_1}{\varepsilon}(\|\nabla \mathbf{u}\|_{L^2}^3 + \|\nabla T\|_{L^2}^3 + \frac{\varepsilon}{3\rho}\|\nabla C\|_{L^2}^3) + \gamma_2(\|U\|_H + 1) \\ &\leq \varepsilon\|\partial\varphi(U)\|_H + \frac{\gamma_3}{\varepsilon}(\varphi^{3/2}(U) + \|U\|_H + 1). \end{aligned} \quad (9)$$

Since $\frac{3}{2} = \frac{1-\alpha}{1-2\alpha} \Leftrightarrow \alpha = \frac{1}{4}$, (9) ensures (A3) $^0_\alpha$ with $\alpha \in [1/4, 1/2)$ and (A4).

A priori estimates

$$(BF) \quad \begin{cases} \mathbf{u}_t = \nu \mathcal{P} \Delta \mathbf{u} - a \mathbf{u} + \mathcal{P} g T + \mathcal{P} h C \\ T_t + \mathbf{u} \cdot \nabla T = \Delta T \quad \text{in } \Omega \times \{t > 0\}, \\ C_t + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T \quad \text{in } \Omega \times \{t > 0\}, \end{cases}$$

Multiplying the second equation of (BF) by T and integrating over Ω , we get

$$\frac{1}{2} \frac{d}{dt} \|T(t)\|_{L^2}^2 + \|\nabla T(t)\|_{L^2}^2 \leq \|f_2(t)\|_{L^2} \|T(t)\|_{L^2}. \quad (10)$$

Here we used the fact that

$$\int_{\Omega} (\mathbf{u} \cdot \nabla T) T \, dx = \int_{\Omega} \mathbf{u} \cdot \nabla \left(\frac{1}{2} T^2 \right) dx = \int_{\Omega} \operatorname{div} \mathbf{u} \frac{1}{2} T^2 \, dx = 0.$$

Hence we can derive the a priori bound for $\|T(t)\|_{L^2}$ and substituting this estimate in (10), we obtain

$$\sup_{0 \leq t \leq S} \|T(t)\|_{L^2}^2 + \int_0^S \|\nabla T(t)\|_{L^2}^2 \, dt \leq \gamma (\|T_0\|_{L^2(\Omega)}, \|f_2\|_{L^2(Q)}). \quad (11)$$

Similarly we get

$$\sup_{0 \leq t \leq S} \|C(t)\|_{L^2}^2 + \int_0^S \|\nabla C(t)\|_{L^2}^2 \, dt \leq \gamma (\|T_0\|_{L^2}, \|C_0\|_{L^2}, \|f_2\|_{L^2(Q)}, \|f_3\|_{L^2(Q)}). \quad (12)$$

Multiplying the first equation of (BF) by $\mathbf{u}_t$, we get

$$\begin{aligned} \|\mathbf{u}_t\|_{\mathbb{L}_\sigma^2}^2 + \frac{\nu}{2} \frac{d}{dt} \int_{\Omega} |\nabla \mathbf{u}|^2 dx + \frac{a}{2} \frac{d}{dt} \int_{\Omega} |\mathbf{u}|^2 dx \\ \leq \|\mathbf{u}_t\|_{\mathbb{L}_\sigma^2} (|\mathbf{g}| \|T\|_{L^2} + |\mathbf{h}| \|C\|_{L^2} + \|\mathbf{f}_1\|_{\mathbb{L}^2}) \\ \leq \frac{1}{2} \|\mathbf{u}_t\|_{\mathbb{L}_\sigma^2}^2 + (|\mathbf{g}| \|T\|_{L^2} + |\mathbf{h}| \|C\|_{L^2})^2 + \|\mathbf{f}_1\|_{\mathbb{L}^2}^2. \end{aligned} \quad (13)$$

Then integrating (13) over $[0, t]$ with $t \in (0, S]$ and using (11) and (12), we obtain

$$\begin{aligned} \sup_{0 \leq t \leq S} (\nu \|\nabla \mathbf{u}(t)\|_{\mathbb{L}^2}^2 + a \|\mathbf{u}(t)\|_{\mathbb{L}_\sigma^2}^2) + \int_0^S \|\mathbf{u}_t\|_{\mathbb{L}_\sigma^2}^2 dt \\ \leq \gamma (\|\nabla \mathbf{u}_0\|_{L^2}, \|U_0\|_H, \|F\|_{L^2(0,S;H)}), \end{aligned} \quad (14)$$

which also implies

$$\int_0^S \|\mathcal{A}\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 dt \leq \gamma (\|\nabla \mathbf{u}_0\|_{L^2}, \|U_0\|_H, \|F\|_{L^2(0,S;H)}). \quad (15)$$

Multiplying the second equation of (BF) by $-\Delta T$, we obtain, by the same argument as for (8) with $\varepsilon = 1$,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla T\|_{L^2}^2 + \|\Delta T\|_{L^2}^2 &\leq \frac{1}{2} \int_{\Omega} |\mathbf{u}|^2 |\nabla T|^2 dx + \frac{1}{2} \|\Delta T\|_{L^2}^2 + \|f_2\|_{L^2} \|\Delta T\|_{L^2} \\ &\leq \left(\frac{1}{8} + \frac{1}{2} + \frac{1}{4}\right) \|\Delta T\|_{L^2}^2 + 2\gamma_0^2 (\|\nabla \mathbf{u}\|_{L^2} + \|\mathbf{u}\|_{\mathbb{L}_{\sigma}^2})^4 \|\nabla T\|_{L^2}^2 + \|f_2\|_{L^2}^2. \end{aligned}$$

Then Gronwall's inequality with (14) yields

$$\begin{aligned} \sup_{0 \leq t \leq S} \|\nabla T(t)\|_{L^2}^2 + \int_0^S \|\Delta T(t)\|_{L^2}^2 dt \\ \leq \gamma (\|\nabla \mathbf{u}_0\|_{L^2}, \|\nabla T_0\|_{L^2}, \|C_0\|_{L^2}, \|F\|_{L^2(0,S;H)}). \end{aligned} \tag{16}$$

Multiplying the third equation of (BF) by $-\Delta C$ and applying the same argument as for (16), we now have

$$\begin{aligned} \sup_{0 \leq t \leq S} \|\nabla C(t)\|_{L^2}^2 + \int_0^S \|\Delta C(t)\|_{L^2}^2 dt \\ \leq \gamma(\|\nabla \mathbf{u}_0\|_{L^2}, \|T_0\|_{H^1}, \|C_0\|_{H^1}, \|F\|_{L^2(0,S;H)}). \end{aligned} \quad (17)$$

Thus, a priori estimates (14), (16) and (17) assure that every local solutions can be continued globally up to $[0, S]$, provided that

$$U_0 \in D(\varphi) = D(\mathcal{A}^{1/2}) \times D(A_N^{1/2}) \times D(A_N^{1/2}).$$

As for the general case where

$$U_0 \in D(\mathcal{A}^\alpha) \times D(A_N^\alpha) \times D(A_N^\alpha) \quad \text{with } \alpha \in [1/4, 1/2),$$

since $U(t)$ enjoys $(\#)_\alpha$, there exists a $t_0 \in (0, S_0)$ such that $U(t_0) \in D(\varphi)$. Hence, regarding $U(t_0)$ as an initial data and applying the global existence result for the case where $U_0 \in D(\varphi)$, we can derive the global existence result for the general case.

Uniqueness

Let U^1 and U^2 be solutions of (BF) for the same initial data and put

$$U^i = (\mathbf{u}^i, T^i, C^i)^t \quad (i = 1, 2) \quad W = U^1 - U^2 = (\mathbf{w}, \tau, \theta)^t.$$

Then $W = (\mathbf{w}, \tau, \theta)^t$ satisfies

$$(D) \quad \begin{cases} \partial_t \mathbf{w} = \nu \mathcal{P}_\Omega \Delta \mathbf{w} - a \mathbf{w} + \mathcal{P}_\Omega \mathbf{g} \tau + \mathcal{P}_\Omega \mathbf{h} \theta, \\ \partial_t \tau = \Delta \tau - \mathbf{u}^1 \cdot \nabla \tau + \mathbf{w} \cdot \nabla T^2, \\ \partial_t \theta = \Delta \theta + \rho \Delta \tau - \mathbf{u}^1 \cdot \nabla \theta + \mathbf{w} \cdot \nabla C^2. \end{cases}$$

Multiplying the first equation of (D) by $\mathbf{w}$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\mathbf{w}\|_{\mathbb{L}_\sigma^2}^2 + \nu \|\nabla \mathbf{w}\|_{L^2}^2 + a \|\mathbf{w}\|_{\mathbb{L}_\sigma^2}^2 &\leq |\mathbf{g}| \|\tau\|_{L^2} \|\mathbf{w}\|_{\mathbb{L}_\sigma^2} + |\mathbf{h}| \|\theta\|_{L^2} \|\mathbf{w}\|_{\mathbb{L}_\sigma^2} \\ &\leq \frac{1}{2} (|\mathbf{g}| + |\mathbf{h}|) \|\mathbf{w}\|_{\mathbb{L}_\sigma^2}^2 + \frac{1}{2} |\mathbf{g}| \|\tau\|_{L^2}^2 + \frac{1}{2} |\mathbf{h}| \|\theta\|_{L^2}^2. \end{aligned} \quad (18)$$

Multiplying the second equation of (D) by τ and $|\nu|_{L^3}^2 \leq |\nu|_{L^2} |\nu|_{L^6}$, we obtain

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\tau\|_{L^2}^2 + \|\nabla \tau\|_{L^2}^2 &\leq \int |\mathbf{w}| |\nabla T^2| |\tau| dx \\ &\leq \|\tau\|_{L^6} \|\nabla T^2\|_{L^2} \|\mathbf{w}\|_{L^3} \\ &\leq \frac{1}{4} (\|\nabla \tau\|_{L^2}^2 + \|\tau\|_{L^2}^2) + \gamma \|\nabla T^2\|_{L^2}^2 \|\mathbf{w}\|_{\mathbb{L}_\sigma^2} \|\nabla \mathbf{w}\|_{L^2} \end{aligned} \quad (19)$$

By the argument similar to that for (19), we obtain

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \|\theta\|_{L^2}^2 + \|\nabla \theta\|_{L^2}^2 &\leq \int_{\Omega} |\mathbf{w}| |\nabla C^2| |\theta| dx + \rho \int_{\Omega} |\nabla \tau| |\nabla \theta| dx \\
&\leq \|\theta\|_{L^6} \|\nabla C^2\|_{L^2} \|\mathbf{w}\|_{L^3} + \rho \|\nabla \tau\|_{L^2} \|\nabla \theta\|_{L^2} \\
&\leq \frac{1}{4} (\|\nabla \theta\|_{L^2}^2 + \|\theta\|_{L^2}^2) + \gamma \|\nabla C^2\|_{L^2}^2 \|\mathbf{w}\|_{\mathbb{L}_{\sigma}^2} \|\nabla \mathbf{w}\|_{L^2} + \frac{1}{4} \|\nabla \theta\|_{L^2}^2 + \rho^2 \|\nabla \tau\|_{L^2}^2 \\
&\leq \frac{1}{2} \|\nabla \theta\|_{L^2}^2 + \rho^2 \|\nabla \tau\|_{L^2}^2 + \frac{\rho^2 \gamma}{4} \|\nabla \mathbf{w}\|_{L^2}^2 + \frac{\gamma^2}{\rho^2 \gamma} \|\nabla C^2\|_{L^2}^4 \|\mathbf{w}\|_{\mathbb{L}_{\sigma}^2}^2 + \frac{1}{4} \|\theta\|_{L^2}^2. \quad (20)
\end{aligned}$$

Put $y(t) = \|\mathbf{w}(t)\|_{\mathbb{L}_{\sigma}^2}^2 + \|\tau(t)\|_{L^2}^2 + \frac{1}{2\rho^2} \|\theta(t)\|_{L^2}^2$ and sum up (18), (19) and (20) $\times \frac{1}{2\rho^2}$, then we get

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} y(t) &\leq \gamma y(t) + \frac{\gamma^2}{\nu} \|\nabla T^2(t)\|_{L^2}^4 \|\mathbf{w}(t)\|_{\mathbb{L}_{\sigma}^2}^2 + \frac{\gamma^2}{2\rho^4 \nu^2} \|\nabla C^2\|_{L^2}^4 \|\mathbf{w}(t)\|_{\mathbb{L}_{\sigma}^2}^2 \\
&\leq \gamma (\|\nabla T^2\|_{L^2}^4 + \|\nabla C^2\|_{L^2}^4 + 1) y(t).
\end{aligned}$$

Here we note that $(\#)_{\alpha}$ with $\alpha \in [1/4, 1/2]$ implies that

$$t^{1/2-\alpha} \|\nabla T^2\|_{L^2}, t^{1/2-\alpha} \|\nabla C^2\|_{L^2} \in L_*^4(0, S) \Rightarrow \|\nabla T^2\|_{L^2}, \|\nabla C^2\|_{L^2} \in L^4(0, S).$$

Hence, the uniqueness follows from Gronwall's inequality.

Proof of Theorem 4

- Reduce our problem to an Abstract Periodic Problem
- Introduce Approximation Problems
- Apply Theorem A-3 to approximation problems
- Establish some a priori estimates
- Convergence of solutions of approximation problems

Approximate Equations

When one tries to apply Theorem A-3 to (AP), one faces some difficulties. The most serious one arises in checking (A5). In fact, we recall that estimate (9) gives

$$\|B(U)\|_H^2 \leq \varepsilon^2 \|\partial\varphi(U)\|_H^2 + \frac{\gamma}{\varepsilon^2} (\varphi(U)^3 + \|U\|_H^2),$$

but the required growth order in (A5) is quadratic. Moreover, when the constant vectors $\mathbf{g}$, $\mathbf{h}$ are very large, it is difficult to examine whether condition (A6) is satisfied. From these reasons, we are led to introduce relaxed approximate problems. However, the approximate problems introduced in Uchida-Ô (2013) prevents establishing the desirable a priori estimates under the homogeneous Neumann boundary condition. In order to manage with this difficulty, we introduce another approximation procedure.

$$(BF)_{\varepsilon, \lambda} \begin{cases} \partial_t \mathbf{u} = \nu \mathcal{P}_\Omega \Delta \mathbf{u} - a \mathbf{u} + \mathcal{P}_\Omega \mathbf{g}[T]_\varepsilon + \mathcal{P}_\Omega \mathbf{h}[C]_\varepsilon + \mathcal{P}_\Omega \mathbf{f}_1, \\ \partial_t T + \mathbf{u} \cdot \nabla T = \Delta T - \varepsilon |T|^{p-2} T - \lambda T + f_2, \\ \partial_t C + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T - \varepsilon |C|^{p-2} C - \lambda C + f_3, \end{cases} \quad (21)$$

where $\varepsilon, \lambda \in (0, 1)$ are approximation parameters and the cut-off function $[T]_\varepsilon$ is defined by

$$[T]_\varepsilon = \begin{cases} T & \text{if } |T| \leq 1/\varepsilon, \\ (\text{Sgn } T) 1/\varepsilon & \text{if } |T| \geq 1/\varepsilon, \end{cases} \quad \varepsilon \in (0, 1),$$

For the perturbation term, we replace it by

$$B_\varepsilon(U) = (au - \mathcal{P}_\Omega \mathbf{g}[T]_\varepsilon - \mathcal{P}_\Omega \mathbf{h}[C]_\varepsilon, \mathbf{u} \cdot \nabla T, \mathbf{u} \cdot \nabla C - \rho \Delta T)^t.$$

We take $\eta = 1$ and replace φ by $\varphi_{\varepsilon,\lambda}$ given by

$$\begin{aligned} \varphi_{\varepsilon,\lambda}(U) &= \varphi(U) + \psi_{\varepsilon,\lambda}(U), \\ \psi_{\varepsilon,\lambda}(U) &= \begin{cases} \frac{\varepsilon}{p} \|T\|_{L^p}^p + \frac{\varepsilon}{9\rho^2 p} \|C\|_{L^p}^p + \frac{\lambda}{2} \|T\|_{L^2}^2 + \frac{\lambda}{18\rho^2} \|C\|_{L^2}^2 & \text{if } U \in D(\psi_{\varepsilon,\lambda}) \\ +\infty & \text{if } U \in H \setminus D(\psi_{\varepsilon,\lambda}). \end{cases} \end{aligned}$$

where $D(\psi_{\varepsilon,\lambda}) = \mathbb{L}_\sigma^2(\Omega) \times L^p(\Omega) \times L^p(\Omega)$.

Since $(\partial\varphi(U), \partial\psi_{\varepsilon,\lambda}(U))_H \geq 0$, we can deduce that $\partial\varphi + \partial\psi_{\varepsilon,\lambda}$ becomes maximal monotone, and hence we get $\partial(\varphi + \psi_{\varepsilon,\lambda}) = \partial\varphi + \partial\psi_{\varepsilon,\lambda}$ with

$$D(\partial(\varphi + \psi_{\varepsilon,\lambda})) = D(\partial\varphi) \cap D(\partial\psi_{\varepsilon,\lambda}).$$

Thus, we have another abstract problem associated with approximate problems:

$$(\text{AP})_{\varepsilon,\lambda} \begin{cases} \frac{dU(t)}{dt} + \partial\varphi_{\varepsilon,\lambda}(U(t)) + B_\varepsilon(U(t)) = F(t) & t \in [0, S], \\ U(0) = U(S). \end{cases}$$

To apply Theorem A-3 to $(AP)_{\varepsilon,\lambda}$, we are going to check (A5) and (A6).

Check of (A5) By the definition of $B_\varepsilon(U)$ and the inner product of H , we get

$$\|B_\varepsilon(U)\|_H^2 \leq \beta \|U\|_H^2 + 2 \|\mathbf{u} \cdot \nabla T\|_{L^2}^2 + \frac{2}{9\rho^2} \|\mathbf{u} \cdot \nabla C\|_{L^2}^2 + \frac{2}{9} \|\Delta T\|_{L^2}^2,$$

Since $\nabla \cdot \mathbf{u} = 0$, the integration by parts gives

$$\|\mathbf{u} \cdot \nabla T\|_{L^2}^2 = \int_{\Omega} \nabla T \cdot \mathbf{u} (\mathbf{u} \cdot \nabla T) dx = - \int_{\Omega} T \mathbf{u} \nabla (\mathbf{u} \cdot \nabla T) dx \leq \int_{\Omega} |T| |\mathbf{u}| |\nabla (\mathbf{u} \cdot \nabla T)| dx.$$

Then by the elliptic estimate and Hölder's inequality, we have

$$\begin{aligned} 2 \|\mathbf{u} \cdot \nabla T\|_{L^2}^2 &\leq \beta \int_{\Omega} |T| |\mathbf{u}| |\mathbf{u}| |\Delta T| dx + \beta \int_{\Omega} |T| |\mathbf{u}| |\nabla \mathbf{u}| |\nabla T| dx \\ &\leq \beta \|T\|_{L^{12}} \|\mathbf{u}\|_{L^6} \|\mathbf{u}\|_{L^4} \|\Delta T\|_{L^2} + \beta \|T\|_{L^{12}} \|\mathbf{u}\|_{L^6} \|\nabla \mathbf{u}\|_{L^4} \|\nabla T\|_{L^2}. \end{aligned}$$

Hence, by using the inequality $\|w\|_{L^4}^4 \leq \|w\|_{L^2}^2 \|w\|_{L^6}^3$, we get

$$\begin{aligned} 2 \|\mathbf{u} \cdot \nabla T\|_{L^2}^2 &\leq \frac{1}{9} \|\Delta T\|_{L^2}^2 + \beta (\|\nabla \mathbf{u}\|_{L^2}^4 + \|T\|_{L^{12}}^{16} \|\mathbf{u}\|_{L^{\frac{2}{\sigma}}}^4) \\ &\quad + \frac{1}{6} \|\mathcal{A}\mathbf{u}\|_{L^{\frac{2}{\sigma}}}^2 + \beta (\|\nabla \mathbf{u}\|_{L^2}^4 + \|\nabla T\|_{L^2}^4 + \|T\|_{L^{12}}^{16}). \end{aligned}$$

Consequently, by taking $p \geq 12$, we obtain

$$\|B_\varepsilon(U)\|_H^2 \leq \frac{1}{3} \|\partial \varphi_{\varepsilon,\lambda}(U)\|_H^2 + \ell(\|U\|_H)(\varphi_{\varepsilon,\lambda}(U) + 1)^2,$$

whence follows (A5) with $k = 1/3$, provided that $p \geq 12$.

Check of (A6) The definition of the inner product of H gives

$$\begin{aligned}
 (\partial\varphi_{\varepsilon,\lambda}(U), U)_H &= \nu \|\nabla \mathbf{u}\|_{L^2}^2 + \|\nabla T\|_{L^2}^2 + \frac{1}{9\rho^2} \|\nabla C\|_{L^2}^2 \\
 &\quad + \varepsilon \|T\|_{L^p}^p + \lambda \|T\|_{L^2}^2 + \frac{\varepsilon}{9\rho^2} \|C\|_{L^p}^p + \frac{\lambda}{9\rho^2} \|C\|_{L^2}^2 \geq 2\varphi_{\varepsilon,\lambda}(U).
 \end{aligned}$$

Moreover, noting that $(\mathbf{u} \cdot \nabla T, T)_{L^2} = (\mathbf{u} \cdot \nabla C, C)_{L^2} = 0$ and the cut-off function is bounded by $1/\varepsilon$, we get

$$\begin{aligned}
 (B_\varepsilon(U), U)_H &\geq a \|\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 - |\mathbf{g}| \|\mathbf{u}\|_{\mathbb{L}_\sigma^2} \|[T]_\varepsilon\|_{L^2} - |\mathbf{h}| \|\mathbf{u}\|_{\mathbb{L}_\sigma^2} \|[C]_\varepsilon\|_{L^2} - \frac{1}{9\rho} \|\nabla T\|_{L^2} \|\nabla C\|_{L^2} \\
 &\geq a \|\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 - 2 \cdot \frac{a}{2} \|\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 - \frac{|\mathbf{g}|^2 + |\mathbf{h}|^2}{2a} \{ \|[T]_\varepsilon\|_{L^2}^2 + \|[C]_\varepsilon\|_{L^2}^2 \} - \frac{1}{2} \|\nabla T\|_{L^2}^2 - \frac{1}{18\rho^2} \|\nabla C\|_{L^2}^2 \\
 &\geq -\frac{\beta}{\varepsilon^2} - \varphi_{\varepsilon,\lambda}(U),
 \end{aligned}$$

where we used Cauchy's inequality and β is a suitable constant. Hence we get

$$(-\partial\varphi_{\varepsilon,\lambda}(U) - B_\varepsilon(U), U)_H + \varphi_{\varepsilon,\lambda}(U) \leq \frac{\beta}{\varepsilon^2},$$

whence follows (A6) with $K = \frac{\beta}{\varepsilon^2}$ and $\alpha = 1$.

For $\lambda > 0$ fixed, Convergence as $\varepsilon \rightarrow 0$

Since $\lambda > 0$ fixed, the principal part $\partial\varphi_{\varepsilon,\lambda}$ is coercive. Then we can establish the same a priori estimates of $U_{\varepsilon,\lambda}$, solutions of $(\text{AP})_{\varepsilon,\lambda}$, independent of ε and repeat the same standard arguments for the convergence of solutions along $\varepsilon \rightarrow 0$ as the Dirichlet boundary condition case.

In this way, we can assure the existence of solutions of the following equations:

$$(\text{BF})_\lambda \left\{ \begin{array}{l} \partial_t \mathbf{u} = \nu \mathcal{P}_\Omega \Delta \mathbf{u} - a \mathbf{u} + \mathcal{P}_\Omega \mathbf{g} T + \mathcal{P}_\Omega \mathbf{h} C + \mathcal{P}_\Omega \mathbf{f}_1, \\ \partial_t T + \mathbf{u} \cdot \nabla T = \Delta T - \lambda T + f_2, \\ \partial_t C + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T - \lambda C + f_3, \\ \mathbf{u}(0) = \mathbf{u}(S), \quad T(0) = T(S), \quad C(0) = C(S). \end{array} \right.$$

A priori estimates for T, C

Integrate the second equation of $(BF)_\lambda$, then by the boundary condition, we get

$$\frac{d}{dt} \int_{\Omega} T_\lambda(x, t) dx + \lambda \int_{\Omega} T_\lambda(x, t) dx = \int_{\Omega} f_2(x, t) dx \quad \forall t \in [0, S]. \quad (22)$$

Here we used the following facts.

$$\int_{\Omega} \Delta T_\lambda dx = \int_{\partial\Omega} \frac{\partial T_\lambda}{\partial n} dS = 0, \quad \int_{\Omega} \mathbf{u}_\lambda \cdot \nabla T_\lambda dx = \int_{\Omega} \operatorname{div}(\mathbf{u}_\lambda T_\lambda) dx = \int_{\partial\Omega} \mathbf{u}_\lambda T_\lambda n dS = 0.$$

Integrating (22) on $t \in (0, S)$ and using the periodic condition and (1), we have

$$\lambda \int_0^S \int_{\Omega} T_\lambda(x, t) dx dt = 0 \quad \text{so} \quad \exists t_0 \in [0, S] \quad \text{such that} \quad \int_{\Omega} T_\lambda(x, t_0) dx = 0.$$

Hence by (22), we obtain

$$\int_{\Omega} T_\lambda(x, t) dx = \int_{t_0}^t e^{-\lambda(t-s)} \int_{\Omega} f_2(x, s) dx ds \quad \forall t \in [t_0, t_0 + S].$$

Then applying Poincaré-Wirtinger's inequality

$$\|v - \bar{v}\|_{L^2} \leq C_W \|\nabla v\|_{L^2} \quad \forall v \in H^1(\Omega), \quad \bar{v} = \frac{1}{|\Omega|} \int_{\Omega} v(x) dx,$$

we obtain

$$\|T_\lambda\|_{L^2(Q)} \leq C_W \|\nabla T_\lambda\|_{L^2(Q)} + S \|f_2\|_{L^2(Q)}. \quad (23)$$

Multiplying the 2nd equation of $(BF)_\lambda$ by T_λ and recalling $(\mathbf{u}_\lambda \cdot \nabla T_\lambda, T_\lambda)_{L^2} = 0$, we have

$$\frac{1}{2} \frac{d}{dt} \|T_\lambda\|_{L^2}^2 + \|\nabla T_\lambda\|_{L^2}^2 + \lambda \|T_\lambda\|_{L^2}^2 = \int_{\Omega} f_2 T_\lambda dx. \quad (24)$$

Integrating (24) over $[0, S]$ and using the periodic condition, we get

$$\|\nabla T_\lambda\|_{L^2(Q)}^2 + \lambda \|T_\lambda\|_{L^2(Q)}^2 \leq \|f_2\|_{L^2(Q)} \|T_\lambda\|_{L^2(Q)}. \quad (25)$$

Then substituting (23) into (25), we get

$$\begin{aligned} \|\nabla T_\lambda\|_{L^2(Q)}^2 &\leq (C_W^2 + 2S) \|f_2\|_{L^2(Q)}^2, \\ \|T_\lambda\|_{L^2(Q)}^2 &\leq \{2C_W^2(C_W^2 + 2S) + 2S^2\} \|f_2\|_{L^2(Q)}^2. \end{aligned} \quad (26)$$

Since $T_\lambda \in C([0, S]; L^2(\Omega))$, there exists $t_0 \in [0, S]$ such that

$$\|T_\lambda(t_0)\|_{L^2} = \min_{0 \leq t \leq S} \|T_\lambda(t)\|_{L^2}.$$

Hence by (26), we have

$$\|T_\lambda(t_0)\|_{L^2}^2 \leq \frac{1}{S} \{2C_W^2(C_W^2 + 2S) + 2S^2\} \|f_2\|_{L^2(Q)}^2 \leq \gamma.$$

Then, integrating (24) over $[t_0, t]$ ($t_0 \leq t \leq t_0 + S$) and over $[t_0, t_0 + S]$, we obtain

$$\|T_\lambda\|_{C([0, S]; L^2(\Omega))}, \|\nabla T_\lambda\|_{L^2(Q)}, \lambda \|T_\lambda\|_{L^2(Q)}^2 \leq \gamma.$$

Similarly we have

$$\|C_\lambda\|_{C([0, S]; L^2(\Omega))}, \|\nabla C_\lambda\|_{L^2(Q)}, \lambda \|C_\lambda\|_{L^2(Q)}^2 \leq \gamma.$$

A priori estimate for u

$$u_t = \nu \mathcal{P}_\Omega \Delta u - a u + \mathcal{P}_\Omega g[T]_\varepsilon + \mathcal{P}_\Omega h[C]_\varepsilon + \mathcal{P}_\Omega f_1, \quad \times u \int_\Omega dx$$

$$\int_0^S dt \Rightarrow$$

$$\begin{aligned} \int_0^S (a \|u\|_{\mathbb{L}_\sigma^2}^2 + \|\nabla u\|_{\mathbb{L}_\sigma^2}^2) dt &\leq \int_0^S (|g| \| [T]_\varepsilon \|_{L^2} + |h| \| [C]_\varepsilon \|_{L^2} + \|f_1\|_{\mathbb{L}^2}) \|u\|_{\mathbb{L}_\sigma^2} dt \\ &\leq C_0 (\|f_1\|_{\mathbb{L}^2}, \|f_2\|_{L^2(0,S;L^2(\Omega))}, \|f_3\|_{L^2(0,S;L^2(\Omega))}) \end{aligned}$$

$$\exists t_0 \in [0, S] \quad \text{s.t.} \quad K \|u(t_0)\|_{\mathbb{L}_\sigma^2} \leq \|\nabla u(t_0)\|_{\mathbb{L}_\sigma^2}^2 \leq C_0/S$$

$$\int_{t_0}^t dt \Rightarrow \max_{0 \leq t \leq S} \|u(t)\|_{\mathbb{L}_\sigma^2} \leq C_0 (\|f_1\|_{\mathbb{L}^2}, \|f_2\|_{L^2(0,S;L^2(\Omega))}, \|f_3\|_{L^2(0,S;L^2(\Omega))}, S, \rho)$$

A Priori Estimate for $u_t, \nabla u$

$$u_t = \nu \mathcal{P}_\Omega \Delta u - a u + \mathcal{P}_\Omega g[T]_\varepsilon + \mathcal{P}_\Omega h[C]_\varepsilon + \mathcal{P}_\Omega f_1, \quad \times u_t \int_\Omega dx$$

$$\int_0^S dt \Rightarrow \int_0^S \|u_t\|_{\mathbb{L}_\sigma^2}^2 dt \leq C_0(\|f_1\|_{L^2}, \|f_2\|_{L^2}, \|f_3\|_{L^2})$$

$$Eq. \Rightarrow \int_0^S \|\mathcal{P}_\Omega \Delta u\|_{\mathbb{L}_\sigma^2}^2 dt \leq C_0$$

$$\exists t_0 \in [0, S] \quad \text{s.t.} \quad \|\nabla u(t_0)\|_{\mathbb{L}_\sigma^2}^2 \leq C_0/S$$

$$\int_{t_0}^t dt \Rightarrow \max_{0 \leq t \leq S} \|\nabla u(t)\|_{\mathbb{L}_\sigma^2} \leq C_0(\|f_1\|_{L^2}, \|f_2\|_{L^2}, \|f_3\|_{L^2}, S, \rho)$$

A Priori Estimates for ∇T , ΔT

$$T_t + \mathbf{u} \cdot \nabla T = \Delta T - \lambda T + f_2, \quad \times - \Delta T \quad \int_{\Omega} dx$$

$$\frac{1}{2} \frac{d}{dt} \|\nabla T(t)\|_{L^2}^2 + \|\Delta T(t)\|_{L^2}^2 + \lambda \|\nabla T(t)\|_{L^2}^2$$

$$\leq (\|\mathbf{u} \cdot \nabla T\|_{L^2} + \|\mathbf{f}_1\|_{\mathbb{L}^2}) \|\Delta T(t)\|_{L^2}$$

$$\|\mathbf{u} \cdot \nabla T\|_{L^2}^2 \leq \|\nabla T\|_{L^4}^2 \|\mathbf{u}\|_{\mathbb{L}^4}^2 \leq K \|\nabla T\|_{L^2}^{1/2} \|\Delta T\|_{L^2}^{3/2} \|\mathbf{u}\|_{\mathbb{L}_{\sigma}^2}^{1/2} \|\nabla \mathbf{u}\|_{\mathbb{L}_{\sigma}^2}^{3/2}$$

$$\leq \varepsilon \|\Delta T\|_{L^2}^2 + C_{\varepsilon} \|\nabla T\|_{L^2}^2 \|\mathbf{u}\|_{\mathbb{L}_{\sigma}^2}^2 \|\nabla \mathbf{u}\|_{\mathbb{L}_{\sigma}^2}^6$$

$$\exists t_0 \in [0, S] \quad \text{s.t.} \quad \|\nabla T(t_0)\|_{L^2} \leq C_0/S$$

$$\int_{t_0}^t dt \Rightarrow \max_{0 \leq t \leq S} \|\nabla T(t)\|_{L^2} + \int_0^S \|\Delta T(t)\|_{L^2}^2 dt \leq C_0.$$

A Priori Estimate for T_t

$$T_t + \mathbf{u} \cdot \nabla T = \Delta T - \lambda T + f_2, \quad \times T_t \quad \int_{\Omega} dx$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla T(t)\|_{L^2}^2 + \|T_t(t)\|_{L^2}^2 + \frac{\lambda}{2} \frac{d}{dt} \|T(t)\|_{L^2}^2 \\ \leq (\|\mathbf{u} \cdot \nabla T\|_{L^2} + \|\mathbf{f}_1\|_{\mathbb{L}^2}) \|T_t(t)\|_{L^2} \end{aligned}$$

$$\|\mathbf{u} \cdot \nabla T\|_{L^2}^2 \leq \varepsilon \|\Delta T\|_{L^2}^2 + C_\varepsilon \|\nabla T\|_{L^2}^2 \|\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 \|\nabla \mathbf{u}\|_{\mathbb{L}_\sigma^2}^6$$

$$\int_0^S dt \Rightarrow \int_0^S \|T_t(t)\|_{L^2}^2 dt \leq C_0$$

A Priori Estimates for ∇C , ΔC

$$C_t + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T - \lambda C + f_3, \quad \times - \Delta C \quad \int_{\Omega} dx$$

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\nabla C(t)\|_{L^2}^2 + \|\Delta C(t)\|_{L^2}^2 + \lambda \|\nabla C(t)\|_{L^2}^2 \\ & \leq (\|\mathbf{u} \cdot \nabla C\|_{L^2} + \|\mathbf{f}_1\|_{L^2} + \rho \|\Delta T\|_{L^2}) \|\Delta C(t)\|_{L^2} \\ \|\mathbf{u} \cdot \nabla C\|_{L^2}^2 & \leq \|\nabla C\|_{L^4}^2 \|\mathbf{u}\|_{L^4}^2 \leq K \|\nabla C\|_{L^2}^{1/2} \|\Delta C\|_{L^2}^{3/2} \|\mathbf{u}\|_{L^2_{\sigma}}^{1/2} \|\nabla \mathbf{u}\|_{L^2_{\sigma}}^{3/2} \\ & \leq \varepsilon \|\Delta C\|_{L^2}^2 + C_{\varepsilon} \|\nabla C\|_{L^2}^2 \|\mathbf{u}\|_{L^2_{\sigma}}^2 \|\nabla \mathbf{u}\|_{L^2_{\sigma}}^6 \\ \exists t_0 \in [0, S] \quad \text{s.t.} \quad & \|\nabla C(t_0)\|_{L^2} \leq C_0/S \\ \int_{t_0}^t dt \Rightarrow \max_{0 \leq t \leq S} & \|\nabla C(t)\|_{L^2} + \int_0^S \|\Delta C(t)\|_{L^2}^2 dt \leq C_0. \end{aligned}$$

A Priori Estimate for C_t

$$C_t + \mathbf{u} \cdot \nabla C = \Delta C + \rho \Delta T - \lambda C + f_2, \quad \times C_t \quad \int_{\Omega} dx$$

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla C(t)\|_{L^2}^2 + \|C_t(t)\|_{L^2}^2 + \frac{\lambda}{2} \frac{d}{dt} \|C(t)\|_{L^2}^2 \\ \leq (\|\mathbf{u} \cdot \nabla C\|_{L^2} + \|\mathbf{f}_1\|_{\mathbb{L}^2} + \rho \|\Delta T\|_{L^2}) \|C_t(t)\|_{L^2} \end{aligned}$$

$$\|\mathbf{u} \cdot \nabla C\|_{L^2}^2 \leq \varepsilon \|\Delta C\|_{L^2}^2 + C_\varepsilon \|\nabla C\|_{L^2}^2 \|\mathbf{u}\|_{\mathbb{L}_\sigma^2}^2 \|\nabla \mathbf{u}\|_{\mathbb{L}_\sigma^2}^6$$

$$\int_0^S dt \Rightarrow \int_0^S \|C_t(t)\|_{L^2}^2 dt \leq C_0$$

Convergence

A priori estimate

$$\max_{0 \leq t \leq S} |U_\lambda(t)|_H + \max_{0 \leq t \leq S} \varphi_\lambda(U_\lambda(t)) + \int_0^S (|\partial_t U_\lambda(t)|_H^2 + |\partial \varphi_\lambda(U_\lambda(t))|_H^2) dt \leq C_0,$$

$$U_\lambda = (\mathbf{u}_\lambda, T_\lambda, C_\lambda)^t$$

$\max_{0 \leq t \leq S} \varphi_\lambda(U_\lambda(t)) \leq C_0 \Rightarrow \{U_\lambda(t)\}_{\lambda \in (0,1)}$ forms a precompact set in $H^{\forall t \in [0, S]}$

$\|\partial_t U_\lambda(t)\|_{L^2(Q)} \leq C_0 \Rightarrow \{U_\lambda(t)\}_{\lambda \in (0,1)}$ is equi-continuous in $C_\pi([0, S]; H)$

Ascoli's Theorem $\Rightarrow \exists U_n(t) = U_{\lambda_n}(t)$ ($\lambda_n \rightarrow 0$ as $n \rightarrow \infty$) s.t.

$U_n \rightarrow U = (\mathbf{u}, T, C)^t$ strongly in $C_\pi([0, S]; H)$ as $n \rightarrow \infty$

$\lambda_n U_n \rightarrow 0 = (0, 0, 0)^t$ strongly in $C_\pi([0, S]; H)$ as $n \rightarrow \infty$

$\partial_t U_n \rightarrow \partial_t U = (\mathbf{u}_t, T_t, C_t)^t$ weakly in $L^2(0, S; H)$ as $n \rightarrow \infty$

$B(U_n) \rightarrow B(U)$ weakly in $L^2(0, S; H)$ as $n \rightarrow \infty$.

$\Rightarrow U$ gives a solution of (AP).

Remark 3

If we impose the assumption

$$f_2(t), f_3(t) \in L_N^2(\Omega) = \{ f \in L^2(\Omega); \int_{\Omega} f(x) dx = 0 \} \quad \text{for a.e. } t \in (0, S),$$

which is slightly stronger than (1), then the periodic solution given in Theorem 2.2 satisfies $T(t), C(t) \in L_N^2(\Omega)$ for a.e. $t \in (0, S)$.

Remark 4

Repeating exactly the same arguments as those in [4] with Poincaré's inequality replaced by Poincaré-Wirtinger's inequality, we can prove the following uniqueness result.

Theorem 5

Let $\mathbf{f}_1 \in \mathbb{L}^2(0, S; \mathbb{L}_{\sigma}^2(\Omega))$, then there exists a (sufficiently small) constant γ_0 depending on $|\mathbf{g}|, |\mathbf{h}|, \nu, \rho$ and $\|\mathbf{f}_1\|_{\mathbb{L}^2(0, S; \mathbb{L}_{\sigma}^2(\Omega))}$ such that if $\|f_2\|_{L^2(0, S; L^2(\Omega))} \leq \gamma_0$ and $\|f_3\|_{L^2(0, S; L^2(\Omega))} \leq \gamma_0$, then the periodic solution of (BF) satisfying $(\#)_{1/2}$ in Theorem 4 is unique.

Concluding Remarks

- Our main theorem holds true also for unbounded domains.

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Concluding Remarks

- Our main theorem holds true also for unbounded domains.
- For bounded domain case with Dirichlet BC, if the external forces are all zero, then $U(t) = (\mathbf{u}(t), T(t), C(t))^t$ decays exponentially to zero.
- Existence of global attractors and exponential attractors

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