

# Long-time dynamics of a hyperbolic non-isothermal viscous Cahn-Hilliard equation with dynamic boundary conditions

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*CIRM-ERC Workshop DIMO-2013*  
*Diffuse Interface Models*  
*Levico Terme, September 10-13, 2013*

Joint work with Maurizio Grasselli and Hao Wu

# Cahn-Hilliard equation

Cahn-Hilliard equation describes the phenomenon of phase separation in materials like binary alloys or polymer mixtures

At early stage, it is called spinodal decomposition

$$\begin{cases} \chi_t - \Delta\mu = 0 & \Omega \times (0, \infty) \\ \mu = -\Delta\chi + \alpha\chi_t + f(\chi) & \Omega \times (0, \infty) \end{cases}$$

- $\chi$  relative concentration difference of one component (order parameter / phase-field)
- $\mu$  chemical potential
- $f = F'$ ,  $F$  potential accounting for different phases  
Example of smooth double-well potential:  $F(y) = (y^2 - 1)^2$
- $\alpha \geq 0$  viscosity coefficient for microscopic effects
- $\Omega \subset \mathbb{R}^d$  ( $d = 2, 3$ ), bdd domain with smooth boundary  $\Gamma$

# Modified Cahn-Hilliard equation

- In Galenko - Jou (2005) and Galenko - Lebedev (2007) and (2008), the authors proposed a modification of the Cahn-Hilliard equation to account, e.g., for rapid spinodal decomposition in materials like glasses

$$(*) \quad \chi_t(t) - \int_0^\infty k(s) \Delta \mu(t-s) ds = 0, \quad \Omega \times (0, \infty)$$

where  $k : (0, \infty) \rightarrow (0, \infty)$  is a smooth, decreasing and integrable kernel

- If  $\varepsilon > 0$  and  $k(s) = \varepsilon^{-1} e^{-s/\varepsilon}$ , then  $(*)$  is equivalent to

$$(**) \quad \varepsilon \chi_{tt} + \chi_t - \Delta \mu = 0, \quad \Omega \times (0, \infty)$$

( $\varepsilon$  relaxation time)

# Non-isothermal modified Cahn-Hilliard system

- Rapid spinodal decomposition in certain materials
- Thermal effects
- Grasselli - Petzeltová - Schimperna (2007)

$$\left\{ \begin{array}{ll} (\theta + \chi)_t + \nabla \cdot \mathbf{q} = 0 & \Omega \times (0, \infty) \\ \sigma \mathbf{q}_t + \mathbf{q} = -\nabla \theta & \Omega \times (0, \infty) \\ \varepsilon \chi_{tt} + \chi_t - \Delta \mu = 0 & \Omega \times (0, \infty) \\ \mu = -\Delta \chi + \alpha \chi_t + f(\chi) - \theta & \Omega \times (0, \infty) \end{array} \right.$$

- $\theta$  (relative) temperature around a given critical one
- $\mathbf{q}$  heat flux
- $\sigma \in [0, 1]$  (relaxation time)
- $\alpha \geq 0$  viscosity coefficient
- from now on  $\varepsilon = \sigma = 1$

# Initial and boundary conditions

- Dynamic boundary condition for  $\chi$ :  
interaction between binary alloy and container walls
- No flux boundary condition for  $\mu$

$$\left\{ \begin{array}{ll} \mathbf{q} \cdot \nu = \partial_\nu \mu = 0 & \Gamma \times (0, \infty) \\ \chi_t - \Delta_\Gamma \chi + \partial_\nu \chi + g(\chi) = 0 & \Gamma \times (0, \infty) \\ \theta(0) = \theta_0, \quad \mathbf{q}(0) = \mathbf{q}_0 & \Omega \\ \chi(0) = \chi_0, \quad \chi_t(0) = \chi_1 & \Omega \end{array} \right.$$

- $\nu$  outward normal unit vector on  $\Gamma$
- $\Delta_\Gamma$  Laplace-Beltrami operator
- $g = G'$ ,  $G$  smooth boundary potential, e.g.  $G = c_1 y^2 - c_2$   
 $c_1$ : interaction with wall     $c_2$ : attraction/repulsion by wall)

- **Lorenzi - Rocca (2007)**

Existence weak solution in  $3 - D$ ,  $\alpha = 0$ , non-isothermal case, generic memory kernel, standard bdy conds

- **Grasselli - Petzeltová - Schimperna (2007)**

Existence and uniqueness of global energy solution, existence of global attractor, convergence of energy solution to a single equilibrium in  $3 - D$ ,  $\alpha > 0$ , non-isothermal case, standard bdy conds

- **Grasselli - Schimperna - Zelik (2009)**

- Grasselli - Schimperna - Segatti - Zelik (2009)**

Existence and uniqueness of global weak solution and existence global and exponential attractors in  $2 - D$  and  $3 - D$ , for  $\alpha = 0$ , small  $\varepsilon > 0$ , standard bdy conds

- **Cavaterra - Gal - Grasselli (2011)**

Existence of global and exponential attractors for  $\alpha > 0$ , generic memory kernel, dynamic bdy conds  
Existence of trajectory attractor for  $\alpha = 0$

# Our contribution

- inertial term  $\varepsilon \chi_{tt}$ ,  $\varepsilon > 0$
- viscous case  $\alpha > 0$
- non-isothermal case
- dynamic boundary conditions for  $\chi$

## GOALS

- Existence and uniqueness of energy and weak solutions
- Existence of the global attractor in the energy phase space
- Convergence of a weak solution to a single equilibrium

## DIFFICULTIES

- No regularizing effect for  $\chi$  due to dynamic bdry conds
- Łojasiewicz-Simon technique is more complicated due to dynamic bdry conds
- The equation for  $\mathbf{q}$  is hyperbolic

# An equivalent problem

- $\xi := \chi|_{\Gamma} \quad \xi_0 := \chi_0|_{\Gamma}$

$$(1) \left\{ \begin{array}{ll} (\theta + \chi)_t + \nabla \cdot \mathbf{q} = 0 & \Omega \times (0, \infty) \\ \mathbf{q}_t + \mathbf{q} = -\nabla \theta & \Omega \times (0, \infty) \\ \chi_{tt} + \chi_t - \Delta \mu = 0 & \Omega \times (0, \infty) \\ \mu = -\Delta \chi + \alpha \chi_t + f(\chi) - \theta & \Omega \times (0, \infty) \\ \mathbf{q} \cdot \nu = \partial_{\nu} \mu = 0 & \Gamma \times (0, \infty) \\ \xi_t - \Delta_{\Gamma} \xi + \partial_{\nu} \chi + g(\xi) = 0 & \Gamma \times (0, \infty) \\ \theta(0) = \theta_0, \mathbf{q}(0) = \mathbf{q}_0, \chi(0) = \chi_0, \chi_t(0) = \chi_1 & \Omega \\ \xi(0) = \xi_0 & \Gamma \end{array} \right.$$

$$H = L^2(\Omega), \quad V = H^1(\Omega), \quad W = H^2(\Omega)$$

$$H_\Gamma = L^2(\Gamma), \quad V_\Gamma = H^1(\Gamma) \quad W_\Gamma = H^2(\Gamma)$$

$$\mathbf{H} = (H)^d, \quad \mathbf{V} = (V)^d, \quad \mathbf{W} = (W)^d$$

$$H_a = \{v \in H : \langle v \rangle := |\Omega|^{-1} \int_\Omega v = 0\}$$

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{V} : \mathbf{v} \cdot \nu|_\Gamma = 0\}, \quad \mathbf{W}_0 = \{\mathbf{w} \in \mathbf{W} : \mathbf{w} \cdot \nu|_\Gamma = 0\}$$

$$\mathbb{H} = L^2(\Omega) \times L^2(\Gamma)$$

$$\mathbb{H}^r = \{(\chi, \xi) \in H^r(\Omega) \times H^r(\Gamma) : \xi = \chi|_\Gamma\}, \quad \forall r > \frac{1}{2}$$

$$\mathbb{X} = H \times \mathbf{H} \times \mathbb{H}^1 \times V^*, \quad \mathbb{Y} = V \times \mathbf{V}_0 \times \mathbb{H}^3 \times V$$

# The operators $A$ , $A_0$ , $A_0^r$ and the space $V^*$

- $A = -\Delta$  with Neumann boundary condition  
 $A : D(A) \subset H \rightarrow H_a, \quad D(A) = \{v \in H^2(\Omega) : \partial_\nu v = 0 \text{ on } \Gamma\}$
- $A_0$ : restriction of  $A$  to  $H_a \cap D(A)$
- $A_0^r$ :  $r$ -power of  $A_0$  ( $A_0$  is a positive linear operator)
- $V_0^r = D(A_0^{r/2})$  with inner product  $(v_1, v_2)_{V_0^r} = (A_0^{\frac{r}{2}} v_1, A_0^{\frac{r}{2}} v_2)$
- $V^*$ : dual space of  $V$  normed by  
$$\|v\|_{V^*}^2 = \|\nabla A_0^{-1}(v - \langle v \rangle)\|^2 + |\langle v \rangle|^2$$

# Assumptions

$$(H1) \quad f, g \in C^2(\mathbb{R})$$

$$(H2) \quad \liminf_{|s| \rightarrow +\infty} f'(s) > 0, \quad \liminf_{|s| \rightarrow +\infty} g'(s) > 0$$

$$(H3) \quad |f''(y)| \leq c_f(1 + |y|^p), \quad |g''(y)| \leq c_g(1 + |y|^q) \\ 0 \leq p \leq 1 \text{ when } n = 3, \quad p \geq 0 \text{ when } n = 2, \\ q \geq 0$$

## Remark

*Given the double well potential  $F(y) = \frac{1}{4}(y^2 - 1)^2$  and  $g(y)$  polynomial of odd degree with a positive leading coefficient, then  $f(y) = F'(y) = y^3 - y$  and  $g(y)$  satisfy (H1)–(H3)*

# Energy solution

$(\theta, \mathbf{q}, \chi, \xi, \chi_t)$  is an **energy solution** to Problem (1) with initial datum  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{X}$  if it satisfies

- $(\theta, \mathbf{q}, \chi, \xi, \chi_t) \in L^\infty(0, T; \mathbb{X})$
- $\theta_t \in L^\infty(0, T; V^*)$ ,  $\mathbf{q}_t \in L^2(0, T; (\mathbf{V}_0)^*)$
- $\chi_t \in L^2(0, T; V^*)$ ,  $\alpha^{\frac{1}{2}} \chi_t \in L^2(0, T; H)$ ,  $\xi_t \in L^2(0, T; H_\Gamma)$
- $\chi_{tt} + \chi_t \in L^\infty(0, T; D(A_0^{-\frac{3}{2}}))$

and solves, for a.e.  $t \in (0, T)$ ,

$$(2) \left\{ \begin{array}{ll} ((\theta + \chi)_t, w)_{V^*, V} - (\mathbf{q}, \nabla w)_H = 0 & \forall w \in V \\ (\mathbf{q}_t + \mathbf{q}, \mathbf{v})_{\mathbf{V}_0^*, \mathbf{V}_0} - (\theta, \nabla \cdot \mathbf{v})_H = 0 & \forall \mathbf{v} \in \mathbf{V}_0 \\ (A_0^{-1}(\chi_{tt} + \chi_t), \phi)_{V^*, V} + (\mu, \phi)_{V^*, V} = 0 & \forall \phi \in D(A_0^{1/2}) \\ (\mu, u)_{V^*, V} = (\nabla \chi, \nabla u)_H + (\nabla_\Gamma \xi, \nabla_\Gamma \mathbf{v})_{H_\Gamma} + \alpha(\chi_t, u)_H \\ + (\xi_t, \mathbf{v})_{H_\Gamma} + (f(\chi), u)_H + (g(\xi), \mathbf{v})_{H_\Gamma} - (\theta, u)_H & \forall (u, \mathbf{v}) \in \mathbb{H}^1 \end{array} \right.$$

An energy solution  $(\theta, \mathbf{q}, \chi, \xi, \chi_t)$  to Problem (1) with initial datum  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{Y}$  it is said **weak solution** if in addition it satisfies

- $(\theta, \mathbf{q}, \chi, \xi, \chi_t) \in L^\infty(0, T; \mathbb{Y})$
- $\theta_t \in L^\infty(0, T; H), \quad \mathbf{q}_t \in L^2(0, T; \mathbf{H})$
- $\chi_{tt} \in L^2(0, T; V^*), \quad \alpha^{\frac{1}{2}} \chi_{tt} \in L^2(0, T; H)$
- $\xi_{tt} \in L^2(0, T; H_\Gamma)$
- $\chi_{tt} + \chi_t \in L^\infty(0, T; D(A_0^{-\frac{1}{2}}))$

# Time regularity of energy and weak solutions

An energy solution belongs to  $C_w([0, T]; \mathbb{X})$ , where

$$C_w([0, T]; \mathbb{X})$$

$$= \{v \in L^\infty(0, T; \mathbb{X}) : (\phi, v(\cdot))_{\mathbb{X}^*, \mathbb{X}} \in C^0([0, T]), \forall \phi \in \mathbb{X}^*\}$$

Therefore, any energy solution can be evaluated point-wisely in time and initial conditions have a well-defined meaning.

The same property holds for weak solutions.

## Conservative properties

- $\langle \theta(t) + \chi(t) \rangle = \langle \theta_0 + \chi_0 \rangle, \quad \forall t \geq 0$
- $\langle \chi_t(t) + \chi(t) \rangle = \langle \chi_1 + \chi_0 \rangle, \quad \forall t \geq 0$

## Decay properties

- $\langle \chi(t) \rangle = (\langle \chi_0 \rangle + \langle \chi_1 \rangle) - \langle \chi_1 \rangle e^{-t}$
- $\langle \chi_t(t) \rangle = \langle \chi_1 \rangle e^{-t}$

# A new equivalent problem

- $\tilde{\theta} = \theta - \langle \theta \rangle, \quad \tilde{\chi} = \chi - \langle \chi \rangle, \quad \tilde{\xi} = \tilde{\chi}|_{\Gamma} = \xi - \langle \chi \rangle$

$$(3) \left\{ \begin{array}{ll} (\tilde{\theta} + \tilde{\chi})_t + \nabla \cdot \mathbf{q} = 0 & \Omega \times (0, \infty) \\ \mathbf{q}_t + \mathbf{q} = -\nabla \tilde{\theta} & \Omega \times (0, \infty) \\ \tilde{\chi}_{tt} + \tilde{\chi}_t - \Delta \tilde{\mu} = 0 & \Omega \times (0, \infty) \\ \tilde{\mu} = -\Delta \tilde{\chi} + \alpha \tilde{\chi}_t + f(\chi) - \tilde{\theta} & \Omega \times (0, \infty) \\ \mathbf{q} \cdot \nu = \partial_{\nu} \tilde{\mu} = 0 & \Gamma \times (0, \infty) \\ \tilde{\xi}_t - \Delta_{\Gamma} \tilde{\xi} + \partial_{\nu} \tilde{\chi} + g(\xi) + \langle \chi_1 \rangle e^{-t} = 0 & \Gamma \times (0, \infty) \\ \tilde{\theta}(0) = \theta_0 - \langle \theta_0 \rangle, \mathbf{q}(0) = \mathbf{q}_0, \tilde{\chi}(0) = \chi_0 - \langle \chi_0 \rangle, \tilde{\chi}_t(0) = \chi_1 & \Omega \\ \tilde{\xi}(0) = \xi_0 - \langle \chi_0 \rangle, & \Gamma \end{array} \right.$$

# Dissipative estimate in $\mathbb{X}$

## Theorem

*Assume (H1)–(H3) and  $(\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t))$  be a regular solution to Problem (1).*

*Then there exist a non decreasing positive function  $\mathcal{Q}$  and two positive constants  $\rho_1, \rho_2$ , independent of  $t$ , such that*

$$\begin{aligned} & \|(\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t))\|_{\mathbb{X}}^2 \\ & + \int_t^{t+1} (\alpha \|\chi_t(\tau)\|^2 + \|\xi_t\|_{H_T}^2 + \|(\theta(\tau), \mathbf{q}(\tau), \chi(\tau), \xi(\tau), \chi_t(\tau))\|_{\mathbb{X}}^2) d\tau \\ & \leq \mathcal{Q}(\|(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1)\|_{\mathbb{X}}) e^{-\rho_1 t} + \rho_2, \quad \forall t \geq 0 \end{aligned}$$

*where  $\mathcal{Q}, \rho_1, \rho_2$  may depend also on  $\langle \theta_0 \rangle, \langle \chi_0 \rangle, \langle \chi_1 \rangle, |\Omega|, |\Gamma|$ .*

## Proof.

All the calculations can be justified within a Faedo-Galerkin approximation scheme.



## Theorem

*Assume (H1)–(H3) and  $(\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t))$  be a regular solution to Problem (1).*

*Then, for any  $t \geq 0$ , it holds*

$$\|(\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t))\|_{\mathbb{Y}} \leq C \|(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1)\|_{\mathbb{Y}}$$

## Proof.

We formally differentiate Problem (3) with respect to time.  $\square$

# Existence and uniqueness results

## Theorem

*Assume (H1)–(H3). Then we have*

*(a) For any initial datum  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{Y}$ , Problem (1) admits a unique weak solution.*

*(b) For any initial datum  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{X}$ , Problem (1) admits a unique energy solution.*

*Moreover, if  $(\theta_0^i, \mathbf{q}_0^i, \chi_0^i, \xi_0^i, \chi_1^i)$ ,  $i = 1, 2$ , are two sets of data and  $(\theta^i, \mathbf{q}^i, \chi^i, \xi^i, \chi^i)$ ,  $i = 1, 2$ , are the corresponding solutions, then, for any  $t \geq 0$ , it holds*

$$\begin{aligned} & \|((\theta^1 - \theta^2)(t), (\mathbf{q}^1 - \mathbf{q}^2)(t), (\chi^1 - \chi^2)(t), (\xi^1 - \xi^2)(t), (\chi_t^1 - \chi_t^2)(t))\|_{\mathbb{X}} \\ & \leq C_1 e^{C_2 T} \|(\theta_0^1 - \theta_0^2, \mathbf{q}_0^1 - \mathbf{q}_0^2, \chi_0^1 - \chi_0^2, \xi_0^1 - \xi_0^2, \chi_1^1 - \chi_1^2)\|_{\mathbb{X}} \end{aligned}$$

*where  $C_1, C_2$  are two positive constants depending only on the  $\mathbb{X}$ -norms of the initial data,  $\alpha$ ,  $|\Omega|$  and  $|\Gamma|$ .*

# Proof of existence and uniqueness theorem

## Proof.

- Using the uniform high order estimate, we prove the existence of **global weak solutions** to Problem (1) by using a Faedo-Galerkin approximation scheme as in Gilardi - Miranville - Schimperna (2009) and (2010) for Cahn-Hilliard equation with dynamical boundary condition.
- We prove the continuous dependence estimate for weak solutions in  $\mathbb{X}$  and we get uniqueness.
- On account of the continuous dependence estimate in  $\mathbb{X}$ , by standard density argument we prove existence and uniqueness of energy solutions to Problem (1).
- The case  $\alpha = 0$  is an open problem.



## Corollary

*Assume (H1)–(H3).*

*(a) The unique global weak solution to Problem (1) defines a semigroup  $S_1(t) : \mathbb{Y} \rightarrow \mathbb{Y}$  such that*

$$S_1(t)(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) = (\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t)), \quad \forall t \geq 0.$$

*(b) The unique global energy solution to Problem (1) defines a strongly continuous semigroup  $S_2(t) : \mathbb{X} \rightarrow \mathbb{X}$  such that*

$$S_2(t)(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) = (\theta(t), \mathbf{q}(t), \chi(t), \xi(t), \chi_t(t)), \quad \forall t \geq 0.$$

## Remark

*$S_1(t)$  is a closed semigroup in the sense of Pata-Zelik (2007)*

# The global attractor: the phase space

- Conservative and decay properties
- $M, M' \geq 0$
- $\mathbb{X}_{M,M'} = \left\{ (z_1, z_2, z_3, z_4, z_5) \in \mathbb{X} : \right.$   
$$\left. |\langle z_1 + z_3 \rangle| \leq M, \quad |\langle z_3 + z_5 \rangle| \leq M, \quad |z_5| \leq M' \right\}$$
- $\mathbb{X}_{M,M'}$  metric space with the metric induced by the  $\mathbb{X}$ -norm

# The global attractor for energy solution

## Theorem

*Assume (H1)-(H3).*

*Then the semigroup  $S_2(t)$  defined by the global energy solutions to Problem (1) on  $\mathbb{X}_{M,M'}$  possesses a compact connected global attractor  $\mathcal{A}$ , which is bounded in  $\mathbb{Y}$ .*

## Proof.

**Step 1.** The restriction of  $S_2(t)$  on  $\mathbb{X}_{M,M'}$  admits a bounded absorbing set

**Step 2.** The trajectories are precompact sets in  $\mathbb{X}$  □

## Proposition

*Assume (H1)-(H3).*

*Then there exists  $R_0 > 0$  such that the ball  $\mathcal{B}_0$  in  $\mathbb{X}_{M,M'}$  of radius  $R_0$  centered at zero is absorbing for the semigroup  $S_2(t)$ .*

*Namely, for every bounded set  $\mathcal{B} \subset \mathbb{X}_{M,M'}$ , there exists a time  $t_0 = t_0(\mathcal{B}, M, M')$  such that*

$$S_2(t)\mathcal{B} \subset \mathcal{B}_0, \quad \forall t \geq t_0$$

# Proof of Step 1

For any bounded set  $\mathcal{B} \subset \mathbb{X}_{M,M'}$ , consider the initial datum  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathcal{B} \subset \mathbb{X}_{M,M'}$ .

Then there exists a positive constant  $R$  depending on  $\mathcal{B}$  s.t.

$$\|(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1)\|_{\mathbb{X}} \leq R$$

Besides

$$|\langle \theta_0 \rangle| \leq 2M + M', \quad |\langle \chi_0 \rangle| \leq M + M', \quad |\langle \chi_1 \rangle| \leq M'$$

Then there exist  $K_1, K_2 > 0$  independent of  $R$  and  $t$  s.t.

$$\|(\theta, \mathbf{q}, \chi, \xi, \chi_t)(t)\|_{\mathbb{X}}^2 \leq \|(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1)\|_{\mathbb{X}}^2 e^{-K_1 t} + K_2$$

Finally, we find  $t_0 = t_0(R, M, M') > 0$ ,  $R_0 = R_0(M, M') > 0$  s.t.

$$\|(\theta, \mathbf{q}, \chi, \xi, \chi_t)(t)\|_{\mathbb{X}} \leq R_0, \quad \forall t \geq t_0$$

### Proposition

*Assume (H1)-(H3) and  $(\theta, \mathbf{q}, \chi, \xi, \chi_t)$  be the unique energy solution to Problem (1) with  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{X}$ .*

*Then the orbit*

$$\bigcup_{t \geq 0} (\theta, \mathbf{q}, \chi, \xi, \chi_t)(t)$$

*is precompact in  $\mathbb{X}$ .*

There exist two constants  $c_0, c_1 > 0$  such that

$$f'(y) \geq -c_0, \quad g'(y) \geq -c_1, \quad \forall y \in \mathbb{R}$$

Moreover, there exist  $\gamma_0 > c_0, \gamma_1 > c_1$  such that

$$\frac{1}{2} \|\nabla z\|^2 + (\gamma_0 - 2c_0) \|z\|^2 \geq \int_{\Omega} f'(\zeta) z^2 dx, \quad \forall z, \zeta \in V$$

$$\frac{1}{2} \|\nabla z\|_{H_\Gamma}^2 + (\gamma_1 - 2c_1) \|z\|_{H_\Gamma}^2 \geq \int_{\Gamma} g'(\zeta) z^2 dS, \quad \forall z, \zeta \in V_\Gamma$$

We introduce the monotone nondecreasing functions in  $\mathbb{R}$

$$\hat{f}(y) = f(y) + \gamma_0 y, \quad \hat{g}(y) = g(y) + \gamma_1 y, \quad y \in \mathbb{R}$$

We split the solution to Problem (1) as follows

$$(\theta, \mathbf{q}, \chi, \xi, \chi_t) = (\theta^d, \mathbf{q}^d, \chi^d, \xi^d, \chi_t^d) + (\theta^c, \mathbf{q}^c, \chi^c, \xi^c, \chi_t^c)$$

where  $(\theta^d, \mathbf{q}^d, \chi^d, \xi^d, \chi_t^d)$  and  $(\theta^c, \mathbf{q}^c, \chi^c, \xi^c, \chi_t^c)$  are solutions to Problem (Pd) and Problem (Pc), respectively, to be defined.

- $\tilde{\theta}_0 = \theta_0 - \langle \theta_0 \rangle, \quad \tilde{\chi}_0 = \chi_0 - \langle \chi_0 \rangle, \quad \tilde{\chi}_1 = \chi_1 - \langle \chi_1 \rangle$

$$(Pd) \left\{ \begin{array}{ll} (\theta^d + \chi^d)_t + \nabla \cdot \mathbf{q}^d = 0, & \Omega \times (0, \infty) \\ \mathbf{q}_t^d + \mathbf{q}^d + \nabla \theta^d = 0, & \Omega \times (0, \infty) \\ \chi_{tt}^d + \chi_t^d - \Delta \mu^d = 0, & \Omega \times (0, \infty) \\ \mu^d = -\Delta \chi^d + \hat{f}(\chi) - \hat{f}(\chi^c) + \alpha \chi_t^d - \theta^d, & \Omega \times (0, \infty) \\ \mathbf{q}^d \cdot \nu = \partial_\nu \mu^d = 0, & \Gamma \times (0, \infty) \\ \xi_t^d - \Delta_\Gamma \xi^d + \partial_\nu \chi^d + \hat{g}(\xi) - \hat{g}(\xi^c) = 0, & \Gamma \times (0, \infty) \\ \theta^d(0) = \tilde{\theta}_0, \mathbf{q}^d(0) = \mathbf{q}_0, \chi^d(0) = \tilde{\chi}_0, \chi_t^d(0) = \tilde{\chi}_1, & \Omega \\ \xi^d(0) = \xi_0, & \Gamma \end{array} \right.$$

$$(Pc) \left\{ \begin{array}{ll} (\theta^c + \chi^c)_t + \nabla \cdot \mathbf{q}^c = 0, & \Omega \times (0, \infty) \\ \mathbf{q}_t^c + \mathbf{q}^c + \nabla \theta^c = 0, & \Omega \times (0, \infty) \\ \chi_{tt}^c + \chi_t^c - \Delta \mu^c = 0, & \Omega \times (0, \infty) \\ \mu^c = -\Delta \chi^c + \hat{f}(\chi^c) + \alpha \chi_t^c - \theta^c - \gamma_0 \chi, & \Omega \times (0, \infty) \\ \mathbf{q}^c \cdot \nu = \partial_\nu \mu^c = 0, & \Gamma \times (0, \infty) \\ \xi_t^c - \Delta_\Gamma \xi^c + \partial_\nu \chi^c + \hat{g}(\xi^c) + \langle \chi_1 \rangle e^{-t} = \gamma_1 \xi, & \Gamma \times (0, \infty) \\ \theta^c(0) = \langle \theta_0 \rangle, \mathbf{q}^c(0) = \mathbf{0}, \chi^c(0) = \langle \chi_0 \rangle, \chi_t^c(0) = \langle \chi_1 \rangle, & \Omega \\ \xi^c(0) = 0, & \Gamma \end{array} \right.$$

- Problem (Pd) and Problem (Pc) admit unique global solutions satisfying uniform in time estimates
- There exists a constant  $K > 0$  such that, for any  $t \geq 0$ ,  
$$\|(\theta^d, \mathbf{q}^d, \chi^d, \xi^d, \chi_t^d)(t)\|_{\mathbb{X}} \leq \|(\tilde{\theta}_0, \mathbf{q}_0, \tilde{\chi}_0, \xi_0, \tilde{\chi}_1)\|_{\mathbb{X}} e^{-Kt}$$
  
 $(\theta^d, \mathbf{q}^d, \chi^d, \xi^d, \chi_t^d)$  decays to zero exponentially fast as time tends to infinity
- $\|(\theta^c, \mathbf{q}^c, \chi^c, \xi^c, \chi_t^c)(t)\|_{\mathbb{Y}} \leq C, \quad \forall t \geq 0$   
 $(\theta^c, \mathbf{q}^c, \chi^c, \xi^c, \chi_t^c)$  is bounded in a space that can be compactly embedded into  $\mathbb{X}$

# Proof of existence of the global attractor

- The semigroup  $S_2(t)$  has a bounded absorbing set in  $\mathbb{X}_{M,M'}$ .
- Any trajectory starting from  $\mathbb{X}$  can be decomposed into two parts: one part decays exponentially fast to zero in  $\mathbb{X}$  and the other part is uniformly bounded in  $\mathbb{Y}$ . Thus, the trajectory is precompact in  $\mathbb{X}$ .
- Classical results in dynamical systems (Temam (1988)) imply the existence of the global attractor for  $(S_2(t), \mathbb{X}_{M,M'})$  bounded in  $\mathbb{Y}$ .

# The closed semigroup $S_1(t)$

- Existence of the global attractor for the closed semigroup  $S_1(t)$  generated by the weak solution can be proved within the framework of Pata-Zelik (2007).
- There exists an absorbing set in the phase space

$$\mathbb{Y}_{M,M'} = \{(z_1, \mathbf{z}_2, z_3, z_4, z_5) \in \mathbb{Y} :$$

$$|\langle z_1 + z_3 \rangle| \leq M, \quad |\langle z_3 + z_5 \rangle| \leq M, \quad |z_5| \leq M'\}$$

- The trajectories satisfy suitable asymptotic compactness property in  $\mathbb{Y}$ .

# Stationary problem

The steady states  $(\theta_\infty, \chi_\infty, \xi_\infty)$  of Problem (1) satisfy the elliptic boundary value problem

$$(S1) \left\{ \begin{array}{ll} -\Delta \theta_\infty = 0, & \Omega \\ -\Delta(-\Delta \chi_\infty + f(\chi_\infty) - \theta_\infty) = 0, & \Omega \\ \partial_\nu \theta_\infty = 0, & \Gamma \\ \partial_\nu(-\Delta \chi_\infty + f(\chi_\infty)) = 0, & \Gamma \\ -\Delta_\Gamma \xi_\infty + \partial_\nu \chi_\infty + g(\xi_\infty) = 0, & \Gamma \\ \xi_\infty = \chi_\infty|_\Gamma, & \Gamma \end{array} \right.$$

with constraints (related to the initial data)

$$\langle \chi_\infty \rangle = \langle \chi_0 + \chi_1 \rangle, \quad \langle \theta_\infty \rangle = \langle \theta_0 \rangle - \langle \chi_1 \rangle$$

# An equivalent stationary problem

Problem (S1) can be reduced to

$$(S2) \left\{ \begin{array}{ll} \theta_{\infty} = \langle \theta_0 \rangle - \langle \chi_1 \rangle & \\ -\Delta \chi_{\infty} + f(\chi_{\infty}) = \mu_{\infty}, & \Omega \\ -\Delta_{\Gamma} \xi_{\infty} + \partial_{\nu} \chi_{\infty} + g(\xi_{\infty}) = 0, & \Gamma \\ \xi_{\infty} = \chi_{\infty}|_{\Gamma}, & \Gamma \\ \langle \chi_{\infty} \rangle = \langle \chi_0 + \chi_1 \rangle & \end{array} \right.$$

where  $\mu_{\infty} = \langle f(\chi_{\infty}) \rangle + \frac{1}{|\Omega|} \int_{\Gamma} g(\xi_{\infty}) dS$

# Existence of stationary solutions

## Proposition

*Assume (H1)-(H3). Then Problem (S2) admits at least one solution  $(\chi_\infty, \xi_\infty) \in \mathbb{H}^1$*

## Proof.

We extend a result of Wu - Zheng (2004)



## Proposition

*Assume (H1)-(H3). Then an extended Łojasiewicz-Simon inequality holds*

## Proof.

We adapt the arguments used in Sprekels - Wu (2010)



# Convergence to single equilibrium

## Theorem

*Assume (H1)-(H3).*

*Then, for any initial data  $(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1) \in \mathbb{Y}$ , the unique global weak solution to Problem (1) satisfies*

$$\lim_{t \rightarrow +\infty} \|(\theta, \mathbf{q}, \chi, \xi, \chi_t)(t) - (\theta_\infty, \mathbf{0}, \chi_\infty, \xi_\infty, 0)\|_{\mathbb{X}} = 0,$$

*where  $(\theta_\infty, \chi_\infty, \xi_\infty)$  solves the stationary problem (S2).*

## Proof.

- $\omega$ -limit set  $\omega(\theta_0, \mathbf{q}_0, \chi_0, \xi_0, \chi_1)$  is non-empty, compact and connected in the strong topology of  $\mathbb{X}$
- Precompactness of trajectory in  $\mathbb{X}$
- Łojasiewicz-Simon inequality
- Uniform high order estimates



# Decay rate of convergence

## Corollary

*Assume (H1)-(H3). Then, for any  $t \geq 0$ , it holds*

$$\|(\theta, \mathbf{q}, \chi, \xi, \chi)(t) - (\theta_\infty, \mathbf{0}, \chi_\infty, \xi_\infty, 0)\|_{\mathbb{X}} \leq C(1+t)^{-\frac{\rho}{1-2\rho}}$$

*where  $C$  is a positive constant depending on  $\mathbb{X}$ -norm of the initial data and on the coefficients of the system, while  $\rho \in (0, \frac{1}{2})$  may depend on  $(\chi_\infty, \xi_\infty)$ .*

## Proof.

We adapt the arguments of Grasselli - Wu - Zheng (2007). □

**Thanks for the attention!**