

Multi-material structural topology optimization using phase field methods

Luise Blank[#], Hassan Farshbaf-Shaker[†], Harald Garcke[#]

and Vanessa Styles[‡]

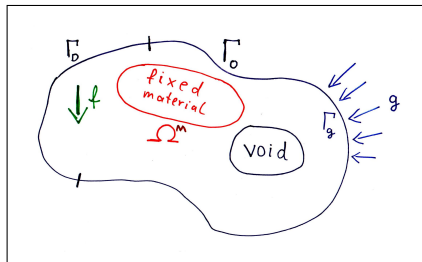
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Talk Outline

- Structural topology optimization
- Multi-material phase field approach
- Structural optimization problem
- Derivation of first order optimality conditions
- Numerical simulations

Structural topology optimization

The goal is to distribute a limited amount of material in a design domain such that an objective functional J is minimized.



- Find domain Ω^M
- Linear elastic material
- Volume forces f given
- Surface forces g given
- Prescribe regions of material and void

Mathematical formulation

Given a design domain $\Omega \subset \mathbb{R}^d$

$$\min_{\Omega^M \in \mathcal{U}_{ad}} J(\Omega^M)$$

Feasible set: $\mathcal{U}_{ad} = \{\Omega^M \subset \Omega \text{ such that } |\Omega^M| = V\}$

May wish to prescribe material and void regions and so require

$$S_0 \subset \Omega^M, \quad \Omega^M \cap S_1 = \emptyset$$

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Possible objective functionals

Compliance (work done by the load)

$$J_1(\Omega^M) = \int_{\Omega^M} \mathbf{f} \cdot \mathbf{u} + \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{u}$$

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Compliance (work done by the load)

$$J_1(\Omega^M) = \int_{\Omega^M} \mathbf{f} \cdot \mathbf{u} + \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{u}$$

Error compared to target displacement

$$J_2(\Omega^M) = \left(\int_{\Omega^M} c(x) |\mathbf{u} - \mathbf{u}_\Omega|^2 \right)^{\frac{1}{2}}$$

$c(x) > 0$: given weighting factor, $\mathbf{u}_\Omega$: target displacement

Structural optimization problem

Minimize $J(\Omega^M) = \alpha J_1(\Omega^M) + \beta J_2(\Omega^M)$ subject to

$$\begin{aligned} -\operatorname{div}(\mathbb{C}^M \mathcal{E}(\mathbf{u})) &= \mathbf{f} && \text{in } \Omega^M \\ (\mathbb{C}^M \mathcal{E}(\mathbf{u}))\mathbf{n} &= \mathbf{0} && \text{on } \Gamma_0 \\ (\mathbb{C}^M \mathcal{E}(\mathbf{u}))\mathbf{n} &= \mathbf{g} && \text{on } \Gamma_g \\ \mathbf{u} &= \mathbf{0} && \text{on } \Gamma_D \end{aligned}$$

$\mathbb{C}^M$ elasticity tensor, $\mathcal{E}(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + \nabla \mathbf{u}^T)$ strain tensor.

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The problem is not well-posed

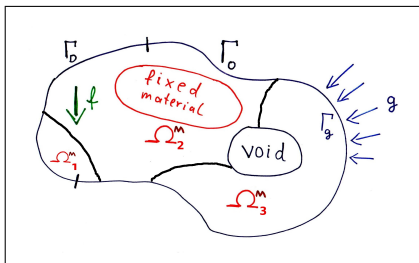
A possible path to well-posedness: Add perimeter: $P(\Omega^M) = \int_{\partial\Omega^M} ds$

This results in the objective functional:

$$J(\Omega^M) = \alpha J_1(\Omega^M) + \beta J_2(\Omega^M) + \gamma P(\Omega^M)$$

Shape derivative: Allaire, Jouve, Toader (2004), Sokolowski & Zolesio (1992).

Multi-material structure optimization



- Heterogeneous structures
- Multi-material shape optimization
- Distribute two or more materials in a given design domain Ω

Multi-material phase field approach

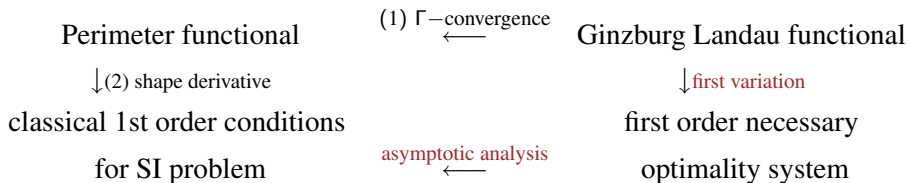
- We consider a phase field method
- Approximate $P(\Omega^M)$ by Ginzburg-Landau functional
- Allows for **topological changes** (nucleation or elimination of holes)
- Easy to construct a **multiphase model**

Multi-material phase field approach

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Phase field approach, earlier work

- The phase field method has been introduced in topology optimization by **Bourdin and Chambolle (2003)**
- Subsequently used by
Burger, Stainko (2006),
Wang, Zhou (2007),
Takezawa, Nishiwaki, Kitamura (2010),
Dedé, Borden, Hughes (2011),
Penzler, Rumpf, Wirth (2013).



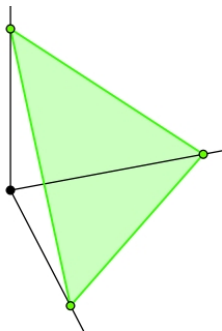
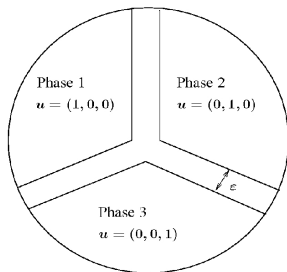
(1) Modica (1987)

(2) Allaire, Jouve, Toader (2004), Sokolowski & Zolesio (1992)

Phase field approach

Fix Notation

- phase field $\varphi := (\varphi^i)_{i=1}^N$; phase-fraction φ^i ($N-1$ materials); void φ^N ; $\varepsilon > 0$ interface width
- Gibbs-Simplex $\mathbf{G} = \{\mathbf{v} \in \mathbb{R}^N \mid v_i \geq 0, \sum_{i=1}^N v_i = 1\}$
- $\mathcal{G} := \{\mathbf{v} \in H^1(\Omega, \mathbb{R}^N) \mid \mathbf{v}(x) \in \mathbf{G} \text{ a.e. in } \Omega\}$
- $\mathcal{G}^m := \{\mathbf{v} \in \mathcal{G} \mid \int_{\Omega} \mathbf{v} = \mathbf{m}\}$



Phase field approach

Mechanical properties of the system

- linear elastic material
- with fourth order elasticity tensors $\mathbb{C}^i, i \in \{1, \dots, N-1\}$
- model **void** with a **very small elasticity tensor** $\mathbb{C}^N = \mathbb{C}^N(\varepsilon) = \varepsilon^2 \tilde{\mathbb{C}}^N$
- elasticity tensor in the interfacial region:

$$\mathbb{C}(\varphi) = \bar{\mathbb{C}}(\varphi) + \mathbb{C}^N(\varepsilon)\varphi^N, \quad \forall \varphi \in \mathbf{G}, \quad \bar{\mathbb{C}}(\varphi) := \sum_{i=1}^{N-1} \mathbb{C}^i \varphi^i$$

Phase field approach

We use a Ginzburg-Landau functional with obstacle type potential [Elliott, Luckhaus \(1999\)](#) to approximate the perimeter

$$E^\varepsilon(\varphi) := \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \varphi|^2 + \frac{1}{\varepsilon} \Psi(\varphi) \right), \quad \varepsilon > 0,$$

with $\Psi(\varphi) = \Psi_0(\varphi) + I_G(\varphi)$ and $\Psi_0(\varphi) = \frac{1}{2}(1 - \varphi \cdot \varphi)$.

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We take

$$\text{mean compliance} \quad J_1(\mathbf{u}, \varphi) := \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \mathbf{u} + \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{u}$$

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$$\text{compliant mechanism} \quad J_2(\mathbf{u}, \varphi) := \left(\int_{\Omega} c(x) (1 - \varphi^N) |\mathbf{u} - \mathbf{u}_{\Omega}|^2 \right)^{\frac{1}{2}}$$

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The objective functional is given by

$$J^\varepsilon(\mathbf{u}, \varphi) := \alpha J_1(\mathbf{u}, \varphi) + \beta J_2(\mathbf{u}, \varphi) + \gamma \hat{E}^\varepsilon(\varphi)$$

Optimal control problem

Similarities to optimal control problem, control: φ , state: $\mathbf{u}$

$$(\mathcal{P}^\varepsilon) \quad \left\{ \begin{array}{ll} \min & J^\varepsilon(\mathbf{u}, \varphi) = \alpha J_1(\mathbf{u}, \varphi) + \beta J_2(\mathbf{u}, \varphi) + \gamma \hat{E}^\varepsilon(\varphi), \\ \text{over} & (\mathbf{u}, \varphi) \in H_D^1(\Omega, \mathbb{R}^d) \times H^1(\Omega, \mathbb{R}^N), \\ \text{s.t.} & \langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\varphi)} = F(\boldsymbol{\eta}, \varphi) \quad \forall \boldsymbol{\eta} \in H_D^1(\Omega, \mathbb{R}^d) \\ & \varphi \in \mathcal{G}^m \cap U_c \end{array} \right.$$

Optimal control problem

Similarities to optimal control problem, control: φ , state: u

$$(\mathcal{P}^\varepsilon) \quad \begin{cases} \min & J^\varepsilon(u, \varphi) = \alpha J_1(u, \varphi) + \beta J_2(u, \varphi) + \gamma \hat{E}^\varepsilon(\varphi), \\ \text{over} & (u, \varphi) \in H_D^1(\Omega, \mathbb{R}^d) \times H^1(\Omega, \mathbb{R}^N), \\ \text{s.t.} & \langle \mathcal{E}(u), \mathcal{E}(\eta) \rangle_{\mathbb{C}(\varphi)} = F(\eta, \varphi) \quad \forall \eta \in H_D^1(\Omega, \mathbb{R}^d) \\ & \varphi \in \mathcal{G}^m \cap U_c \end{cases}$$

Here

$$H_D^1(\Omega, \mathbb{R}^d) := \{\eta \in H^1(\Omega, \mathbb{R}^d) \mid \eta = \mathbf{0} \text{ on } \Gamma_D\}$$

$$\langle \mathcal{E}(u), \mathcal{E}(v) \rangle_{\mathbb{C}} := \int_{\Omega} \mathcal{E}(u) : \mathbb{C} \mathcal{E}(v),$$

$$F(\eta, \varphi) := \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \eta + \int_{\Gamma_g} \mathbf{g} \cdot \eta$$

$$U_c := \{\varphi \in H^1(\Omega, \mathbb{R}^N) \mid \varphi^N = 0 \text{ a.e. on } S_0 \text{ and } \varphi^N = 1 \text{ a.e. on } S_1\}.$$

“Control-to-state” operator

Lemma:

$$S : L^\infty(\Omega, \mathbb{R}^N) \rightarrow H_D^1(\Omega, \mathbb{R}^d), \quad S(\varphi) := \mathbf{u}$$

The control-to-state operator $S : L^\infty(\Omega, \mathbb{R}^N) \rightarrow H_D^1(\Omega, \mathbb{R}^d)$ is Fréchet-differentiable.

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Its directional derivative at φ in the direction $\mathbf{h}$ is given by

$$S'(\varphi)\mathbf{h} = \mathbf{u}^*$$

where $\mathbf{u}^*$ denotes the unique solution of the problem

$$\langle \mathcal{E}(\mathbf{u}^*), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\varphi)} = -\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\varphi)\mathbf{h}} - \int_{\Omega} h^N \mathbf{f} \cdot \boldsymbol{\eta}, \quad \forall \boldsymbol{\eta} \in H_D^1(\Omega, \mathbb{R}^d)$$

Reduced functional

Lemma:

Set

$$j(\varphi) := J^\varepsilon(S(\varphi), \varphi) = \alpha J_1(S(\varphi), \varphi) + \beta J_2(S(\varphi), \varphi) + \gamma \hat{E}^\varepsilon(\varphi)$$

The reduced cost-functional $j : H^1(\Omega, \mathbb{R}^N) \cap L^\infty(\Omega, \mathbb{R}^N) \rightarrow \mathbb{R}$ is Fréchet-differentiable.

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First-order necessary optimality conditions

Let φ denote a minimizer of the problem $(\mathcal{P}^\varepsilon)$ and $\mathbf{u} = S(\varphi)$ be the associated state variable. $(\mathcal{P}^\varepsilon)$ can be reformulated to

$$\min_{\varphi \in \mathcal{G}^m \cap U_c} j(\varphi).$$

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$$\min_{\varphi \in \mathcal{G}^m \cap U_c} j(\varphi).$$

The following variational inequality is fulfilled:

$$j'(\varphi)(\tilde{\varphi} - \varphi) \geq 0 \quad \forall \tilde{\varphi} \in \mathcal{G}^m \cap U_c.$$

First order optimality conditions

Recall

$$J^\varepsilon(\mathbf{u}, \boldsymbol{\varphi}) = \alpha \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \mathbf{u} + \alpha \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{u} + \beta \left(\int_{\Omega} c(x) (1 - \varphi^N) |\mathbf{u} - \mathbf{u}_{\Omega}|^2 \right)^{\frac{1}{2}} + \gamma \int_{\Omega} \left(\frac{\varepsilon}{2} |\nabla \boldsymbol{\varphi}|^2 + \frac{1}{\varepsilon} \Psi_0(\boldsymbol{\varphi}) \right)$$

Introducing the adjoint equation

$$\begin{aligned} \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{q}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} &= \beta J_2(\mathbf{u}, \boldsymbol{\varphi})^{-1} \int_{\Omega} c(x) (1 - \varphi^N) (\mathbf{u} - \mathbf{u}_{\Omega}) \cdot \mathbf{q} \\ &\quad + \alpha \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \mathbf{q} + \alpha \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{q} \quad \forall \mathbf{q} \in H_D^1(\Omega, \mathbb{R}^d) \end{aligned}$$

we have

$$\begin{aligned} j'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) &= J_{\mathbf{u}}^\varepsilon(\mathbf{u}, \boldsymbol{\varphi}) \underbrace{\mathbf{u}^*}_{S'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi})} + J_{\boldsymbol{\varphi}}^\varepsilon(\mathbf{u}, \boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \\ &= \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{u}^*) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + J_{\boldsymbol{\varphi}}^\varepsilon(\mathbf{u}, \boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \end{aligned}$$

Recall that $\mathbf{u}^*$ satisfies $\langle \mathcal{E}(\mathbf{u}^*), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} = -\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})h} - \int_{\Omega} h^N \mathbf{f} \cdot \boldsymbol{\eta}$ we have

$$\left\{ \begin{aligned} &\gamma \varepsilon \int_{\Omega} \nabla \boldsymbol{\varphi} : \nabla (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\boldsymbol{\varphi}) \cdot (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) - \frac{\beta}{2} J_2(\mathbf{u}, \boldsymbol{\varphi})^{-1} \int_{\Omega} c(\tilde{\varphi}^N - \varphi^N) |\mathbf{u} - \mathbf{u}_{\Omega}|^2 \\ &\quad - \int_{\Omega} (\tilde{\varphi}^N - \varphi^N) \mathbf{f} \cdot (\alpha \mathbf{u} + \mathbf{p}) - \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})} (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \geq 0, \forall \tilde{\boldsymbol{\varphi}} \in \mathcal{G}^m \cap U_c. \end{aligned} \right.$$

Theorem (optimality system)

The functions $(\mathbf{u}, \boldsymbol{\varphi}, \mathbf{p}) \in H_D^1(\Omega, \mathbb{R}^d) \times (\mathcal{G}^m \cap U_c) \times H_D^1(\Omega, \mathbb{R}^d)$ fulfill the state equation

$$(SE) \quad \langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\boldsymbol{\eta}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} = \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \quad \forall \boldsymbol{\eta} \in H_D^1(\Omega, \mathbb{R}^d)$$

the adjoint equation (AE)

$$(AE) \quad \begin{cases} \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{q}) \rangle_{\mathbb{C}(\boldsymbol{\varphi})} &= \beta J_2(\mathbf{u}, \boldsymbol{\varphi})^{-1} \int_{\Omega} c(x) (1 - \varphi^N) (\mathbf{u} - \mathbf{u}_{\Omega}) \cdot \mathbf{q} \\ &+ \alpha \int_{\Omega} (1 - \varphi^N) \mathbf{f} \cdot \mathbf{q} + \alpha \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{q} \quad \forall \mathbf{q} \in H_D^1(\Omega, \mathbb{R}^d). \end{cases}$$

and the gradient inequality (GI)

$$(GI) \quad \begin{cases} \gamma \varepsilon \int_{\Omega} \nabla \boldsymbol{\varphi} : \nabla (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + \frac{\gamma}{\varepsilon} \int_{\Omega} \boldsymbol{\Psi}'_0(\boldsymbol{\varphi}) \cdot (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) - \frac{\beta}{2} J_2(\mathbf{u}, \boldsymbol{\varphi})^{-1} \int_{\Omega} c(\tilde{\varphi}^N - \varphi^N) |\mathbf{u} - \mathbf{u}_{\Omega}|^2 \\ \quad - \int_{\Omega} (\tilde{\varphi}^N - \varphi^N) \mathbf{f} \cdot (\alpha \mathbf{u} + \mathbf{p}) - \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})} (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) \geq 0, \\ \quad \forall \tilde{\boldsymbol{\varphi}} \in \mathcal{G}^m \cap U_c. \end{cases}$$

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- In the case $\beta = 0$ we have $\mathbf{p} = \alpha \mathbf{u}$ and there is no adjoint equation.

Gradient flow dynamics

Consider L^2 -gradient flow dynamics of the reduced cost functional $j(\boldsymbol{\varphi})$ (+ mass conservation)

$$\begin{aligned} \varepsilon \int_{\Omega} \boldsymbol{\varphi}_t \cdot (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + \gamma \varepsilon \int_{\Omega} \nabla \boldsymbol{\varphi} : \nabla (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\boldsymbol{\varphi}) \cdot (\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi}) - \frac{\beta}{2} J_2^{-1} \int_{\Omega} c(\tilde{\boldsymbol{\varphi}}^N - \boldsymbol{\varphi}^N) |\boldsymbol{u} - \boldsymbol{u}_{\Omega}|^2 \\ - \int_{\Omega} (\tilde{\boldsymbol{\varphi}}^N - \boldsymbol{\varphi}^N) \boldsymbol{f} \cdot (\alpha \boldsymbol{u} + \boldsymbol{p}) - \langle \mathcal{E}(\boldsymbol{p}), \mathcal{E}(\boldsymbol{u}) \rangle_{\mathbb{C}'(\boldsymbol{\varphi})(\tilde{\boldsymbol{\varphi}} - \boldsymbol{\varphi})} \geq 0 \quad \forall \tilde{\boldsymbol{\varphi}} \in \mathcal{G}^m \cap U_c. \end{aligned}$$

Gradient flow dynamics

Consider L^2 -gradient flow dynamics of the reduced cost functional $j(\varphi)$ (+ mass conservation)

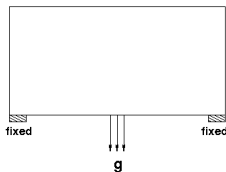
$$\varepsilon \int_{\Omega} \varphi_t \cdot (\tilde{\varphi} - \varphi) + \gamma \varepsilon \int_{\Omega} \nabla \varphi : \nabla (\tilde{\varphi} - \varphi) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\varphi) \cdot (\tilde{\varphi} - \varphi) - \frac{\beta}{2} J_2^{-1} \int_{\Omega} c(\tilde{\varphi}^N - \varphi^N) |u - u_{\Omega}|^2 \\ - \int_{\Omega} (\tilde{\varphi}^N - \varphi^N) f \cdot (\alpha u + p) - \langle \mathcal{E}(p), \mathcal{E}(u) \rangle_{\mathbb{C}'(\varphi)(\tilde{\varphi} - \varphi)} \geq 0 \quad \forall \tilde{\varphi} \in \mathcal{G}^m \cap U_c.$$

Numerical approximation

- We discretise the system using a standard finite element approximation
- Solve variational inequality with primal dual active set method by [Hintermüller, Ito, Kunisch \(2003\)](#)
- PDAS method analysed for Allen–Cahn systems by [Blank, Garcke, Sarbu, Styles \(2013\)](#)

Numerical simulations

Mean compliance



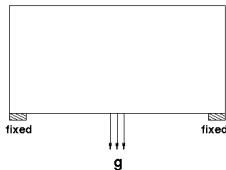
Michell type structure



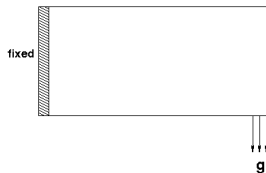
Cantilever beam configuration

Numerical simulations

Mean compliance

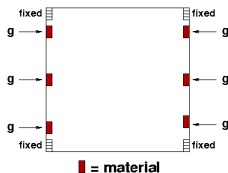


Michell type structure



Cantilever beam configuration

Minimize error to
given target
displacement $u_{\Omega} \equiv \mathbf{0}$



Push configuration

Michell type structure $d = 2, N = 2, \beta = 0, \mathbf{f} = \mathbf{0}$



$$J^\varepsilon(\mathbf{u}, \varphi) = \alpha J_1(\mathbf{u}, \varphi) + \gamma E^\varepsilon(\varphi)$$

$$\alpha = 1, \gamma = 1, \varepsilon = 0.02$$

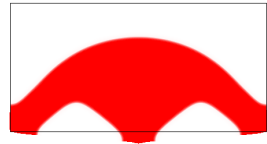
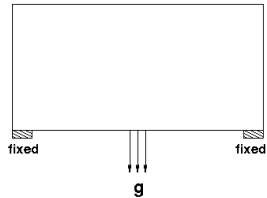
$$\mathbf{g} = (0, -1000)^T, \lambda_1 = \mu_1 = 1000$$

$$\varepsilon \int_{\Omega} \varphi_i \cdot (\tilde{\varphi} - \varphi) + \gamma \varepsilon \int_{\Omega} \nabla \varphi : \nabla (\tilde{\varphi} - \varphi) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\varphi) \cdot (\tilde{\varphi} - \varphi) - \alpha \langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\varphi)} (\tilde{\varphi} - \varphi) \geq 0$$

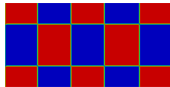
$$\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{v}) \rangle_{\mathbb{C}(\varphi)} = \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_D^1(\Omega, \mathbb{R}^d)$$

- Since $N = 2$, the vector valued VI can be replaced by a scalar VI for φ .
- No adjoint equation

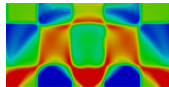
Michell type structure $d = 2, N = 2, \beta = 0, \mathbf{f} = \mathbf{0}$



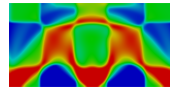
Deformed state



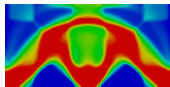
$t = 0.0$



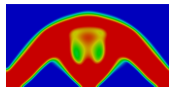
$t = 0.0002$



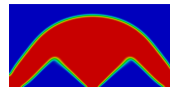
$t = 0.0003$



$t = 0.0005$



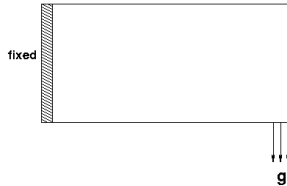
$t = 0.001$



$t = 0.006$

Computation starting with checker-board (red: material, blue: void)

Cantilever beam construction $d = 2, N = 3, \beta = 0, \mathbf{f} = \mathbf{0}$



$$J^\varepsilon(\mathbf{u}, \varphi) = \alpha J_1(\mathbf{u}, \varphi) + \gamma E^\varepsilon(\varphi)$$

$$\alpha = 1, \gamma = 1, \varepsilon = 0.02$$

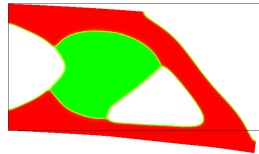
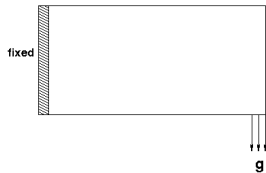
$$\mathbf{g} = (0, -250)^T, \lambda_1 = \mu_1 = 5000$$

$$\varepsilon \int_{\Omega} \varphi_i \cdot (\tilde{\varphi} - \varphi) + \gamma \varepsilon \int_{\Omega} \nabla \varphi : \nabla (\tilde{\varphi} - \varphi) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\varphi) \cdot (\tilde{\varphi} - \varphi) - \alpha \langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\varphi)} (\tilde{\varphi} - \varphi) \geq 0$$

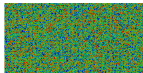
$$\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{v}) \rangle_{\mathbb{C}(\varphi)} = \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_D^1(\Omega, \mathbb{R}^d)$$

- No adjoint equation

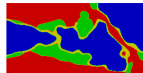
Cantilever beam construction $d = 2, N = 3, \beta = 0, \mathbf{f} = \mathbf{0}$



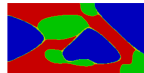
Deformed state



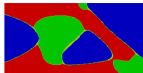
$t = 0.000$



$t = 0.0015$



$t = 0.01$



$t = 0.02$



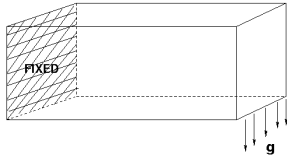
$t = 0.04$



$t = 0.3$

Computation starting with random initial data (red: hard material, green: soft material, blue: void)

Cantilever beam construction $d = 3, N = 2, \beta = 0, \mathbf{f} = \mathbf{0}$



$$J^\varepsilon(\mathbf{u}, \varphi) = \alpha J_1(\mathbf{u}, \varphi) + \gamma E^\varepsilon(\varphi)$$

$$\alpha = 1, \gamma = 1, \varepsilon = 0.02$$

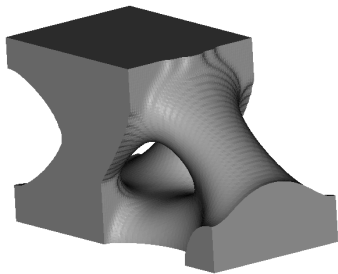
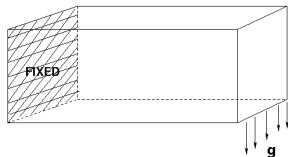
$$\mathbf{g} = (0, -165)^T, \lambda_1 = \mu_1 = 5000$$

$$\varepsilon \int_{\Omega} \varphi_t \cdot (\tilde{\varphi} - \varphi) + \gamma \varepsilon \int_{\Omega} \nabla \varphi : \nabla (\tilde{\varphi} - \varphi) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\varphi) \cdot (\tilde{\varphi} - \varphi) - \alpha \langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\varphi)} (\tilde{\varphi} - \varphi) \geq 0$$

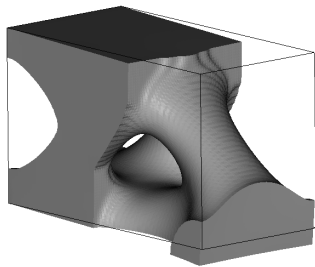
$$\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{v}) \rangle_{\mathbb{C}(\varphi)} = \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_D^1(\Omega, \mathbb{R}^d)$$

- Since $N = 2$, the vector valued VI can be replaced by a scalar VI for φ .
- No adjoint equation

Cantilever beam construction $d = 3, N = 2, \beta = 0, \mathbf{f} = \mathbf{0}$

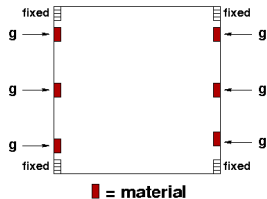


final state



deformed state

Push construction $d = 2, N = 2, \alpha = 0, \mathbf{f} = \mathbf{0}$



$$J^\varepsilon(\mathbf{u}, \varphi) = \beta J_2(\mathbf{u}, \varphi) + \gamma E^\varepsilon(\varphi)$$

$$\beta = 10, \gamma = 0.2, \varepsilon = 0.02$$

$$\mathbf{g} = (\pm 7, 0)^T, \lambda_1 = \mu_1 = 10$$

$$\varepsilon \int_{\Omega} \varphi_t \cdot (\tilde{\varphi} - \varphi) + \gamma \varepsilon \int_{\Omega} \nabla \varphi : \nabla (\tilde{\varphi} - \varphi) + \frac{\gamma}{\varepsilon} \int_{\Omega} \Psi'_0(\varphi) \cdot (\tilde{\varphi} - \varphi)$$

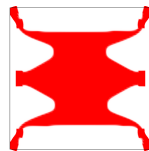
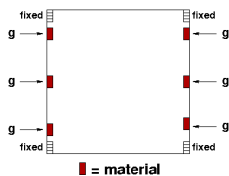
$$-\frac{\beta}{2} J_2^{-1} \int_{\Omega} c(\tilde{\varphi}^N - \varphi^N) |\mathbf{u} - \mathbf{u}_{\Omega}|^2 - \langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{u}) \rangle_{\mathbb{C}'(\varphi)(\tilde{\varphi} - \varphi)} \geq 0$$

$$\langle \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{v}) \rangle_{\mathbb{C}(\varphi)} = \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \quad \forall \mathbf{v} \in H_D^1(\Omega, \mathbb{R}^d)$$

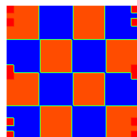
$$\langle \mathcal{E}(\mathbf{p}), \mathcal{E}(\mathbf{q}) \rangle_{\mathbb{C}(\varphi)} = \int_{\Omega} \beta J_0(\mathbf{u}, \varphi)^{-1} c(x) (1 - \varphi^N) (\mathbf{u} - \mathbf{u}_{\Omega}) \cdot \mathbf{q} + \alpha \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{q} \quad \forall \mathbf{q} \in H_D^1(\Omega, \mathbb{R}^d).$$

- Since $N = 2$, the vector valued VI can be replaced by a scalar VI for φ .

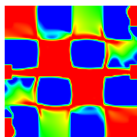
Push construction $d = 2, N = 2, \alpha = 0, f = 0$



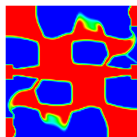
Deformed state



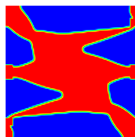
$t = 0.0$



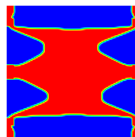
$t = 0.01$



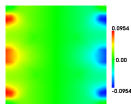
$t = 0.05$



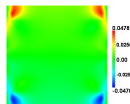
$t = 1.0$



$t = 2.6$



x component of $\mathbf{u}$



y component of $\mathbf{u}$