

On the weak solutions to the Euler-Fourier system

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Basic field equations

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0$$

Internal energy balance

$$\partial_t(\varrho e(\varrho, \vartheta)) + \operatorname{div}_x(\varrho e(\varrho, \vartheta) \mathbf{u}) + \operatorname{div}_x \mathbf{q} = -p(\varrho, \vartheta) \operatorname{div}_x \mathbf{u}$$

Constitutive relations

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Gibbs' equation

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D\left(\frac{1}{\varrho}\right)$$

Thermodynamics stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Boundary and initial conditions

Periodic boundary conditions

$$x \in \mathcal{T} = ([-1, 1] |_{\{-1, 1\}})^N, \quad \boxed{N = 3}$$

Initial conditions

$$\varrho(0, \cdot) = \varrho_0 > 0, \quad \vartheta(0, \cdot) = \vartheta_0 > 0, \quad \mathbf{u}(0, \cdot) = \mathbf{u}_0$$

Local well posedness

Regularity of the data

$$\varrho_0, \vartheta_0, \mathbf{u}_0 \in W^{m,2}, \quad m \geq 3$$

Local in time existence

T. Alazard [2006] - local existence for large data

D. Serre [2008, 2012]

Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2} \left[\partial_t(\varrho \vartheta) + \operatorname{div}_x(\varrho \vartheta \mathbf{u}) \right] - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \mathbf{u}$$

Existence of weak solutions

Initial data

$$\varrho_0, \vartheta_0, \mathbf{u}_0 \in C^3, \varrho_0 > 0, \vartheta_0 > 0$$

Global existence

For any (smooth) initial data $\varrho_0, \vartheta_0, \mathbf{u}_0$ the Euler-Fourier system admits infinitely many weak solutions on a given time interval $(0, T)$

Regularity class

$$\varrho \in C^2, \partial_t \vartheta, \nabla_x^2 \vartheta \in L^p \text{ for any } 1 \leq p < \infty$$

$$\mathbf{u} \in C_{\text{weak}}([0, T]; L^2) \cap L^\infty, \operatorname{div}_x \mathbf{u} \in C^1$$

Incompressible Euler system

Equations

$$\operatorname{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

Initial conditions

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0$$

DeLellis, Székelyhidi [2008]

There exist (infinite) set of initial data $\mathbf{v}_0$ in $L^2 \cap L^\infty(\Omega; \mathbb{R}^3)$ such that the incompressible Euler system possesses infinitely many global in time solutions, with prescribed energy

$$\frac{1}{2} |\mathbf{v}|^2 = e, \quad e(0, \cdot) = \frac{1}{2} |\mathbf{v}_0|^2 \text{ and the pressure } \Pi = -\frac{1}{3} |\mathbf{v}|^2$$

Method of convex integration, I

Set of subsolutions

$$\begin{aligned} X_{0,e}[0, T] = \Big\{ & \mathbf{v} \in C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)) \mid \\ & \mathbf{v}(0, \cdot) = \mathbf{v}_0, \mathbf{v}(T, \cdot) = \mathbf{v}_T, \operatorname{div}_x \mathbf{v} = 0, \\ & \mathbf{v} \in C^1((0, T) \times \Omega; \mathbb{R}^3), \partial_t \mathbf{v} + \operatorname{div}_x \mathbb{U} = 0 \\ & \text{for a certain } \mathbb{U} \in C^1((0, T) \times \Omega; \mathbb{R}^{3 \times 3}_{\text{sym},0}), \\ & \frac{3}{2} \lambda_{\max} [\mathbf{v} \otimes \mathbf{v} - \mathbb{U}] < e \text{ in } (0, T) \times \Omega \Big\}, \end{aligned}$$

Distance function

$$\begin{aligned} \frac{3}{2} \lambda_{\max} [\mathbf{v} \otimes \mathbf{v} - \mathbb{U}] &\geq \frac{1}{2} |\mathbf{v}|^2 \\ e = \frac{1}{2} |\mathbf{v}|^2 &\Rightarrow \mathbb{U} + \mathbf{v} \otimes \mathbf{v} - \frac{1}{3} |\mathbf{v}|^2 \mathbb{I} \end{aligned}$$

Method of convex integration, II

Oscillatory lemma [DeLellis, Székelyhidi]

For any $\mathbf{v} \in X_{0,e}[0, T]$ there exist sequences

$$\{\mathbf{w}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0, T) \times \Omega; \mathbb{R}^3), \quad \{\mathbb{U}_n\}_{n=1}^{\infty} \subset C_c^{\infty}((0, T) \times \Omega; \mathbb{R}^3)$$

such that the functions $\mathbf{v} + \mathbf{w}_n$ belong to $X_{0,e}[0, T]$, with the associated tensor fields $\mathbb{U} + \mathbb{U}_n$,

$$\mathbf{w}_n \rightarrow 0 \text{ in } C_{\text{weak}}([0, T]; L^2(\Omega; \mathbb{R}^3)),$$

and

$$\liminf_{n \rightarrow \infty} \|\mathbf{w}_n\|_{L^2((0, T) \times \Omega)}^2 \geq \Lambda \int_{\tau_1}^{\tau_2} \int_{\Omega} \left(e - \frac{1}{2} |\mathbf{v}|^2 \right)^2 dx dt, \quad \Lambda > 0,$$

where the constant Λ depends only on the norm of e in $L^{\infty}((0, T) \times \Omega)$.

Application of convex integration, I

Ansatz

$$\varrho \mathbf{u} = \mathbf{v} + \nabla_x \Psi, \operatorname{div}_x \mathbf{v} = 0$$

Equations

$$\partial_t \varrho + \Delta \Psi = 0$$

$$\partial_t \mathbf{v} + \operatorname{div}_x \left(\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} \right) + \nabla_x (\partial_t \Psi + \varrho \vartheta) = 0$$

$$\frac{3}{2} \left(\partial_t (\varrho \vartheta) + \operatorname{div}_x \left(\vartheta (\mathbf{v} + \nabla_x \Psi) \right) \right) - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

“Energy”

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta[\mathbf{v}]$$

Application of convex integration, II

Fixing the density and the acoustic potential

$$\varrho = \varrho(t, x), \quad \Psi = \Psi(t, x), \quad \Psi = -\Delta^{-1}[\partial_t \varrho]$$

Computing the temperature as a function of $\mathbf{v}$

$$\frac{3}{2} \left(\partial_t(\varrho \vartheta) + \operatorname{div}_x \left(\vartheta (\mathbf{v} + \nabla_x \Psi) \right) \right) - \Delta \vartheta = -\varrho \vartheta \operatorname{div}_x \left(\frac{\mathbf{v} + \nabla_x \Psi}{\varrho} \right)$$

$$\|\vartheta\|_{L^\infty((0, \tau) \times \Omega)} \leq c(\varrho, \Psi) \text{ independent of } \mathbf{v}$$

Kinetic energy

$$e = \chi(t) - \frac{3}{2} \varrho \vartheta[\mathbf{v}]$$

Application of convex integration, III

Set of subsolutions

$$\begin{aligned} X_{0,e}[0, T] = \Big\{ & \mathbf{v} \in C_{\text{weak}}([0, T]; L^2(\Omega; R^3) \mid \\ & \mathbf{v}(0, \cdot) = \mathbf{v}_0, \mathbf{v}(T, \cdot) = \mathbf{v}_T, \operatorname{div}_x \mathbf{v} = 0, \\ & \mathbf{v} \in C^1((0, T) \times \Omega; R^3), \partial_t \mathbf{v} + \operatorname{div}_x \mathbb{U} = 0 \\ & \text{for a certain } \mathbb{U} \in C^1((0, T) \times \Omega; R_{\text{sym},0}^{3 \times 3}), \\ & \frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] < e[\vartheta] \text{ in } (0, T) \times \Omega \Big\}, \end{aligned}$$

Distance function

$$\frac{3}{2} \lambda_{\max} \left[\frac{(\mathbf{v} + \nabla_x \Psi) \otimes (\mathbf{v} + \nabla_x \Psi)}{\varrho} - \mathbb{U} \right] \geq \frac{1}{2} \frac{|\mathbf{v} + \nabla_x \Psi|^2}{\varrho},$$

Energy solutions

Energy conservation

$$\frac{d}{dt} \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) dx = 0$$

Relative entropy (energy)

Relative entropy functional

$$\begin{aligned} & \mathcal{E}(\varrho, \vartheta, \mathbf{u} \mid r, \Theta, \mathbf{U}) \\ &= \int_{\Omega} \left(\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right) dx \end{aligned}$$

Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho \left(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \right)$$

Coercivity of the ballistic free energy

$$\varrho \mapsto H_{\Theta}(\varrho, \Theta) \text{ strictly convex}$$

$$\vartheta \mapsto H_{\Theta}(\varrho, \vartheta) \text{ decreasing for } \vartheta < \Theta \text{ and increasing for } \vartheta > \Theta$$

Dissipative solutions

Relative entropy inequality

$$\begin{aligned} \left[\mathcal{E} \left(\varrho, \vartheta, \mathbf{u} \middle| r, \Theta, \mathbf{U} \right) \right]_{t=0}^{\tau} - \int_0^{\tau} \int_{\Omega} \frac{\Theta}{\vartheta} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \, dx \, dt \\ \leq \int_0^{\tau} \mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U}) \, dt \end{aligned}$$

for any $r > 0$, $\Theta > 0$, $\mathbf{U}$ satisfying relevant boundary conditions

Remainder

$$\begin{aligned}
 & \boxed{\mathcal{R}(\varrho, \vartheta, \mathbf{u}, r, \Theta, \mathbf{U})} \\
 &= \int_{\Omega} \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \right) \cdot (\mathbf{U} - \mathbf{u}) \, dx \\
 &+ \int_{\Omega} \left[\left(p(r, \Theta) - p(\varrho, \vartheta) \right) \operatorname{div} \mathbf{U} + \frac{\varrho}{r} (\mathbf{U} - \mathbf{u}) \cdot \nabla_x p(r, \Theta) \right] \, dx \\
 &- \int_{\Omega} \left(\varrho \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \partial_t \Theta + \varrho \left(s(\varrho, \vartheta) - s(r, \Theta) \right) \mathbf{u} \cdot \nabla_x \Theta \right. \\
 &\quad \left. + \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \cdot \nabla_x \Theta \right) \, dx \\
 &+ \int_{\Omega} \frac{r - \varrho}{r} \left(\partial_t p(r, \Theta) + \mathbf{U} \cdot \nabla_x p(r, \Theta) \right) \, dx
 \end{aligned}$$

Dissipative solutions to the Euler-Fourier system

Initial data

$$\varrho_0 \in C^2, \vartheta_0 \in C^2, \varrho_0 > 0, \vartheta_0 > 0$$

Infinitely many dissipative weak solutions

For any regular initial data ϱ_0, ϑ_0 , there exists a velocity field $\mathbf{u}_0$ such that the Euler-Fourier problem admits infinitely many dissipative weak solutions in $(0, T)$