

Nonlocal Cahn-Hilliard-Navier-Stokes systems with nonconstant mobility

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Diffuse Interface Models

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jointly with Sergio Frigeri and Elisabetta Rocca

- Cahn-Hilliard-Navier-Stokes systems (model **H**)
- CHNS systems with nonlocal interactions
- $\exists$ of a global weak solution
- some properties of weak sols
- global attractor in dimension two
- nonlocal CH equation with degenerate mobility

- isothermal motion of an incompressible homogeneous binary mixture of immiscible fluids (model **H**: Siggia, Halperin & Hohenberg '76, Halperin & Hohenberg '77)
 - Lagrangian description for the interfaces evolution
 - Eulerian description for the bulk fluid flow
- the phase-field approach postulates $\exists$ **a diffuse interface**
 - φ order parameter which varies from one phase to another ($\nabla\varphi$ **is large on the diffuse interface**)
 - F a double well mixing potential energy depending on φ
- φ satisfies a (convective) Cahn-Hilliard equation
- the fluid velocity $\mathbf{u}$ is influenced through a capillarity force (Korteweg force) proportional to $\Delta\varphi\nabla\varphi$

Model H: the evolution system

- *rigorous* derivation: Gurtin, Polignone & Viñals '96, Jasnow & Viñals '96, Morro '10

$$\begin{aligned}\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi &= -\varepsilon \mu \nabla \varphi \\ \nabla \cdot \mathbf{u} &= 0 \\ \varphi_t + \mathbf{u} \cdot \nabla \varphi &= \nabla \cdot (m(\varphi) \nabla \mu) \\ \mu &= -\varepsilon \Delta \varphi + \varepsilon^{-1} F'(\varphi)\end{aligned}$$

- $\mathbf{u}$ (averaged) fluid velocity, density = 1
- φ (relative) difference of the two concentrations
- viscosity $\nu > 0$, mobility m , diffuse interface thickness $\varepsilon > 0$
- μ **chemical potential**, F **potential energy** density

Regular and singular potentials

- **regular**: the typical polynomial double-well potential

$$F(s) = (s^2 - 1)^2$$

for all $s \in \mathbb{R}$

- **singular**: the logarithmic potential

$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$

for all $s \in (-1, 1)$, $\theta < \theta_c$

CHNS systems: theoretical results

- well-posedness, stability of equilibria: Starovoitov '97 [$\Omega = \mathbb{R}^2$, regular F , spatially decaying sols]
- existence and uniqueness, local stability of constant solutions: Boyer '99 [degenerate m]
- existence and uniqueness: Abels '09 [constant m , singular F]
- unmatched densities: Boyer '01, Abels '09 and '12, Abels et al. '12, Abels et al. '13 [degenerate m]
- compressible case: Abels & Feireisl '08 [$\exists$ weak sols]

convergence to equilibrium of single trajectories

- Abels '09 [singular F]
- Gal & G '09 [2D, regular F , conv. rate estimates]
- Zhao, Wu & Huang '09 [regular F , nonconstant m , conv. rate estimates]

attractors

- Abels '09 [singular F , global attractor]
- Gal & G '10 [3D, regular F , nonconstant m , time-dependent ext. force, trajectory attractor]
- Gal & G '09 and '11 [2D, regular F , ext. force, smooth global attractor, exp. attractors, dim. bounds]
- Bosia & Gatti '13 [2D, pullback attractor]

- taking the limit as $\varepsilon \searrow 0$ one gets a sharp interface model: the **Navier-Stokes-Mullins-Sekerka** system (matched densities Abels & Röger '09, unmatched densities Abels & Lengeler '12)
- **numerical approximation**
Badalassi, Cenicerros & Banerjee '03, Liu & Shen '03; Kay, Styles & Welford '08; Kim, Kang & Lowengrub '04; Shen & Yang '10, Boyer et al. '11, ...

Free energies: local vs. nonlocal

- standard CH eq can be derived **phenomenologically** as the conserved gradient flow generated by μ i.e. is by the first variation of the (local) free energy

$$E(\varphi) = \int_{\Omega} \left(\frac{\xi}{2} |\nabla \varphi(x)|^2 + \eta F(\varphi(x)) \right) dx$$

- a **nonlocal** CH eq can be **rigorously justified** as a macroscopic limit of microscopic of suitable phase segregation models with particle conserved dynamics (Giacomin & Lebowitz '97, '98)
- the **nonlocal** free energy is

$$\mathbf{E}(\varphi) = \frac{1}{4} \int_{\Omega} \int_{\Omega} K(x-y) (\varphi(x) - \varphi(y))^2 dx dy + \eta \int_{\Omega} F(\varphi(x)) dx$$

where $K : \mathbb{R}^N \rightarrow \mathbb{R}$ s.t. $K(x) = K(-x)$ (e.g. Gaussian or Newtonian kernel)

Nonlocal chemical potential

The chemical potential given by the **nonlocal** free energy is

$$\mu = a\varphi - K * \varphi + \eta F'(\varphi)$$

where

$$(K * \varphi)(x) := \int_{\Omega} K(x-y)\varphi(y)dy, \quad a(x) := \int_{\Omega} K(x-y)dy$$

Remark

The term

$$\int_{\Omega} \frac{\xi}{2} |\nabla \varphi(x)|^2 dx$$

can be viewed as the **first approximation** of

$$\int_{\Omega} \int_{\Omega} K(x-y)(\varphi(x) - \varphi(y))^2 dx dy$$

Nonlocal interactions: some math literature

- **nonlocal Cahn-Hilliard eqs**: Giacomini & Lebowitz '97 and '98; Chen & Fife '00; Gajewski '02; Gajewski & Zacharias '03; Han '04; Bates & Han '05; Colli, Krejčí, Rocca & Sprekels '07; Londen & Petzeltová '11; Gal & G '12
- **binary fluids with long range segregating interactions**: Bastea et al. '00
- **Navier-Stokes-Korteweg systems** (liquid-vapour phase transitions): Rohde '05, Haspot '10
- **nonlocal Allen-Cahn eqs and phase-field systems**: Bates et al.; Sprekels et al.; Feireisl, Issard-Roch & Petzeltová '04; G & Schimperna '13

Nonlocal CHNS system

$\Omega \subset \mathbb{R}^d$ bdd ($d = 2, 3$)

$$\mathbf{u}_t + \mathbf{u} \cdot \nabla \mathbf{u} - \nu \Delta \mathbf{u} + \nabla \pi = \mu \nabla \varphi + \mathbf{g}(t)$$

$$\nabla \cdot \mathbf{u} = 0$$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \nabla \cdot (m(\varphi) \nabla \mu)$$

$$\mu = -K * \varphi + a\varphi + F'(\varphi)$$

$$\text{in } \Omega \times (0, +\infty)$$

subject to

$$\mathbf{u} = \mathbf{0}, \quad \partial_{\mathbf{n}} \mu = 0 \quad \text{on } \partial\Omega \times (0, +\infty)$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0 \quad \text{in } \Omega$$

Regular potential

- $\exists$ of a global sol (Colli, Frigeri & G '12)
- global attractor (2D) and trajectory attractor (3D) (Frigeri & G '12)
- 2D: strong sols in 2D, smooth global attractor, convergence to single equilibria (Frigeri, G & Krejčí '13)
- 2D: uniqueness, regularity and exponential attractors (Frigeri, Gal & G *in progress*)

Singular potential

- $\exists$ global attractor (2D) and trajectory attractor (3D) (Frigeri & G '12)

Nonconstant mobility and reg. F : basic assumptions

- $m \in C_{loc}^{0,1}(\mathbb{R})$ s.t. $0 < m_1 \leq m(s) \leq m_2, \forall s \in \mathbb{R}$

- $J(\cdot - x) \in W^{1,1}(\Omega)$ for a.a $x \in \Omega$, $J(x) = J(-x)$,

$$a(x) := \int_{\Omega} J(x - y) dy \geq 0 \text{ and}$$

$$a^* := \sup_{x \in \Omega} \int_{\Omega} |J(x - y)| dy < \infty, \quad b := \sup_{x \in \Omega} \int_{\Omega} |\nabla J(x - y)| dy < \infty$$

- $F \in C_{loc}^{2,1}(\mathbb{R})$ and $\exists c_0 > 0$ s.t.

$$F''(s) + a(x) \geq c_0, \quad \forall s \in \mathbb{R}, \quad \text{a.e. } x \in \Omega$$

- $\exists c_1 > (a^* - a_*)/2$ and $c_2 \in \mathbb{R}$ s.t.

$$F(s) \geq c_1 s^2 - c_2, \quad \forall s \in \mathbb{R}$$

- $\exists c_3 > 0, c_4 \geq 0$ and $r \in (1, 2]$ s.t.

$$|F'(s)|^r \leq c_3 |F(s)| + c_4, \quad \forall s \in \mathbb{R}$$

Nonconstant mobility and reg. F : weak solution 1

- $\mathbf{u}_0 \in (L^2(\Omega))_{div}$
- $\varphi_0 \in L^2(\Omega)$ s.t. $F(\varphi_0) \in L^1(\Omega)$
- $\mathbf{h} \in L^2(0, T; H^1(\Omega)_{div}^*)$

For any given $T > 0$, a pair $[\mathbf{u}, \varphi]$ is a **weak sol** on $[0, T]$ if

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)_{div}) \cap L^2(0, T; H^1(\Omega)_{div})$$

$$\varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$$

$$\mathbf{u}_t \in L^{4/3}(0, T; H^1(\Omega)_{div}^*), \quad \varphi_t \in L^{4/3}(0, T; H^1(\Omega)^*), \quad d = 3$$

$$\mathbf{u}_t \in L^{2-\gamma}(0, T; H^1(\Omega)_{div}^*), \quad \varphi_t \in L^{2-\delta}(0, T; H^1(\Omega)^*), \quad d = 2$$

$$\mu := \mathbf{a}\varphi - \mathbf{J} * \varphi + F'(\varphi) \in L^2(0, T; H^1(\Omega))$$

where $\gamma, \delta \in (0, 1)$

and $[\mathbf{u}, \varphi]$ satisfies

$$\langle \varphi_t, \psi \rangle + (m(\varphi) \nabla \mu, \nabla \psi) = (\mathbf{u} \varphi, \nabla \psi), \quad \forall \psi \in H^1(\Omega)$$

$$\langle \mathbf{u}_t, \mathbf{v} \rangle + \nu(\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = -(\varphi \nabla \mu, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle, \quad \forall \mathbf{v} \in H^1(\Omega)_{div}$$

for a.a. $t \in (0, T)$ with

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0, \quad \bar{\varphi}(t) = \bar{\varphi}_0, \quad \forall t \in [0, T]$$

where $\bar{\varphi} = \frac{1}{|\Omega|} \int_{\Omega} \varphi$

Remark

The regularity of φ is lower w.r.t. to the local CHNS ($L^\infty(L^2)$ vs. $L^\infty(H^1)$) so that the coupling terms need to be handled with more care

Nonconstant mobility and reg. F : $\exists$ weak solution

$\exists$ a weak sol $[u, \varphi]$ satisfying the energy inequality

$$\begin{aligned}\mathcal{E}(\mathbf{u}(t), \varphi(t)) + \int_0^t \left(\nu \|\nabla \mathbf{u}\|^2 + \|\sqrt{m(\varphi)} \nabla \mu\|^2 \right) d\tau \\ \leq \mathcal{E}(\mathbf{u}_0, \varphi_0) + \int_0^t \langle \mathbf{h}(\tau), \mathbf{u} \rangle d\tau\end{aligned}$$

for every $t \in [0, T]$, where

$$\begin{aligned}\mathcal{E}(\mathbf{u}(t), \varphi(t)) = \frac{1}{2} \|\mathbf{u}(t)\|^2 + \frac{1}{4} \int_{\Omega} \int_{\Omega} J(x-y) (\varphi(x, t) - \varphi(y, t))^2 dx dy \\ + \int_{\Omega} F(\varphi(x, t)) dx\end{aligned}$$

Furthermore, assume the p -coercivity, that is,

- $F \in C_{loc}^{2,1}(\mathbb{R})$ and $\exists c_5 > 0, c_6 > 0$ and $p > 2$ s.t.

$$F''(s) + a(x) \geq c_5 |s|^{p-2} - c_6, \quad \forall s \in \mathbb{R}, \text{ a.e. } x \in \Omega$$

then, for every $T > 0$, $\exists$ a weak sol $[u, \varphi]$ satisfying

$$\varphi \in L^\infty(0, T; L^p(\Omega))$$

$$\varphi_t \in L^2(0, T; H^1(\Omega)^*), \quad \text{if } d = 2 \quad \text{or} \quad d = 3 \text{ and } p \geq 3$$

$$\mathbf{u}_t \in L^2(0, T; H^1(\Omega)_{div}^*), \quad \text{if } d = 2$$

Nonconstant mobility and reg. F : $d = 2$

Suppose once again that F satisfies the p -coercivity, then

- any weak sol satisfies the energy identity

$$\frac{d}{dt}\mathcal{E}(\mathbf{u}, \varphi) + \nu \|\nabla \mathbf{u}\|^2 + \|\sqrt{m(\varphi)}\nabla \mu\|^2 = \langle \mathbf{h}(t), \mathbf{u} \rangle, \quad t > 0$$

- $\mathbf{u} \in C([0, \infty); L^2(\Omega)_{div})$, $\varphi \in C([0, \infty); L^2(\Omega))$
- $\int_{\Omega} F(\varphi) \in C([0, \infty))$
- if $\mathbf{h} \in L^2_{tb}(0, \infty; H^1(\Omega)^*_{div})$ then any weak sol also satisfies

$$\mathcal{E}(\mathbf{u}(t), \varphi(t)) \leq \mathcal{E}(\mathbf{u}_0, \varphi_0)e^{-kt} + F(\bar{\varphi}_0)|\Omega| + K, \quad \forall t \geq 0$$

where $k, K > 0$ are independent of the initial data, with K depending on Ω , ν , J , F and $\|\mathbf{h}\|_{L^2_{tb}(0, \infty; H^1(\Omega)^*_{div})}$

Degenerate mobility and sing. F : preliminaries

- previous results can be proven arguing as for the constant mobility case (Colli, Frigeri & G '12; Frigeri & G '12)
- they are helpful to construct a sequence of approximating sols which yields a weak sol in the **degenerate mobility and singular potential case**
- however, an **alternative weak formulation** is needed (see Elliott & Garcke '96 for the local CH eq. with degenerate mobility and singular F)

Degenerate mobility and sing. F : assumptions

On account of [Elliott & Garcke, '96] and [Giacomin & Lebowitz, '97 & '98]

- $m \in C^1([-1, 1])$, $m \geq 0$, $m(s) = 0$ if and only if $s = -1$ or $s = 1$
- $F \in C^2(-1, 1)$ s.t.

$$mF'' \in C([-1, 1])$$

- $F = F_1 + F_2$, $F_2 \in C^2([-1, 1])$ and $\exists a_2 > 4(a^* - a_* - b_2)$ with $b_2 = \min F_2''$ and $\varepsilon_0 > 0$ s.t.

$$F_1''(s) \geq a_2, \quad \forall s \in (-1, -1 + \varepsilon_0] \cup [1 - \varepsilon_0, 1)$$

- F_1'' is non-decreasing in $[1 - \varepsilon_0, 1)$, non-increasing in $(-1, -1 + \varepsilon_0]$
- $\exists c_0 > 0$ s.t.

$$F''(s) + a(x) \geq c_0, \quad \forall s \in (-1, 1), \quad \text{a.e. } x \in \Omega$$

Degenerate mobility and sing. F : example 1

$$m(s) = k_1(1 - s^2)$$

$$F(s) = \frac{\theta}{2}((1+s)\log(1+s) + (1-s)\log(1-s)) - \frac{\theta_c}{2}s^2$$

- $0 < \theta < \theta_c$
- $\inf_{\Omega} a > \theta_c - \theta$

Remark

Setting

$$F_1(s) := (\theta/2)((1+s)\log(1+s) + (1-s)\log(1-s))$$

$$F_2(s) = -(\theta_c/2)s^2$$

we have

$$mF_1'' = k_1\theta > 0$$

Degenerate mobility and sing. F : example 2

$$m(s) = k(s)(1 - s^2)^n$$
$$F(s) = -k_2 s^2 + F_1(s)$$

- $n \geq 2$
- $k \in C^1([-1, 1])$ s.t. $0 < k_3 \leq k(s) \leq k_4$ for all $s \in [-1, 1]$
- F_1 is a $C^2(-1, 1)$ convex function s.t.

$$F_1''(s) = \ell(s)(1 - s^2)^{-n}, \quad \forall s \in (-1, 1)$$

where $\ell \in C^1([-1, 1])$

Degenerate mobility and sing. F : weak solution 1

- $M \in C^2(-1, 1)$ s.t. $m(s)M''(s) = 1$, $M(0) = 0$, $M'(0) = 0$
- $u_0 \in L^2(\Omega)_{div}$
- $\varphi_0 \in L^2(\Omega)$ s.t. $F(\varphi_0) \in L^1(\Omega)$ and $M(\varphi_0) \in L^1(\Omega)$
- $\mathbf{h} \in L^2(0, T; H^1(\Omega)_{div}^*)$

For any given $T > 0$, a pair $[\mathbf{u}, \varphi]$ is a **weak solution** on $[0, T]$ if

$$\mathbf{u} \in L^\infty(0, T; L^2(\Omega)_{div}) \cap L^2(0, T; H^1(\Omega)_{div})$$

$$\mathbf{u}_t \in L^{4/3}(0, T; H^1(\Omega)_{div}^*), \quad d = 3$$

$$\mathbf{u}_t \in L^2(0, T; H^1(\Omega)_{div}^*), \quad d = 2$$

$$\varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega)), \quad \varphi_t \in L^2(0, T; H^1(\Omega)^*)$$

$$\varphi \in L^\infty(Q_T), \quad |\varphi(x, t)| \leq 1 \quad \text{a.e. } (x, t) \in Q_T := \Omega \times (0, T)$$

and $[\mathbf{u}, \varphi]$ satisfies

$$\begin{aligned} \langle \varphi_t, \psi \rangle + \int_{\Omega} m(\varphi) F''(\varphi) \nabla \varphi \cdot \nabla \psi + \int_{\Omega} m(\varphi) \mathbf{a} \nabla \varphi \cdot \nabla \psi \\ + \int_{\Omega} m(\varphi) (\varphi \nabla \mathbf{a} - \nabla J * \varphi) \cdot \nabla \psi = (\mathbf{u} \varphi, \nabla \psi) \\ \langle \mathbf{u}_t, \mathbf{v} \rangle + \nu (\nabla \mathbf{u}, \nabla \mathbf{v}) + b(\mathbf{u}, \mathbf{u}, \mathbf{v}) = ((\mathbf{a} \varphi - J * \varphi) \nabla \varphi, \mathbf{v}) + \langle \mathbf{h}, \mathbf{v} \rangle \end{aligned}$$

$$\mathbf{u}(0) = \mathbf{u}_0, \quad \varphi(0) = \varphi_0$$

$$\overline{\varphi}(t) = \overline{\varphi_0}, \quad \forall t \in [0, T]$$

for every $\psi \in H^1(\Omega)$, every $\mathbf{v} \in H^1(\Omega)_{div}$ and a.a. $t \in (0, T)$

Degenerate mobility and sing. $F: \exists$ weak sol $d = 3$

- $\exists$ a weak solution $[\mathbf{u}, \varphi]$
- $\varphi \in L^\infty(0, T; L^p(\Omega))$ with $p \leq 6$
- $[\mathbf{u}, \varphi]$ satisfies the following inequality

$$\begin{aligned} & \frac{1}{2}(\|\mathbf{u}(t)\|^2 + \|\varphi(t)\|^2) + \int_0^t \int_\Omega m(\varphi) F''(\varphi) |\nabla \varphi|^2 \\ & + \int_0^t \int_\Omega a m(\varphi) |\nabla \varphi|^2 + \nu \int_0^t \|\nabla \mathbf{u}\|^2 \leq \frac{1}{2}(\|\mathbf{u}_0\|^2 + \|\varphi_0\|^2) \\ & + \int_0^t \int_\Omega m(\varphi) (\nabla J * \varphi - \varphi \nabla a) \cdot \nabla \varphi \\ & + \int_0^t \int_\Omega (a\varphi - J * \varphi) \mathbf{u} \cdot \nabla \varphi + \int_0^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau \end{aligned}$$

Degenerate mobility and sing. $F: \exists$ weak sol $d = 2$

- $\varphi \in L^\infty(0, T; L^p(\Omega))$ with $2 \leq p < \infty$
- $[\mathbf{u}, \varphi]$ satisfies the identity

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} (\|\mathbf{u}\|^2 + \|\varphi\|^2) + \int_{\Omega} m(\varphi) F''(\varphi) |\nabla \varphi|^2 \\ & + \int_{\Omega} a m(\varphi) |\nabla \varphi|^2 + \nu \|\nabla \mathbf{u}\|^2 \\ & = \int_{\Omega} m(\varphi) (\nabla J * \varphi - \varphi \nabla a) \cdot \nabla \varphi \\ & + \int_{\Omega} (a \varphi - J * \varphi) \mathbf{u} \cdot \nabla \varphi + \langle \mathbf{h}, \mathbf{u} \rangle \end{aligned}$$

Strongly degenerate mobility and singular F

In addition, **suppose that $m(\pm 1) = 0$ with order ≥ 2** , then, the weak solution $z = [\mathbf{u}, \varphi]$ also fulfills the inequality

$$\mathcal{E}(z(t)) + \int_0^t \left(\nu \|\nabla \mathbf{u}\|^2 + \left\| \frac{\mathcal{J}}{\sqrt{m(\varphi)}} \right\|^2 \right) d\tau \leq \mathcal{E}(z_0) + \int_0^t \langle \mathbf{h}, \mathbf{u} \rangle d\tau$$

for all $t > 0$, where the mass flux

$$\mathcal{J} = -m(\varphi) \nabla (a\varphi - J * \varphi) - m(\varphi) F''(\varphi) \nabla \varphi$$

is s.t.

$$\mathcal{J}, \frac{\mathcal{J}}{\sqrt{m(\varphi)}} \in L^2(\Omega \times (0, T))$$

Remark

*In this case it can be proven that the sets $\{x \in \Omega : \varphi(x, t) = 1\}$ and $\{x \in \Omega : \varphi(x, t) = -1\}$ have both zero measure for a.a. $t > 0$ (**no pure phases are allowed**)*

Degenerate mobility and $\exists$ of pure phases

- If $m(\pm 1) = 0$ with order $[1, 2)$ then F and M are bdd in $[-1, 1]$
- the alternative weak formulation allows the existence of sols which are pure phases even on positive measure
- if, for instance, F is bounded and at $t = 0$ the fluid is in a pure phase, i.e. $\varphi_0 \equiv 1$, and $\mathbf{u}(0) = \mathbf{u}_0 \in L^2(\Omega)_{div}$, then the pair

$$\mathbf{u} = \mathbf{u}(x, t), \quad \varphi = \varphi(x, t) = 1, \quad \text{a.e. in } \Omega \times (0, T)$$

where $\mathbf{u}$ is a sol to the NS eqs with non-slip boundary condition explicitly satisfies the weak formulation

- $\exists$ of pure phases on positive measure sets is excluded in the model with constant (or strongly degenerate) mobility

Degenerate mobility and sing. F : a regularity result

Suppose in addition

- $F_1 \in C^3(-1, 1)$
- $\exists \alpha_0, \beta_0 > 0$ and $\rho \in [0, 1)$ s.t.

$$m(s)F_1''(s) \geq \alpha_0 > 0, \quad |m^2(s)F_1'''(s)| \leq \beta_0, \quad \forall s \in [-1, 1]$$

$$F_1'(s)F_1'''(s) \geq 0 \quad \forall s \in (-1, 1)$$

$$\rho F_1''(s) + F_2''(s) + a(x) \geq 0 \quad \forall s \in (-1, 1), \quad \text{for a.e. } x \in \Omega$$

- $F'(\varphi_0) \in L^2(\Omega)$

then the weak sol given by the $\exists$ result is s.t.

$$\mu \in L^\infty(0, T; L^2(\Omega)) \quad \nabla \mu \in L^2(0, T; L^2(\Omega))$$

thus $[u, \varphi]$ also satisfies a weak formulation like in the non-degenerate mobility case as well as the energy inequality and, for $d = 2$, the energy identity

Global attractor in dimension 2: setting

- $\mathbf{h} \in H^1(\Omega)_{div}^*$ time independent
- $\mathcal{G}_{m_0}$ is the set of all weak sols corresponding to all initial data $\mathbf{z}_0 = [\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$, where

$$\mathcal{X}_{m_0} := L^2(\Omega)_{div} \times \mathcal{Y}_{m_0}$$

with

$$\mathcal{Y}_{m_0} := \{\varphi \in L^\infty(\Omega) : |\varphi| \leq 1, F(\varphi), M(\varphi_0) \in L^1(\Omega), |\overline{\varphi}| \leq m_0\}$$

and $m_0 \in [0, 1]$ is fixed

- metric in $\mathcal{X}_{m_0}$

$$\mathbf{d}(\mathbf{z}_2, \mathbf{z}_1) := \|\mathbf{u}_2 - \mathbf{u}_1\| + \|\varphi_2 - \varphi_1\|$$

- $\exists \alpha_0 > 0$ and $\rho \in [0, 1)$ such that

$$m(s)F_1''(s) \geq \alpha_0, \quad \forall s \in [-1, 1]$$

$$\rho F_1''(s) + F_2''(s) + \mathbf{a}(\mathbf{x}) \geq 0, \quad \forall s \in (-1, 1) \quad \text{a.e. in } \Omega$$

Global attractor in dimension 2: $\exists$

- $\mathcal{G}_{m_0} = \{g : [0, \infty) \rightarrow \mathcal{X}_{m_0}\}$
is a generalized semiflow on $\mathcal{X}_{m_0}$ (Ball '97 & '98)
- $\exists$ of a g for each $[\mathbf{u}_0, \varphi_0] \in \mathcal{X}_{m_0}$
- translation of sols are sols
- concatenation
- upper semicontinuity w.r.t. initial data
- $\mathcal{G}_{m_0}$ is point dissipative and compact

Thus $\mathcal{G}_{m_0}$ possesses a global attractor (i.e. a compact, invariant set that attracts all bdd sets of initial data)

Remark

$\exists$ of the global attractor does not require that $|\overline{\varphi}| < 1$ and, in particular, a (strict) separation property is not needed

Nonlocal CH eq. with deg. mobility: $\exists$

$$\varphi_t + \mathbf{u} \cdot \nabla \varphi = \nabla \cdot (m(\varphi)(\mathbf{a} + F''(\varphi))\nabla \varphi - m(\varphi)(\nabla K * \varphi + \varphi \nabla \mathbf{a}))$$

- $\mathbf{u} \in L^2_{loc}([0, \infty); H^1(\Omega)_{div} \cap L^\infty(\Omega)^d)$ given
- $\varphi_0 \in L^\infty(\Omega)$ s.t. $F(\varphi_0), M(\varphi_0) \in L^1(\Omega)$

Then, for every $T > 0$, $\exists \varphi \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$ s.t.

$$\varphi \in L^\infty(\Omega \times (0, T)), \quad |\varphi| \leq 1$$

$$\begin{aligned} \langle \varphi_t, \psi \rangle + \int_{\Omega} m(\varphi) F''(\varphi) \nabla \varphi \cdot \nabla \psi + \int_{\Omega} m(\varphi) \mathbf{a} \nabla \varphi \cdot \nabla \psi \\ + \int_{\Omega} m(\varphi) (\varphi \nabla \mathbf{a} - \nabla J * \varphi) \cdot \nabla \psi = (\mathbf{u} \varphi, \nabla \psi) \\ \varphi(0) = \varphi_0, \quad \overline{\varphi}(t) = \overline{\varphi_0}, \quad \forall t \in [0, T] \end{aligned}$$

Energy identity and uniqueness also hold for $d = 3$

- the weak sol also satisfies the energy identity

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\varphi\|^2 + \int_{\Omega} m(\varphi) F''(\varphi) |\nabla \varphi|^2 + \int_{\Omega} a m(\varphi) |\nabla \varphi|^2 \\ + \int_{\Omega} m(\varphi) (\varphi \nabla a - \nabla J * \varphi) \cdot \nabla \varphi = 0 \end{aligned}$$

for a.a. $t > 0$ and in $\mathcal{D}'(0, \infty)$

- if $\exists \alpha_0 > 0$ and $\rho \in [0, 1)$ s.t.

$$m(s) F_1''(s) \geq \alpha_0 > 0 \quad \forall s \in [-1, 1]$$

$$\rho F_1''(s) + F_2''(s) + a(x) \geq 0 \quad \forall s \in (-1, 1), \quad \text{for a.e. } x \in \Omega$$

then the weak sol is unique (more general than Giacomini & Lebowitz '98 and Gajewski & Zacharias '03)

The global attractor also $\exists$ for $d = 3$

- suppose $\mathbf{u} \in H^1(\Omega)_{div} \cap L^\infty(\Omega)^d$ time independent
- a semigroup $S(t)$ can be defined on $\mathcal{Y}_{m_0}$, $m_0 \in [0, 1]$, endowed with the metric induced by the L^2 -norm
- the dynamical system $(\mathcal{Y}_{m_0}, S(t))$ possesses a connected global attractor

Remark

*Uniqueness of solutions as well as $\exists$ of global attractors are still **open issues** for the local CH equation with degenerate mobility*

Ongoing work and future plans

$d = 2$ (Frigeri, Gal & G)

- uniqueness
- strong-weak uniqueness with variable viscosity, constant mobility, regular potential
- $\exists$ exponential attractors (const. mobility and reg. F)

some open issues

- unmatched densities (Abels, Garcke & Grün '12)
- Cahn-Hilliard-Hele-Shaw systems (Wang & Zhang '12; Wang & Wu '12; Lowengrub, Titi & Zhao '13)
- sharp interface limits
- non-isothermal models
- compressible case
- singular interaction kernels (Abels, Bosia & G for CH only, in progress)