



Weierstrass Institute for
Applied Analysis and Stochastics



Local versus energetic solutions in rate-independent brittle delamination

Marita Thomas

(jointly with T. Roubíček & C. Panagiotopoulos)

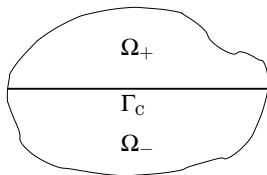
Rate-independent brittle delamination

Crack initiation and growth along prescribed interface Γ_c

Modeling ansatz: Generalized Standard Materials

$\rightsquigarrow$ for delamination [Frémond82,87].

delamination variable $z : [0, T] \times \Gamma_c \rightarrow [0, 1]$ volume fraction of *active* bonds

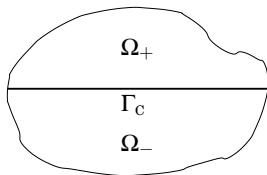


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Rate-independent system described by $(\mathcal{Q}, \mathcal{E}, \mathcal{R}_1)$

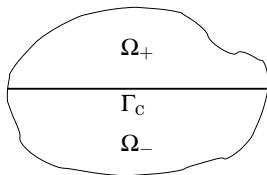
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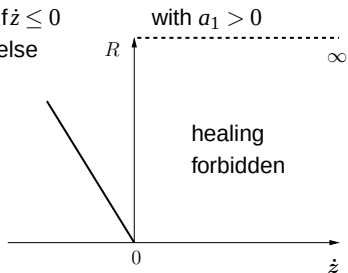
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Dissipation potential:

$$\mathcal{R}_1(\dot{z}) = \int_{\Gamma_C} R_1(\dot{z}(x)) ds \quad R_1(\dot{z}) := \begin{cases} a_1 |\dot{z}| & \text{if } \dot{z} \leq 0 \\ \infty & \text{else} \end{cases}$$

Rate-independence $\Leftrightarrow$ 1-homogeneity:

$$\mathcal{R}_1(0) = 0 \text{ and } \forall \lambda > 0 \forall v : \mathcal{R}_1(\lambda v) = \lambda \mathcal{R}_1(v)$$

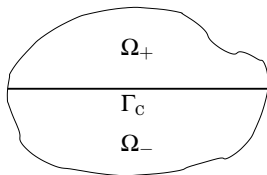


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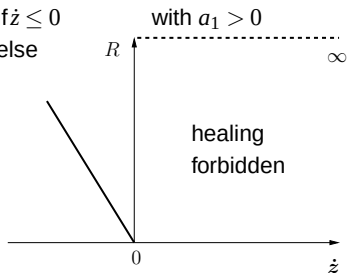
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$\Downarrow$

Dissipation distance: $\mathcal{D}_1(z_1, z_2) = \mathcal{R}_1(z_2 - z_1)$

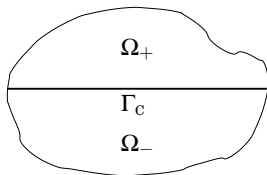


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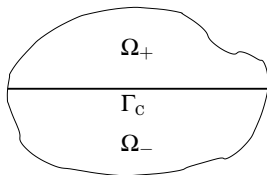
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energy functional: $\mathcal{E}(t, u, z) := \mathcal{E}^{\text{bulk}}(t, u) + \int_{\Gamma_C} I_{[z[[u]]=0]} ([[u(t)]], z(t)) + I_{[0,1]}(z) \, dx$

displacement $u : [0, T] \times (\Omega_+ \cup \Omega_-) \rightarrow \mathbb{R}^d$, $[[u]]$: jump of u across Γ_C

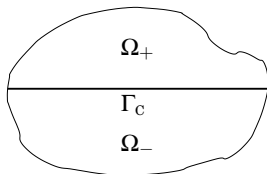
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Regularization of the **brittle constraint** by

adhesive contact: $\forall t \in [0, T] : \quad \underbrace{z(t) [[u(t)]] \neq 0}_{\text{penalized by energy term}} \text{ allowed on } \Gamma_C$

$$J_k([[u]], z) := \frac{k}{2} z |[[u]]|^2$$

Goal: Model for brittle delamination displaying driving force for crack growth

Ansatz: Approximation adhesive contact $\xrightarrow{\Gamma}$ brittle delamination

This result is known for *energetic* solutions [Roubíček/Scardia/Zanini09].

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But general drawback of energetic solutions: too early jumps

Thus alternative notion of solution: local solutions

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Plan of the talk:

1. 2 notions of solution for rate-independent systems: energetic and local solutions
2. 1D-comparison of their behavior for **adhesive** contact
3. Alternative scaling for the local **adhesive** model
Motivation as a stress-driven delamination model
4. Analytical results & challenges
5. Numerical experiment

1. Energetic & local solutions for rate-independent systems

Definition [Mielke&Co]: $q : [0, T] \rightarrow \mathcal{Q}$ is an **energetic solution** to $(\mathcal{Q}, \mathcal{E}, \mathcal{R}_1)$, if for all $t \in [0, T]$ it holds $\partial_t \mathcal{E}(\cdot, q(\cdot)) \in L^1((0, T))$, $\mathcal{E}(t, q(t)) < \infty$ and:

$$\begin{cases} \text{(S) Stability :} & \text{for all } \tilde{q} \in \mathcal{Q} : \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{R}_1(\tilde{z} - z(t)), \\ \text{(E) Energy balance :} & \mathcal{E}(t, q(t)) + \text{Diss}_{\mathcal{R}_1}(z, [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_t \mathcal{E}(\xi, q(\xi)) d\xi, \end{cases}$$

where $\text{Diss}_{\mathcal{R}_1}(z, [s, t]) := \sup_{\text{all part. of } [s, t]} \sum_{j=1}^N \mathcal{R}_1(z(\xi_j) - z(\xi_{j-1}))$.

Results in mathematical literature:

Rate-independent evolution of delamination (energetic solutions): e.g.

[Kočvara/Mielke/Roubíček06, Roubíček/Scardia/Zanini09, Freddi/Paroni/Roubíček/Zanini11,12, Mielke/Roubíček/Th12]

Rate-dependent evolution of delamination: e.g.

[Frémond82,87, Point88, Raous/Cangémi/Cocu99, Bonetti/Bonfanti/Rossi07,08,09,11,12]

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$$\begin{cases} \text{For a.a. } t \in [0, T] : \\ \quad (\mathcal{S}_{\text{loc}}^u) \text{ Minimality :} & \text{for all } \tilde{u} \in \mathcal{U} : \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, \tilde{u}, z(t)), \\ \quad (\mathcal{S}_{\text{loc}}^z) \text{ Semistability :} & \text{for all } \tilde{z} \in \mathcal{Z} : \mathcal{E}(t, u(t), z(t)) \leq \mathcal{E}(t, u(t), \tilde{z}) + \mathcal{R}_1(\tilde{z} - z(t)), \\ \text{For all } 0 \leq t_1 < t_2 \leq T : \\ \quad (\mathcal{E}_{\text{loc}}) \text{ Energy ineq. :} & \mathcal{E}(t_2, q(t_2)) + \text{Diss}_{\mathcal{R}_1}(z, [t_1, t_2]) \leq \mathcal{E}(t_1, q(t_1)) + \int_{t_1}^{t_2} \partial_t \mathcal{E}(\xi, q(\xi)) d\xi. \end{cases}$$

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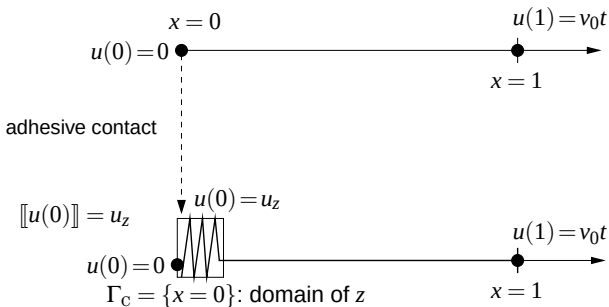
Hence: Energetic solutions are particular local solutions.

Ansatz: Approximation **adhesive** contact $\xrightarrow{\Gamma}$ **brittle** delamination

Thus:

2. 1D-comparison of energetic and local solutions for **adhesive** contact

3. 1D-comparison of energetic and local solutions for adhesive contact



$$\mathcal{E}_k(t, u, z) = \underbrace{\int_0^1 \frac{\mathbb{C}}{2} |d_x u|^2 dx}_{\mathcal{E}^{\text{bulk}}(t, u)} + \frac{k}{2} z u_z^2 \quad (1)$$

Minimizer of $\mathcal{E}^{\text{bulk}}(t, \cdot)$ & BCs: $u(x) = v_0 t x - u_z(x-1)$. $\Rightarrow$ (1) reduces to:

$$\tilde{\mathcal{E}}_k(t, u_z, z) = \frac{\mathbb{C}}{2} (v_0 t - u_z)^2 + \frac{k}{2} z u_z^2$$

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$$\tilde{\mathcal{E}}_k(t, u_z, z) = \frac{\mathbb{C}}{2} (v_0 t - u_z)^2 + \frac{k}{2} z u_z^2, \quad \mathcal{R}_1(\dot{z}) = \begin{cases} a_1 \dot{z} & \text{if } \dot{z} \leq 0, \\ \infty & \text{otherwise.} \end{cases} \quad (2)$$

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Energetic solutions:

$$\begin{aligned} \text{Stability (S): } \forall (\tilde{u}, \tilde{z}) \in \mathcal{U} \times \mathcal{Z}: \quad & \tilde{\mathcal{E}}_k(t, u_z, z) \leq \tilde{\mathcal{E}}_k(t, \tilde{u}, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z) \\ \Leftrightarrow \quad & (u_z, z) = \operatorname{argmin} \left(\frac{\mathbb{C}}{2}(v_0 t - \tilde{u})^2 + \frac{k}{2} \tilde{z} \tilde{u}^2 - a_1 \tilde{z} \right) \end{aligned}$$

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$$\text{Reduced energy: } J_k(t, z) = \frac{\mathbb{C}}{2} \left(\frac{v_0 t k z}{\mathbb{C} + k z} \right)^2 + \frac{k}{2} \left(\frac{v_0 t \mathbb{C}}{\mathbb{C} + k z} \right)^2$$

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For the unbroken material, $z = 1$, rupture occurs when $\mathcal{J}_k(t, 1) = a_1$.

$\Rightarrow$ Rupture time for energetic adhesive contact model:

$$t_{\text{ES}}^k = \sqrt{\frac{2a_1(\mathbb{C} + k)}{\mathbb{C} k v_0^2}}$$

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Local solutions:

$$\text{Minimality } (S_{\text{loc}}^u): \quad \forall \tilde{u} \in \mathcal{U}: \quad \tilde{\mathcal{E}}_k(t, u_z, z) \leq \tilde{\mathcal{E}}_k(t, \tilde{u}, z).$$

$$\text{Semistability } (S_{\text{loc}}^z): \quad \forall \tilde{z} \in \mathcal{Z}: \quad \tilde{\mathcal{E}}_k(t, u_z, z) \leq \tilde{\mathcal{E}}_k(t, u_z, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z).$$

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$$\Leftrightarrow \frac{k}{2} z \left(\frac{\mathbb{C} v_0 t}{\mathbb{C} + k z} \right)^2 - a_1 z \leq \frac{k}{2} \tilde{z} \left(\frac{\mathbb{C} v_0 t}{\mathbb{C} + k \tilde{z}} \right)^2 - a_1 \tilde{z}$$

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Local solutions:

Minimality (S_{loc}^u): $\forall \tilde{u} \in \mathcal{U}: \quad \tilde{\mathcal{E}}_k(t, u_z, z) \leq \tilde{\mathcal{E}}_k(t, \tilde{u}, z).$

$$\Leftrightarrow u_z = \operatorname{argmin} \left(\frac{\mathbb{C}}{2}(v_0 t - \tilde{u})^2 + \frac{k}{2} \tilde{z} \tilde{u}^2 - a \tilde{z} \right) = \frac{\mathbb{C} v_0 t}{\mathbb{C} + k z}.$$

Semistability (S_{loc}^z): $\forall \tilde{z} \in \mathcal{Z}: \quad \tilde{\mathcal{E}}_k(t, u_z, z) \leq \tilde{\mathcal{E}}_k(t, u_z, \tilde{z}) + \mathcal{R}_1(\tilde{z} - z).$

$$\Leftrightarrow \frac{k}{2} z \left(\frac{\mathbb{C} v_0 t}{\mathbb{C} + k z} \right)^2 - a_1 z \leq \frac{k}{2} \tilde{z} \left(\frac{\mathbb{C} v_0 t}{\mathbb{C} + k \tilde{z}} \right)^2 - a_1 \tilde{z}$$

For the unbroken material, $z = 1$, rupture occurs when $\frac{k}{2} \left(\frac{\mathbb{C} v_0 t}{\mathbb{C} + k} \right)^2 = a_1$.

$\Rightarrow$ Rupture time for local adhesive contact model:

$$t_{\text{LS}}^k = \frac{\mathbb{C} + k}{\mathbb{C} v_0} \sqrt{\frac{2a}{k}}.$$

3. 1D-comparison of energetic and local solutions for adhesive contact

Comparison of rupture times:

Energetic solution: $t_{\text{ES}}^k = \sqrt{\frac{2a_1(\mathbb{C}+k)}{\mathbb{C}kv_0^2}}$

Local solution: $t_{\text{LS}}^k = \frac{\mathbb{C}+k}{\mathbb{C}v_0} \sqrt{\frac{2a}{k}}$

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$$\text{Energetic solution: } t_{\text{ES}}^k = \sqrt{\frac{2a_1(\mathbb{C}+k)}{\mathbb{C}k v_0^2}} \xrightarrow{k \rightarrow \infty} t_{\text{ES}}^{\text{brittle}} = \sqrt{\frac{2a_1}{\mathbb{C}v_0^2}}$$

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No rupture in the local brittle limit model!

Alternative scaling needed to obtain rupture in the local model as $k \rightarrow \infty$!

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Ansatz: rescale $\mathcal{R}_1$ by $\frac{1}{k}$:

$$\tilde{\mathcal{E}}_k(t, u_z, z) = \frac{\mathbb{C}}{2}(v_0 t - u_z)^2 + \frac{k}{2} z u_z^2, \quad \mathcal{R}_1^k(\dot{z}) = \begin{cases} \frac{a_1}{k} \dot{z} & \text{if } \dot{z} \leq 0, \\ \infty & \text{otherwise.} \end{cases} \quad (2)$$

$\Rightarrow$ Rupture time of rescaled local adhesive model:

$$t_{\text{LS}^*}^k = \frac{\mathbb{C}+k}{\mathbb{C}v_0 k} \sqrt{2a_1} \xrightarrow{k \rightarrow \infty} t_{\text{LS}^*}^{\text{brittle}} = \frac{\sqrt{2a_1}}{\mathbb{C}v_0}.$$

3. Rescaled local adhesive model

$$\mathcal{E}_k(t, u, z) = \mathcal{E}^{\text{bulk}}(t, u) + \int_{\Gamma_C} \frac{k}{2} z |[[u]]|^2 + I_{[0,1]}(z) \, dx, \quad \mathcal{R}_1^k(v) = \frac{1}{k} \mathcal{R}_1(v) \quad (3)$$

Clearly $\frac{1}{k} \mathcal{R}_1(v) \rightarrow \mathcal{R}_1^\infty(v) = \begin{cases} 0 & \text{if } v \leq 0, \\ \infty & \text{otherwise.} \end{cases}$

Semistability for the brittle limit:

$$\int_{\Gamma_C} I_{[z[[u]]]}(z, [[u]]) + I_{[0,1]}(z) \, dx \leq \int_{\Gamma_C} I_{[\tilde{z}[[u]]]}(\tilde{z}, [[u]]) + I_{[0,1]}(\tilde{z}) \, dx + \mathcal{R}_1^\infty(\tilde{z} - z)$$

Meaning?! ...Where is the driving force for brittle delamination?!

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Semistability of the adhesive models, multiplied by k :

$$\int_{\Gamma_C} \frac{k^2}{2} z_k |\llbracket u_k \rrbracket|^2 \, dx \leq \underbrace{\int_{\Gamma_C} \frac{k^2}{2} \tilde{z} |\llbracket u_k \rrbracket|^2 \, dx}_{\text{adhesive}} + \mathcal{R}_1(\tilde{z} - z_k)$$

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Hence, if $(u_k, z_k) \rightsquigarrow (u, z)$ and $\sigma(u_k) \rightsquigarrow \sigma(u)$, then traction stress-driven evolution of delamination in the brittle limit.

3. Rescaled local models

Existence of local solutions for brittle delamination via

- adhesive contact approximation with the rescaled local models
- time-discrete scheme
- simultaneous limit, i.e. time-step size $\tau = \tau(k)$
- additional regularization for z
- additional nonpenetration property included: $\llbracket u \rrbracket \cdot \mathbf{n} \geq 0$ a.e. on Γ_C .

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Final functionals for adhesive model:

$$\begin{aligned}\mathcal{R}_1^k(v) &= \frac{1}{k} \mathcal{R}_1(v) \\ \mathcal{E}_k(t, u, z) &= \mathcal{E}^{\text{bulk}}(t, u) + \int_{\Gamma_C} \frac{k}{2} z |[[u]]|^2 + I_{[0,1]}(z) + I_{[[u]] \cdot n \geq 0}([u]) \, dx + \frac{b}{k} P(Z, \Gamma_C), \quad (4) \\ \mathcal{Z} &= \{\tilde{z} : \Gamma_C \rightarrow \{0, 1\}, \tilde{z} \text{ char. fct. of set } \tilde{Z} \text{ with finite perimeter } P(\tilde{Z}, \Gamma_C) < \infty\}\end{aligned}$$

Enforce that $z \in \{0, 1\}$ a.e. on Γ_C .

Assumptions on $\mathcal{E}^{\text{bulk}}$: strictly convex, coercive in $e(u)$, p -growth, $p \in (1, \infty)$, ext. load f
e.g. linear elasticity $\frac{1}{2} e(u) : \mathbb{C} : e(u) - f \cdot u$.

4. Analytical results & challenges:

- 4.a) Time-discrete version of the local model
- 4.b) Convergence result
- 4.c) Main challenge: limit passage in energy inequality & mechanical force balance via Mosco-convergence of the energy functionals

4.a) Time-discrete version of the local model

Nested partitions of $[0, T]$:

$$t_0^{N(k)} = 0 \quad \text{and for all } i \in \{1, \dots, N(k)\}: \quad t_i^{N(k)} = t_0^{N(k)} + \frac{i}{N(k)}.$$

Choose e.g. $N(k) = 2^k$.

Time-discrete scheme:

Given $(u_0, z_0) = (u(0), z(0))$, for all $k \in \mathbb{N}$, for all $i \in \{1, \dots, N(k)\}$, find

$$u_i^{N(k)} = \underset{u \in \mathcal{U}}{\operatorname{Argmin}} \mathcal{E}_k(t_i^{N(k)}, u, z_{i-1}^{N(k)}), \quad (5a)$$

$$z_i^{N(k)} \in \underset{z \in \mathcal{Z}}{\operatorname{argmin}} (\mathcal{E}_k(t_i^{N(k)}, u_i^{N(k)}, z) + \mathcal{R}_1^k(z - z_{i-1}^{N(k)})). \quad (5b)$$

Existence of $(u_i^{N(k)}, z_i^{N(k)})$ by direct method.

4.a) Time-discrete version of the local model

Piecewise constant interpolants:

$$\left. \begin{aligned} \bar{\mathcal{E}}_k(t, u, z) &:= \mathcal{E}_k(t_i^{N(k)}, u, z), \\ \bar{w}_k(t) &:= w_i^{N(k)} \quad \& \quad \underline{w}_k(t) := w_{i-1}^{N(k)}, \end{aligned} \right\} \text{ for } t \in [t_{i-1}^{N(k)}, t_i^{N(k)}), \quad (6)$$

where w stands for u, z or f .

Time-discrete version of the local formulation:

$$\text{For all } t \in [0, T]: \quad \partial_u \bar{\mathcal{E}}_k(t, \bar{u}_k(t), \bar{z}_k(t)) \ni 0, \quad (7a)$$

$$\text{for all } t \in [0, T] \forall \tilde{z} \in \mathcal{Z}: \quad \bar{\mathcal{E}}_k(t, \bar{u}_k(t), \bar{z}_k(t)) \leq \bar{\mathcal{E}}_k(t, \bar{u}_k(t), \tilde{z}) + \mathcal{R}_k(\tilde{z} - \bar{z}_k(t)), \quad (7b)$$

$$\text{for all } t_1 \in (t_{l-1}^{N(k)}, t_l^{N(k)}] \text{ and } t_2 \in (t_{m-1}^{N(k)}, t_m^{N(k)}] \text{ with } l < m \in \{1, \dots, N(k)\}: \quad (7c)$$

$$\bar{\mathcal{E}}_k(t_2, \bar{u}_k(t_2), \bar{z}_k(t_2)) + \text{Diss}_{\mathcal{R}_1^k}(\bar{z}_k; [t_1, t_2]) \leq \bar{\mathcal{E}}_k(t_1, \bar{u}_k(t_1), \bar{z}_k(t_1)) + \int_{t_l^{N(k)}}^{t_m^{N(k)}} \partial_t \mathcal{E}(t, \underline{u}_k(t), \underline{z}_k(t)) dt.$$

4.b) Convergence result

Proposition ($\exists$ time-discr. local sol. & a priori bounds): Let $k \in \mathbb{N}$ fixed, $\Omega_{\pm}, \Gamma_c$ Lipschitz, $(\mathcal{U} \times \mathcal{Z}, \mathcal{E}_k, \mathcal{Z}_k)$ as in (4), given data suff. smooth.

Then, $(\bar{u}_k, \underline{u}_k, \bar{z}_k, \underline{z}_k)$ satisfy the time-discrete local formulation and the a priori bounds:

$$\text{For all } t \in [0, T]: \quad \|u_k(t)\|_{\mathcal{U}} \leq C \quad \text{and} \quad P(Z_k(t), \Gamma_c) \leq C \quad (8a)$$

$$\text{For all } t < t_* \in [0, T]: \quad \mathcal{R}_1(z_k(t) - z_k(t_*)) \leq C, \quad (8b)$$

where z_k is char. fct. of Z_k , $u_k \in \{\bar{u}_k, \underline{u}_k\}$, $z_k \in \{\bar{z}_k, \underline{z}_k\}$.

$$\mathcal{R}_1^k(v) = \frac{1}{k} \mathcal{R}_1(v) \quad (4)$$

$$\mathcal{E}_k(t, u, z) = \mathcal{E}^{\text{bulk}}(t, u) + \int_{\Gamma_c} \frac{k}{2} z |\llbracket u \rrbracket|^2 + I_{[0,1]}(z) + I_{[\llbracket u \rrbracket \cdot n \geq 0]}(\llbracket u \rrbracket) \, dx + \frac{b}{k} P(Z, \Gamma_c),$$

and $\mathcal{E}^{\text{bulk}}$: strictly convex, coercive in $e(u)$, p -growth, $p \in (1, \infty) \rightsquigarrow \mathcal{U} = W^{1,p}(\Omega \setminus \Gamma_c, \mathbb{R}^d)$.

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Observe: $\|u_k\|_{\mathcal{U}} \leq C$ by class. arguments,

$P(Z_k(t), \Gamma_c) \leq C$ by testing minimality of $Z_i^{N(k)}$ with $\tilde{z} = 0$.

Bound for z_k : $\text{Diss}_{\mathcal{R}_1^k}(z_k, [0, t_m^{N(k)}]) \leq C \Rightarrow z_k \in \text{BV}((0, T), L^1(\Gamma_c))$ & mon. $\searrow$ in time

$$\Rightarrow \text{Diss}_{\mathcal{R}_1}(z_k, [0, t_m^{N(k)}]) \leq a_1 \mathcal{L}^{d-1}(\Gamma_c).$$

4.b) Convergence result

Proposition ($\exists$ time-discr. local sol. & a priori bounds): Let $k \in \mathbb{N}$ fixed, $\Omega_{\pm}, \Gamma_c$ Lipschitz, $(\mathcal{U} \times \mathcal{Z}, \mathcal{E}_k, \mathcal{Z}_k)$ as in (4), given data suff. smooth.

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Theorem (Convergence result): Assumptions as above. Then, for $k \rightarrow \infty$:

$$\text{for all } t \in [0, T]: \quad \bar{z}_k(t) \xrightarrow{*} z(t), \quad \underline{z}_k(t) \xrightarrow{*} \underline{z}(t) \text{ in } \mathcal{Z}_{\text{SBV}} \text{ and } \bar{u}_k(t) \rightharpoonup u(t) \text{ in } \mathcal{U}, \quad (9a)$$

$$\text{for a.a. } t \in [0, T]: \quad z(t) = \underline{z}(t) \quad \text{and} \quad \underline{u}_k(t) \rightharpoonup u(t) \text{ in } \mathcal{U} \quad (9b)$$

and (u, z) is a local solution of the brittle limit system $(\mathcal{U} \times L^\infty(\Gamma_c), \mathcal{E}_\infty, \mathcal{R}_\infty)$.

$$\mathcal{E}_\infty(t, q) = \begin{cases} \mathcal{E}^{\text{bulk}}(t, u) + \mathcal{E}_{\text{surf}}^{\text{brittle}}(q) & \text{if } q \in \mathcal{U} \times L^\infty(\Gamma_c), \\ \infty & \text{otw.,} \end{cases} \quad \mathcal{R}_\infty(v) = \begin{cases} 0 & \text{if } v \leq 0 \text{ a.e.,} \\ \infty & \text{otw.} \end{cases}$$

$$\text{with } \mathcal{E}_{\text{surf}}^{\text{brittle}}(z, u) = \int_{\Gamma_c} I_{[\![u]\!]_{\text{supp } z}=0}(z, [\![u]\!]) + I_{[\![u]\!] \cdot n \geq 0}([\![u]\!]) + I_{[0,1]}(z) \, dx.$$

4.b) Convergence result

Theorem (Convergence result): Assumptions as above. Then, for $k \rightarrow \infty$:

$$\text{for all } t \in [0, T]: \bar{z}_k(t) \overset{*}{\rightharpoonup} z(t), \underline{z}_k(t) \overset{*}{\rightharpoonup} \underline{z}(t) \text{ in } \mathcal{Z}_{\text{SBV}} \text{ and } \bar{u}_k(t) \rightharpoonup u(t) \text{ in } \mathcal{U}, \quad (10a)$$

$$\text{for a.a. } t \in [0, T]: z(t) = \underline{z}(t) \text{ and } \underline{u}_k(t) \rightharpoonup u(t) \text{ in } \mathcal{U} \quad (10b)$$

and (u, z) is a local solution of the brittle limit system $(\mathcal{U} \times L^\infty(\Gamma_C), \mathcal{E}_\infty, \mathcal{R}_\infty)$.

Comments on the proof:

- Limit passage in semistability ineq.: Adaption of construction in [Th12:DCDS-S, Rossi/Th12].
- Difficulties in energy ineq. & mech. force balance since **strong convergence** needed!

for all $t_1 \in (t_{l-1}^{N(k)}, t_l^{N(k)})$ and $t_2 \in (t_{m-1}^{N(k)}, t_m^{N(k)})$ with $l < m \in \{1, \dots, N(k)\}$:

$$\bar{\mathcal{E}}_k(t_2, \bar{u}_k(t_2), \bar{z}_k(t_2)) + \text{Diss}_{\mathcal{R}_1^k}(\bar{z}_k; [t_1, t_2]) \leq \bar{\mathcal{E}}_k(t_1, \bar{u}_k(t_1), \bar{z}_k(t_1)) + \int_{t_l^{N(k)}}^{t_m^{N(k)}} \partial_t \mathcal{E}(t, \underline{u}_k(t), \underline{z}_k(t)) dt.$$

4.b) Convergence result

Theorem (Convergence result): Assumptions as above. Then, for $k \rightarrow \infty$:

$$\text{for all } t \in [0, T]: \bar{z}_k(t) \overset{*}{\rightharpoonup} z(t), \underline{z}_k(t) \overset{*}{\rightharpoonup} \underline{z}(t) \text{ in } \mathcal{Z}_{\text{SBV}} \text{ and } \bar{u}_k(t) \rightharpoonup u(t) \text{ in } \mathcal{U}, \quad (10a)$$

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Γ -convergence is not enough! We need **Mosco**-convergence of $(\tilde{\mathcal{E}}_k(t, \cdot, z_k))_k$!!!

where $\tilde{\mathcal{E}}_k(t, \tilde{u}, z_k) := \mathcal{E}^{\text{bulk}}(t, \tilde{u}) + \int_{\Gamma_C} \frac{k}{2} z_k |\llbracket \tilde{u} \rrbracket|^2 + I_{[\llbracket \tilde{u} \rrbracket \cdot n \geq 0]}(\llbracket \tilde{u} \rrbracket) dx$

$\Rightarrow$ **G**-convergence of $(\partial \tilde{\mathcal{E}}_k(t, \cdot, z_k))_k \Rightarrow$ convergence in mech. force balance $\checkmark$

4.c) Mosco-convergence of the functionals $(\tilde{\mathcal{E}}_k(t, \cdot, z_k))_k$

WANTED: $\tilde{\mathcal{E}}_k(t, \tilde{u}, z_k) \xrightarrow{\text{Mosco}} \mathcal{E}(t, \cdot, z)$, for z_k semistable wrt. $\mathcal{E}_k$ and $z_k \xrightarrow{*} z$ in $\text{SBV}(\Gamma_C, \{0, 1\})$.

i.e.: for all $v \in \mathcal{U}$ with $v|_{\text{supp } z} = 0$ a.e. construct recovery sequence $(v_k)_k$ s.th.:

$$v_k \rightarrow v \text{ strongly in } \mathcal{U} \quad \text{and} \quad \int_{\Gamma_C} \frac{k}{2} z_k |\llbracket v_k \rrbracket|^2 dx \xrightarrow{!} 0 \quad \text{for all } k \in \mathbb{N}.$$

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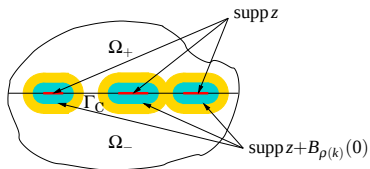
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Tool: Support convergence: [Rossi/Th.12]

$\text{supp } z_k \subset \text{supp } z + B_{\rho(k)}(0)$ for all $k \in \mathbb{N}$ and $\rho(k) \rightarrow 0$ as $k \rightarrow \infty$



$v_k \in W^{1,p}(\Omega \setminus \Gamma_C, \mathbb{R}^d)$ s.th.

$v_k = 0$ in $\text{supp } z + B_{\rho(k)}(0)$,

$v_k = v$ in $\Omega \setminus \text{supp } z + B_{2\rho(k)}(0)$

$\llbracket v_k \rrbracket \cdot n \geq 0$ on Γ_C

[Mielke/Roubíček/Th10]: Let $\rho(k) \rightarrow 0$ and Hardy's inequality hold true.

Then, $v_k \rightarrow v$ strongly in $W^{1,p}(\Omega \setminus \Gamma_C, \mathbb{R}^d)$.

[Lewis88]: Hardy's inequality for bad closed sets $M(=\text{supp } z) \subset \partial\Omega_{\pm}$ true for $p > d$.

$$\exists C_M > 0 \forall v \in W^{1,p}(\Omega_{\pm}, \mathbb{R}^d): \quad \|v/d_M\|_{L^p(\Omega_{\pm}, \mathbb{R}^d)} \leq C_M \|\nabla v\|_{L^p(\Omega_{\pm}, \mathbb{R}^{d \times d})}$$

4.c) Mosco-convergence of the functionals $(\tilde{\mathcal{E}}_k(t, \cdot, z_k))_k$

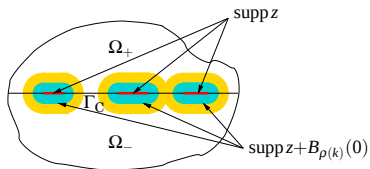
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[Haller-Dintelmann/Kaiser/Rehberg13]: Hardy's inequality for $p \in (1, \infty)$

for closed sets $M \subset \partial\Omega_{\pm}$ with **Property a**.

4.c) Property α & support convergence

[Campanato63,64]: A (open) set $M \subset \mathbb{R}^m$ has **Property α** iff there exists $\alpha > 0$ s.th.

$$\forall y \in M \forall \rho_\star > 0 : \mathcal{L}^m(M \cap B_{\rho_\star}(y)) \geq \alpha \rho_\star^m.$$

Show **Property α** for semistable z : [Rossi/Th.12]

Let $\mathcal{Z} := \{z : \Gamma_c \rightarrow \{0, 1\} \text{ is char. fct. of set } Z \text{ with } P(Z, \Gamma_c) < \infty\}$, $\Gamma_c \subset \mathbb{R}^m$, $m = d - 1$

Consider: $\mathcal{E}(t, u, z) + bP(Z, \Gamma_c)$, $\mathcal{E}(t, u, \cdot)$ monotonously increasing, z semistable

$$\begin{array}{c} \downarrow \\ \mathcal{E}(t, u, z) + bP(Z, \Omega) \leq \mathcal{E}(t, u, \tilde{z}) + bP(\tilde{Z}, \Omega) + \mathcal{R}(\tilde{z} - z) \quad \forall \tilde{z} \leq z \end{array}$$

$$\begin{array}{c} \Downarrow \\ bP(Z, \Omega) \leq bP(\tilde{Z}, \Omega) + a_1 \mathcal{L}^d(Z \setminus \tilde{Z}) \quad \forall \tilde{Z} \subset Z \quad (\star) \end{array}$$

$$\Downarrow$$

Property α : Let Z stable as in $(\star)$. Then, $\exists R, \alpha(\Gamma_c) > 0$ depending on Γ_c , m , b and a_1 , s.th.

$$\forall y \in \text{supp } z \forall \rho_\star \in (0, R) : \mathcal{L}^m(Z \cap B_{\rho_\star}(y)) \geq \alpha(\Gamma_c) \rho_\star^m, \quad (11a)$$

$$\forall y \in \text{supp } z \quad \forall \rho_\star \geq R : \mathcal{L}^m(Z \cap B_{\rho_\star}(y)) \geq \alpha(\Gamma_c) R^m, \quad (11b)$$

4.c) Property α & support convergence

Property α : Let Z stable as in $(\star)$. Then, $\exists R, \alpha(\Gamma_c) > 0$ depending on Γ_c, m, b and a_1 , s.th.

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Support convergence of $(Z_k)_k$ stable as in $(\star)$, $z_k \rightarrow z$ in $L^1(\Gamma_c)$:

$\text{supp } z_k \subset \text{supp } z + B_{\rho(k)}(0)$ for all $k \in \mathbb{N}$ and $\rho(k) \rightarrow 0$ as $k \rightarrow \infty$

$\Rightarrow$ construction of recovery sequence $(v_k)_k$ s.th.:

$$v_k \rightarrow v \text{ strongly in } \mathcal{U} \quad \text{and} \quad \int_{\Gamma_c} \frac{k}{2} z_k |v_k|^2 dx \stackrel{!}{=} 0 \quad \text{for all } k \in \mathbb{N}. \quad \checkmark$$

4.c) Property α & support convergence

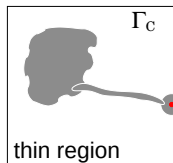
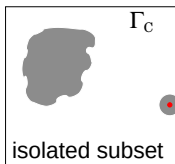
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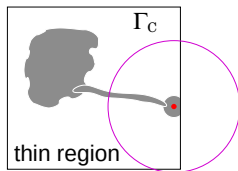
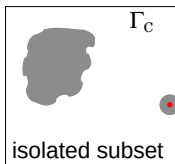
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Isolated subsets and thin regions of very small volume excluded by $(\star)$:



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Exclude isolated subsets by $(\star)$:

Tool: Isoperimetric inequality relative to Γ_C .

Let Z satisfy $(\star)$, z charact. fct. of Z , $A \subset Z$ isolated.

Test $(\star)$ with $\tilde{z}$, charact. fct. of $Z \setminus A$:

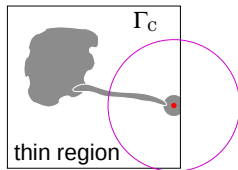
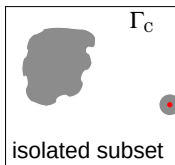
$$bP(Z, \Gamma_C) \leq bP(Z \setminus A, \Gamma_C) + \mathcal{R}(\tilde{z} - z)$$

$$\Rightarrow bP(A, \Gamma_C) \leq a_1 \mathcal{L}^m(A)$$

$$\frac{b}{a_1} \leq \frac{\mathcal{L}^m(A)}{P(A, \Gamma_C)} \leq c_{\Gamma_C} \mathcal{L}^m(A)^{1/m}$$

isoperimetric inequality

Isolated subsets and thin regions of very small volume excluded by $(\star)$:



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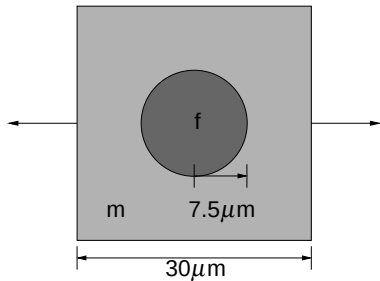
isoperimetric inequality

Exclude general thin subsets by $(\star)$:

[Th13]

Tool: Relative isoperimetric inequality for $\Gamma_C \cap B_r(y)$ with unif. const. for all $y \in \Gamma_C$, $r > 0$.

5. Numerical experiment – setup



Material	Young's modulus	Poisson's ratio
Glass	$E_f=70.8\text{ GPa}$	$\nu_f=0.22$
Epoxy	$E_m=2.79\text{ GPa}$	$\nu_m=0.33$

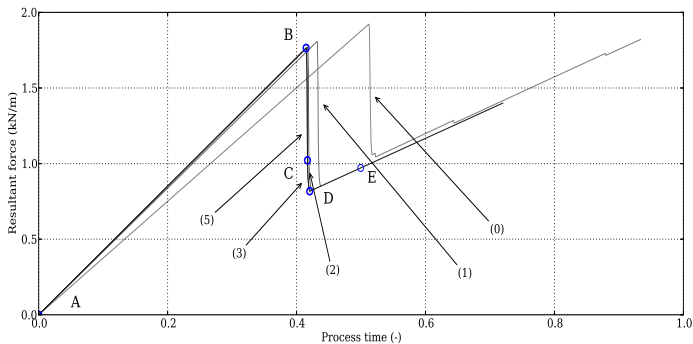
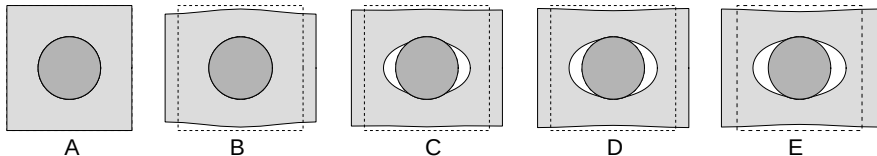
$$k_t := k_t^0 k, \quad k_t^0 := 2025\text{ TPa/m},$$

$$k_n := k_n^0 k, \quad k_n^0 := k_t^0/3$$

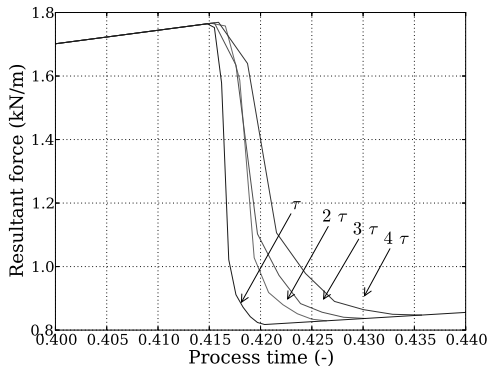
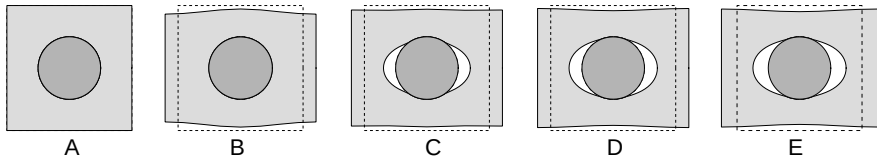
plane strain: 2D

$$\int_{\Omega \setminus \Gamma_C} \frac{1}{2} e(u+g(t)) : \mathbb{C} : e(u+g(t)) \, dx + \int_{\Gamma_C} \frac{1}{2} (k_t z | \llbracket u \rrbracket \cdot t |^2 + k_n z | \llbracket u \rrbracket \cdot n |^2) \, dx$$

g mon. ↗ in time.



increasing stiffness $k_n = 2025 \times 10^i$ TPa/m with $i = 0, 1, 2, 3, 4, 5$, while $k_t = \text{const.}$



$\tau = 35 \times 10^{-5}$, k_n , k_t fixed.

- **adhesive** contact approximation of **brittle** delamination in local formulation well-suited for simulations
- but scaling of dissipation by $1/k$ needed, otherwise no rupture
- the rescaled model can be interpreted as a stress-driven delamination model
- Hope: this property carries over from **adhesive** contact to **brittle** delamination
- analytical challenge: limit passage in energy inequality & weak mechanical force balance:
Mosco-convergence of energy functionals needed!
- this requires additional information on fine properties of semistable sets:
 - regularity **Property α**
 - results from semistability wrt. perimeter & unidir. 1-hom. dissipation
 - minimal regularity requirement for Hardy's inequality with $p \in (1, \infty)$
 - **support convergence** (implied by **Property α**)

Thank You!