

Sharp Interface Limit for the Cahn-Larché System

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Overview

- 1 Sharp Interface Limit for the Cahn-Hilliard Equation
- 2 Sharp Interface Limit for the Cahn-Larché System

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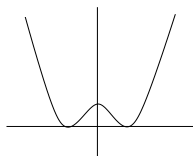
Energy of a Two-Component Mixture

Let $c = c_1 - c_2$ be **concentration difference** of a two-component mixture.

$$E_\varepsilon(c) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla c(x)|^2 dx + \varepsilon^{-1} \int_{\Omega} f(c(x)) dx$$

be the **free energy** of the mixture, where $\Omega \subseteq \mathbb{R}^d$, $d = 1, 2, 3$, $\varepsilon > 0$ and

$$f: \mathbb{R} \rightarrow [0, \infty) \text{ with } f(c) = 0 \Leftrightarrow c = \pm 1.$$



Example:
 $f(c) = \frac{1}{8}(1 - c^2)^2$

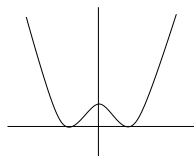
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Theorem (Modica-Mortola '77, Modica '87)

$$E_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} \sigma P \quad \text{w.r.t. } L^1 - \Gamma - \text{convergence}$$

where $\sigma = \int_{-1}^1 \sqrt{2f(s)} ds$ and

$$P(v) = \begin{cases} \mathcal{H}^{d-1}(\partial^* E) & \text{if } v = 2\chi_E - 1 \\ +\infty & \text{else.} \end{cases}$$

Sharp Interface Limit of Cahn-Hilliard Equation

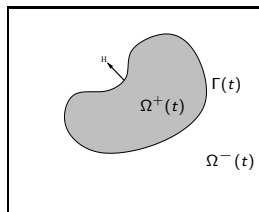
We consider

$$\partial_t c = m \Delta \mu, \quad (1)$$

$$\mu = -\varepsilon \Delta c + \varepsilon^{-1} f'(c) \quad (2)$$

together with suitable boundary and initial conditions. Then (1)-(2) converges to the **Mullins-Sekerka equation**:

$$\begin{aligned} V &= -\frac{m}{2} [\mathbf{n}_{\Gamma(t)} \cdot \nabla \mu] && \text{on } \Gamma(t) \\ \mu|_{\Gamma(t)} &= \sigma H && \text{on } \Gamma(t) \\ \Delta \mu &= 0 && \text{on } \Omega^\pm(t) \end{aligned}$$



due to

- Alikakos et al. '94 (local strong solutions)
- X. Chen '96 (global varifold solutions).

Here H and V are the mean curvature and normal velocity of $\Gamma(t)$.

$$[f](x) = \lim_{h \rightarrow 0} (f(x + h \mathbf{n}_{\Gamma(t)}) - f(x - h \mathbf{n}_{\Gamma(t)})).$$

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Two-Component Mixture with Elastic Energy

To take the **anisotropic structure of a metal** into account we consider

$$E_\varepsilon(c, \mathbf{u}) = \frac{\varepsilon}{2} \int_{\Omega} |\nabla c(x)|^2 dx + \varepsilon^{-1} \int_{\Omega} f(c(x)) dx + \int_{\Omega} W(c, \mathcal{E}(\mathbf{u})) dx$$

where Ω, f are as before, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$,

$$W(c, \mathcal{E}) = \frac{1}{2}(\mathcal{E} - \mathcal{E}^*(c)) : \mathcal{C}(c)(\mathcal{E} - \mathcal{E}^*(c))$$

$\mathcal{C}(c) : \mathbb{R}_{\text{sym}}^d \rightarrow \mathbb{R}_{\text{sym}}^d$ is symmetric and positive definite, and $\mathcal{E}^*(c)$ is the **eigenstrain** at concentration c .

Theorem (Garcke '00)

$$E_\varepsilon \rightarrow_{\varepsilon \rightarrow 0} E_0 \quad \text{w.r.t. } (L^1 \times H^1) - \Gamma - \text{convergence}$$

where

$$E_0(v, \mathbf{u}) = \begin{cases} \sigma \mathcal{H}^{d-1}(\partial^* E) + \int_{\Omega} W(v, \mathcal{E}(u)) dx & \text{if } v = 2\chi_E - 1 \\ +\infty & \text{else.} \end{cases}$$

Cahn-Larché System

Assuming a **quasi-stationary evolution** for the elastic response one derives

$$\partial_t c = \Delta \mu \quad \text{in } \Omega \quad (3)$$

$$\mu = -\varepsilon \Delta c + \varepsilon^{-1} f'(c) + W_{,c}(c, \mathcal{E}(u)) \equiv \frac{\delta E_\varepsilon}{\delta c} \quad \text{in } \Omega \quad (4)$$

$$0 = \operatorname{div}(\mathcal{C}(c)(\mathcal{E}(u) - \mathcal{E}^*(c))) \equiv \operatorname{div}(W_{,\mathcal{E}}(c, \mathcal{E}(u))) \quad \text{in } \Omega \quad (5)$$

together with

$$\mathbf{n}_{\partial\Omega} \cdot \nabla c|_{\partial\Omega} = \mathbf{n}_{\partial\Omega} \cdot \nabla \mu|_{\partial\Omega} = \mathbf{u}|_{\partial\Omega} = 0 \quad \text{on } \partial\Omega \quad (6)$$

$$c|_{t=0} = c_0 \quad \text{in } \Omega \quad (7)$$

plus suitable boundary conditions for $c, \mu, \mathbf{u}$ and an initial condition for c .

Garcke '03: Existence of weak solutions, uniqueness if $\mathcal{C}$ is independent of c and $\mathcal{E}^*(c)$ if affine linear.

Sharp Interface Limit System

By formally matched asymptotic calculations it was shown by [Fried and Gurtin '94](#) and [Leo, Lowengrub and Jou '98](#) that the limit system is

$$V = -\frac{1}{2}[\mathbf{n}_{\Gamma(t)} \cdot \nabla \mu] \quad \text{on } \Gamma(t) \quad (8)$$

$$\mu|_{\Gamma(t)} = \sigma H + \frac{1}{2}\mathbf{n}_{\Gamma(t)}^T [W \text{Id} - (\nabla \mathbf{u})^T \mathbf{S}] \mathbf{n}_{\Gamma(t)} \quad \text{on } \Gamma(t) \quad (9)$$

$$[\mathbf{S} \mathbf{n}_{\Gamma(t)}] = [\mathbf{u}] = [\mu] = 0 \quad \text{on } \Gamma(t) \quad (10)$$

$$\Delta \mu = 0 \quad \text{on } \Omega^\pm(t) \quad (11)$$

$$\text{div } \mathbf{S} = 0 \quad \text{on } \Omega^\pm(t) \quad (12)$$

$$\Gamma(0) = \Gamma_0 \quad (13)$$

where $\mathbf{S} = W_{,\mathcal{E}}(\pm 1, \mathcal{E}(\mathbf{u}))$ in $\Omega^\pm(t)$.

[Garcke, Kwak '06](#): Proof of convergence on the level of varifold solutions similar to [Chen '96](#).

Convergence in Strong Norms

We assume that $\mathcal{C}$ is independent of c and $\mathcal{E}^*(c)$ is linear.

Theorem (A., Schaubeck '13)

Let $\Gamma_0 \subseteq \Omega$ be a smooth hypersurface and *assume that the limit system (8)-(13) has a smooth solution* $(\Gamma(t)_{t \in [0, T]}, \mu, \mathbf{u})$ on $[0, T]$ for some $T > 0$. Moreover, let $(c_\varepsilon, \mu_\varepsilon, \mathbf{u}_\varepsilon)$ be the solution of (3)-(7) with suitably well-prepared initial data $c_0 \equiv c_{0, \varepsilon}$. Then

$$\lim_{\varepsilon \rightarrow 0} c_\varepsilon(t) = \pm 1 \quad \text{in } \Omega^\pm(t)$$

$$\lim_{\varepsilon \rightarrow 0} \mu_\varepsilon(t) = \mu(t) \quad \text{in } \overline{\Omega}$$

$$\lim_{\varepsilon \rightarrow 0} \mathbf{u}_\varepsilon(t) = \mathbf{u}(t) \quad \text{in } \overline{\Omega}$$

uniformly (on compact subsets) for every $t \in [0, T]$.

Schaubek '13: Existence of smooth solutions for short times.

Sketch of the Proof

- ① **Estimate of approximate and exact solution:** Assuming that $(c_{A,\varepsilon}, \mu_{A,\varepsilon}, \mathbf{u}_{A,\varepsilon})$ is a suitable approximate solution up to order $O(\varepsilon^{\kappa k})$, $\kappa = \frac{d+4}{d+2}$, one proves

$$\|c_\varepsilon - c_{A,\varepsilon}\|_{L^p(\Omega \times (0,T))} + \|\mathbf{u}_\varepsilon - \mathbf{u}_{A,\varepsilon}\|_{L^2(0,T;W_2^1)} \leq C\varepsilon^k$$

provided $k \geq K(d)$.

- ② **Interpolation with higher order norms** yields convergence of $c_\varepsilon - c_{A,\varepsilon}$ in arbitrarily high norms if k is sufficiently large.
- ③ **Construction of approximate solution** using finitely many terms calculated by formally matched asymptotics.

To construct the expansion rigorously one solves certain ODEs in normal direction. Solvability conditions yield e.g.

$$V = -\frac{1}{2}[\mathbf{n}_{\Gamma(t)} \cdot \nabla \mu] \quad \text{on } \Gamma(t)$$

$$\mu|_{\Gamma(t)} = \sigma H + \frac{1}{2}\mathbf{n}_{\Gamma(t)}^T [W \text{Id} - (\nabla \mathbf{u})^T \mathbf{S}] \mathbf{n}_{\Gamma(t)} \quad \text{on } \Gamma(t)$$

and corresponding linearized equations.

Spectral Estimate of Linearized Operator

Let $\Gamma \subset \Omega$ be a smooth interface, $d(x) = \text{sdist}(x, \Gamma)$, $s(x) = P_\Gamma(x)$, and

$$\phi^\varepsilon(x) = \theta_0\left(\frac{d(x)}{\varepsilon}\right) + \varepsilon p^\varepsilon(s(x))\theta_1\left(\frac{d(x)}{\varepsilon}\right) + \varepsilon^2 q^\varepsilon(x),$$

close to Γ , where θ_0 is the **optimal profile** that solves

$$-\theta_0'' + f'(\theta_0) = 0 \text{ in } \mathbb{R}, \quad \theta_0(0) = 0, \quad \theta_0(\pm\infty) = \pm 1, \quad (14)$$

$\theta_1 \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is any function satisfying

$$\int_{\mathbb{R}} \theta_1(\theta_0')^2 f'''(\theta_0) ds = 0,$$

and $p^\varepsilon(x)$, $q^\varepsilon(x)$ are bounded suitably.

Theorem (Chen '94)

There exist constants $\varepsilon_0 > 0$ and $C_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, $w \in H_{(0)}^1(\Omega)$

$$\int_{\Omega} \left(\varepsilon |\nabla w|^2 + \varepsilon^{-1} f''(\phi^\varepsilon) w^2 \right) dx \geq -C_0 \|w\|_{H_{(0)}^{-1}(\Omega)}^2$$

Estimate of Approximate and Exact Solution (I)

Let $d_\varepsilon = c_\varepsilon - c_{A,\varepsilon}$, $\mathbf{v}_\varepsilon = \mathbf{u}_\varepsilon - \mathbf{u}_{A,\varepsilon}$, where $(c_{A,\varepsilon}, \mu_{A,\varepsilon}, \mathbf{u}_{A,\varepsilon})$ solve (3)-(7) up to order κk and

$$L_\varepsilon d_\varepsilon := -\varepsilon \Delta d_\varepsilon + \varepsilon^{-1} f''(c_{A,\varepsilon}) d_\varepsilon \quad (\text{linearized CH operator}).$$

Then

$$\partial_t d_\varepsilon = \Delta \left(L_\varepsilon d_\varepsilon + W_{,c}(d_\varepsilon, \mathbf{v}_\varepsilon) \right) + \frac{1}{\varepsilon} \mathcal{N}(c_\varepsilon, c_{A,\varepsilon}) + r_\varepsilon \quad (15)$$

since $W_{,c}(\cdot, \cdot)$ is linear, where $r_\varepsilon = O(\varepsilon^{\kappa k})$ and

$$\mathcal{N}(c_\varepsilon, c_{A,\varepsilon}) = f'(c_\varepsilon) - f'(c_{A,\varepsilon}) - f''(c_{A,\varepsilon}) d_\varepsilon = O(d_\varepsilon^2).$$

Multiplication of (15) with d_ε in $H_{(0)}^{-1}(\Omega)$ yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|d_\varepsilon\|_{H_{(0)}^{-1}(\Omega)}^2 + (L_\varepsilon d_\varepsilon, d_\varepsilon)_{L^2(\Omega)} \\ &= - \int_\Omega W_{,c}(d_\varepsilon, \mathbf{v}_\varepsilon) d_\varepsilon \, dx + \frac{1}{\varepsilon} \int_\Omega \mathcal{N}(c_\varepsilon, c_{A,\varepsilon}) d_\varepsilon \, dx + \int_\Omega d_\varepsilon r_\varepsilon \, dx \end{aligned}$$

Estimate of Approximate and Exact Solution (II)

To estimate

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|d_\varepsilon\|_{H_{(0)}^{-1}}^2 + (L_\varepsilon d_\varepsilon, d_\varepsilon)_{L^2} \\ &= - \int_{\Omega} W_{,c}(d_\varepsilon, \mathbf{v}_\varepsilon) d_\varepsilon \, dx + \frac{1}{\varepsilon} \int_{\Omega} \mathcal{N}(c_\varepsilon, c_{A,\varepsilon}) d_\varepsilon \, dx + \int_{\Omega} d_\varepsilon r_\varepsilon \, dx \end{aligned}$$

one uses:

$$(L_\varepsilon d_\varepsilon, d_\varepsilon)_{L^2(\Omega)} = \int_{\Omega} \varepsilon |\nabla d_\varepsilon|^2 + \varepsilon^{-1} f''(c_{A,\varepsilon}) d_\varepsilon^2 \, dx \geq -C \|d_\varepsilon\|_{H_{(0)}^{-1}(\Omega)}^2$$

due to the spectral estimate by Chen.

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one uses:

$$\begin{aligned} & \int_{\Omega} W_{,c}(d_\varepsilon, \mathbf{v}_\varepsilon) d_\varepsilon \, dx \\ &= \int_{\Omega} \mathcal{C}(\mathcal{E}(\mathbf{v}_\varepsilon) - \mathcal{E}^*(d_\varepsilon)) : (\mathcal{E}(\mathbf{v}_\varepsilon) - \mathcal{E}^*(d_\varepsilon)) \, dx - \int_{\Omega} \mathbf{v}_\varepsilon \cdot \mathbf{s}_{A,\varepsilon} \, dx \\ &= c_0 \|\mathcal{E}(\mathbf{v}_\varepsilon) - \mathcal{E}^*(d_\varepsilon)\|_{L^2}^2 - \int_{\Omega} \mathbf{v}_\varepsilon \cdot \mathbf{s}_{A,\varepsilon} \, dx \end{aligned}$$

since

$$\operatorname{div} \mathcal{C}(\mathcal{E}(\mathbf{v}_\varepsilon) - \mathcal{E}^*(d_\varepsilon)) = \mathbf{s}_{A,\varepsilon}.$$

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one uses:

That

$$\frac{1}{\varepsilon} \int_{\Omega} \mathcal{N}(c_\varepsilon, c_{A,\varepsilon}) d_\varepsilon dx \approx O(\varepsilon^{3k-1})$$

compared to the left-hand side which is of order $O(\varepsilon^{2k-m})$ for some $m > 0$ independent of k . Hence for k large enough this term can be absorbed by the left-hand side.

Thank you for your attention!