

DIMO-2013
Diffuse Interface Models

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**A PDE system related to solid-solid phase transition
in shape memory alloys**

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1 - Plan of the talk

Presentation of a joint work with

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- **Mauro Fabrizio** (Bologna)

Subject: an existence result for a complicated PDE system.

3D model close to the 1D model introduced by
Berti, Fabrizio, Grandi, Phys. D 2010

2 - The equations

The mathematical model

- $\Omega = \text{body} \subset \mathbb{R}^3$ and $\Gamma = \text{boundary of } \Omega$
 - Ω open, bounded, connected, smooth
 - ϑ = absolute temperature
 - u = displacement (a scalar quantity)
 - σ = stress (a vector)
 - χ = order parameter
- (1) energy balance for the evolution of ϑ
 - (2) momentum balance for u and σ accounting for acceleration
+ stress-strain relationship between σ and ∇u
 - (3) Ginzburg-Landau type equation for χ
- proper IC (initial cond's) and BC (boundary cond's)

3 - Structure of the system

The PDE system

$$\begin{aligned} & (c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ & = R_\Omega + |\partial_t \chi|^2 \quad \text{and} \quad \vartheta > 0 \\ & \partial_t^2 u - \operatorname{div} \sigma = B_\Omega \quad \text{and} \quad \sigma = \kappa \nabla u - \gamma(\chi) \mathbf{e} \\ & \partial_t \chi - \Delta \chi + \vartheta_c F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot \mathbf{e} \gamma'(\chi) \underset{(\text{or } \ni)}{=} 0 \end{aligned}$$

- c_0 , κ and ϑ_c positive constants; $\mathbf{e}$ = fixed unit vector
- $\alpha : (0, +\infty) \rightarrow \mathbb{R}$ bounded ... with $\alpha(0^+) = 0$ (**crucial**)
- $G : \mathbb{R} \rightarrow \mathbb{R}$ nonnegative bounded ... with $G(0) = 0$
- $\gamma(\chi) := G(\chi) \operatorname{sign} \chi$
- F = general “double” well potential (“=” $\rightarrow$ “ $\ni$ ” in 3rd eq’n)
- **parabolicity**: $c_0 - \vartheta \alpha''(\vartheta) G(\chi) \geq \delta_0 > 0 \quad \forall \vartheta, \chi$

4 - Examples of nonlinearities

- An example of a non-trivial α (bounded, etc)

$$\alpha(\vartheta) = \alpha_0 \tanh \vartheta, \quad \alpha_0 > 0 \quad (\text{parabolicity for } \alpha_0 \text{ small})$$

- An example of G (nonnegative, bounded, etc) and γ

$$G(\chi) = \frac{\chi^2}{\chi^2 + 1} \quad \text{and} \quad \gamma(\chi) := G(\chi) \operatorname{sign} \chi = \frac{\chi|\chi|}{\chi^2 + 1}$$

- Examples of $F = \text{convex} + (\text{e.g., concave}) \text{ perturbation}$

$$F(\chi) = (1 - \chi^2)^2, \quad D(F) = \mathbb{R}, \quad \text{classical smooth dwp}$$

$$F(\chi) = \chi \ln \chi + (1 - \chi) \ln(1 - \chi) + C \chi(1 - \chi)$$

$$C \gg 1, \quad D(F) = [0, 1], \quad \text{logarithmic dwp}$$

$$F(\chi) = \frac{1}{1 - \chi^2} - C \chi^4, \quad C \gg 1, \quad D(F) = (-1, 1)$$

a more singular triple well potential

$$F(\chi) = \text{Indicator}_{[-1,1]}(\chi) - \chi^2, \quad D(F) = [-1, 1]$$

a very singular double well potential

5 - Source terms and IC + BC

Recall

$$(\mathbf{c}_0 - \vartheta \alpha''(\vartheta) \mathbf{G}(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) \mathbf{G}'(\chi) \partial_t \chi - \Delta \vartheta \\ = \mathbf{R}_\Omega + |\partial_t \chi|^2$$

$$\partial_t^2 \mathbf{u} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{B}_\Omega$$

$$\boldsymbol{\sigma} = \kappa \nabla \mathbf{u} - \gamma(\chi) \mathbf{e}$$

$$\partial_t \chi - \Delta \chi + \vartheta_c \mathbf{F}'(\chi) + \alpha(\vartheta) \mathbf{G}'(\chi) - \boldsymbol{\sigma} \cdot \mathbf{e} \gamma'(\chi) = 0$$

- $\mathbf{R}_\Omega$ and $\mathbf{B}_\Omega$ given source terms, $\mathbf{R}_\Omega \geq 0$ (**crucial**)
- Initial conditions

$$\vartheta(0) = \vartheta_0 > 0$$

$$\mathbf{u}(0) = \mathbf{u}_0 \quad \text{and} \quad \partial_t \mathbf{u}(0) = \mathbf{u}'_0$$

$$\chi(0) = \chi_0$$

- Boundary conditions

$$\partial_n \vartheta = 0, \quad \boldsymbol{\sigma} \cdot \mathbf{n} = \mathbf{b}_\Gamma = \text{source term} \quad \text{and} \quad \partial_n \chi = 0$$

6 - Existence

Provided that everything is made precise, we have

Theorem (BCFG). *There exists a solution in a proper space.*

Next slides:

- sketch of the proof

7 - Strategy for existence

Recall (+ IC)

$$(1) \quad (c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ = R_\Omega + |\partial_t \chi|^2, \quad \vartheta > 0 \quad \text{and} \quad \text{BC}$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B_\Omega \quad \text{and} \quad \sigma \cdot n = b_\Gamma, \quad \sigma = \kappa \nabla u - \gamma(\chi) e$$

$$(3) \quad \partial_t \chi - \Delta \chi + \vartheta_c F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot e \gamma'(\chi) = 0 \quad + \text{BC}$$

- Main difficulty (among others, like $\vartheta > 0$): $|\partial_t \chi|^2$ in (1)
whence, an ε -problem

-
- Regularize the nonlinearities α , G , γ and F and the initial data
 - Add a higher order term in (3) that controls $|\partial_t \chi|^2$
 - Just require $\vartheta \geq 0$ and modify the form of (1) a little

Moreover, replace (2) by simpler eq'n + BC.....

8 - Strategy for existence (cont'd)

Recall (+ IC)

$$(1) \quad (c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ = R_\Omega + |\partial_t \chi|^2, \quad \vartheta > 0 \quad \text{and} \quad \text{BC}$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B_\Omega \quad \text{and} \quad \sigma \cdot n = b_\Gamma, \quad \sigma = \kappa \nabla u - \gamma(\chi) e$$

$$(3) \quad \partial_t \chi - \Delta \chi + \vartheta F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot e \gamma'(\chi) = 0 \quad + \text{BC}$$

..... moreover, replace (2) by..... Here is how

- Write $(2_1) + \text{BC}$ (a.e. in $(0, T)$) as: for every $v \in V := H^1(\Omega)$

$$(2'_1) \quad \int_\Omega \partial_t^2 u v + \int_\Omega \sigma \cdot \nabla v = \langle B, v \rangle := \int_\Omega B_\Omega v + \int_\Gamma b_\Gamma v$$

- Regularize $B : (0, T) \rightarrow V^*$ by $B_\varepsilon : (0, T) \rightarrow H := L^2(\Omega)$.
(the concentrated b_Γ spreads to the whole of Ω)
- This means that $(2_1) + \text{BC}$ is replaced by

$$\partial_t^2 u - \operatorname{div} \sigma = B_\varepsilon \quad \text{and} \quad \sigma \cdot n = 0$$

9 - The ε -problem

We write ϑ etc instead of ϑ_ε etc

$$\begin{aligned}(1) \quad & (c_0 - (\vartheta + \varepsilon)\alpha_\varepsilon''(\vartheta)G_\varepsilon(\chi))\partial_t \vartheta \\ & - (\vartheta + \varepsilon)\alpha_\varepsilon'(\vartheta)G_\varepsilon'(\chi)\partial_t \chi - \Delta \vartheta \\ & = R_\Omega + |\partial_t \chi|^2, \quad \vartheta \geq 0 \quad \text{and} \quad \text{BC}\end{aligned}$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B_\varepsilon \quad \text{and} \quad \sigma \cdot \mathbf{n} = 0, \quad \sigma = \kappa \nabla u - \gamma_\varepsilon(\chi) \mathbf{e}$$

$$\begin{aligned}(3) \quad & \partial_t \chi - \Delta \chi - \varepsilon \partial_t \Delta \chi \quad (\leftarrow \text{higher order perturbation}) \\ & + \vartheta_c F_\varepsilon'(\chi) + \alpha_\varepsilon(\vartheta)G_\varepsilon'(\chi) - \sigma \cdot \mathbf{e} \gamma_\varepsilon'(\chi) = 0 \quad + \text{BC}\end{aligned}$$

◇ **Good point:**

as $\vartheta \geq 0$, $1/(\vartheta + \varepsilon)$ = good test fnc't'n and modified (1) helps

♡ **We would like:**

boundedness, parabolicity etc to be uniform with respect to ε

♡ **We would like:**

to have $\vartheta \geq 0$ only a posteriori (by maximum principle)

10 - The ε -problem (cont'd)

Recall

$$\begin{aligned}(1) \quad & (c_0 - (\vartheta + \varepsilon)\alpha_\varepsilon''(\vartheta)G_\varepsilon(\chi))\partial_t\vartheta \\ & - (\vartheta + \varepsilon)\alpha_\varepsilon'(\vartheta)G_\varepsilon'(\chi)\partial_t\chi - \Delta\vartheta \\ & = R_\Omega + |\partial_t\chi|^2, \quad \vartheta \geq 0 \quad \text{and} \quad \text{BC}\end{aligned}$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B_\varepsilon \quad \text{and} \quad \sigma \cdot \mathbf{n} = 0, \quad \sigma = \kappa \nabla u - \gamma_\varepsilon(\chi) \mathbf{e}$$

$$\begin{aligned}(3) \quad & \partial_t\chi - \Delta\chi - \varepsilon\partial_t\Delta\chi \quad (\leftarrow \text{higher order perturbation}) \\ & + \vartheta_c F_\varepsilon'(\chi) + \alpha_\varepsilon(\vartheta)G_\varepsilon'(\chi) - \sigma \cdot \mathbf{e} \gamma_\varepsilon'(\chi) = 0 \quad + \text{BC}\end{aligned}$$

♠ **Bad point:**

as *sign* ϑ might change, we must set $\alpha_\varepsilon(\vartheta) = 0$ for $\vartheta < 0$
and keep C^2 + uniform boundedness and parabolicity
→ tricky construction of α_ε

♠ **Bad point:**

No L^2 estimates for $F'(\chi)$ due to $-\partial_t\Delta\chi$, whence,
monotonicity argument in the pair (V, V^*) , $V = H^1(\Omega)$

♠ **Worst point:**

Existence for the ε -problem is not obvious !!!

11 - Outline

Recall

$$\begin{aligned}(1) \quad & (c_0 - (\vartheta + \varepsilon)\alpha_\varepsilon''(\vartheta)G_\varepsilon(\chi))\partial_t\vartheta \\ & - (\vartheta + \varepsilon)\alpha_\varepsilon'(\vartheta)G_\varepsilon'(\chi)\partial_t\chi - \Delta\vartheta \\ & = R_\Omega + |\partial_t\chi|^2, \quad \vartheta \geq 0 \quad \text{and} \quad \text{BC}\end{aligned}$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B_\varepsilon \quad \text{and} \quad \sigma \cdot \mathbf{n} = 0, \quad \sigma = \kappa \nabla u - \gamma_\varepsilon(\chi) \mathbf{e}$$

$$\begin{aligned}(3) \quad & \partial_t\chi - \Delta\chi - \varepsilon\partial_t\Delta\chi \\ & + \vartheta_c F_\varepsilon'(\chi) + \alpha_\varepsilon(\vartheta)G_\varepsilon'(\chi) - \sigma \cdot \mathbf{e} \gamma_\varepsilon'(\chi) = 0 \quad + \text{BC}\end{aligned}$$

- Time discretization with some time step τ
- A priori estimates uniform with respect to τ , only
- Existence of a very smooth solution to the ε -problem
- A priori estimates uniform with respect to ε
- Existence for the original problem
- Too technical, hence just formal estimates and basic ideas here

12 - Starting point for the exposition

Original system with one more equation and the modification “↓”

$$(1) \quad (c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ = R + |\partial_t \chi|^2, \quad \vartheta > 0$$

$$(1') \quad \partial_t w - \alpha(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta = R + |\partial_t \chi|^2 \\ w := c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi) \\ = \int_0^\vartheta \underbrace{(c_0 - r \alpha''(r) G(\chi))}_{\geq \delta_* \text{ (by parabolicity)}} dr \geq \delta_* \vartheta$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B \quad \text{and} \quad \sigma = \kappa \nabla u - \gamma(\chi) \mathbf{e} \quad \text{with} \quad \kappa = 1$$

$$(3) \quad \partial_t \chi - \Delta \chi + \vartheta_c F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot \mathbf{e} \gamma'(\chi) = 0$$

$$BC \quad \partial_n \vartheta = 0, \quad \sigma \cdot \mathbf{n} = 0 \quad \text{and} \quad \partial_n \chi = 0 \quad (\text{no bndry terms !}) \\ \swarrow \text{true only for the } \varepsilon\text{-pbl}$$

$$IC \quad \vartheta(0), w(0), u(0), \partial_t u(0), \chi(0) \quad \text{given}$$

Equation (1') and IC for w added, for convenience: $(1') \approx (1)$

13 - The basic first a priori estimate

Recall

$$(1) \quad (c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ = R + |\partial_t \chi|^2, \quad \vartheta > 0$$

$$(1') \quad \partial_t w - \alpha(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta = R + |\partial_t \chi|^2 \\ w := c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi)$$

$$(2) \quad \partial_t^2 u - \operatorname{div} \sigma = B \quad \text{and} \quad \sigma = \nabla u - \gamma(\chi) \mathbf{e}$$

$$(3) \quad \partial_t \chi - \Delta \chi + \vartheta_c F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot \mathbf{e} \gamma'(\chi) = 0$$

- Even though $c_0 - \vartheta \alpha''(\vartheta) G(\chi) \geq \delta_* > 0$ (parabolicity)

(1) $\times \partial_t \vartheta$ **cannot** be done: RHS of (1) $\rightarrow |\partial_t \chi|^2 \partial_t \vartheta$!!!

- **This can** be (formally) done, instead

$$\int_0^t \int_{\Omega} \left\{ (1) \times \left(-\frac{1}{\vartheta} \right) + (1') \times 1 + (2_1) \times \underset{\uparrow}{2} \partial_t u + (3) \times \underset{\uparrow}{2} \partial_t \chi \right\}$$

14 - The basic first a priori estimate (cont'd)

Recall

$$(1) \quad \underbrace{c_0 \partial_t \vartheta - \vartheta \alpha''(\vartheta) \partial_t \vartheta G(\chi)}_{(c_0 - \vartheta \alpha''(\vartheta) G(\chi)) \partial_t \vartheta} - \vartheta \alpha'(\vartheta) G'(\chi) \partial_t \chi - \Delta \vartheta \\ = R + |\partial_t \chi|^2, \quad \vartheta > 0 \quad (\text{recall: } R \geq 0 !!!)$$

$$(1') \quad \partial_t w - \Delta \vartheta = R + |\partial_t \chi|^2 + \overbrace{\alpha(\vartheta) G'(\chi)}^{\text{bounded}} \partial_t \chi \\ w := c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi), \quad w \geq \delta_* \vartheta$$

- $(1) \times (-1/\vartheta) \longrightarrow$

LHS: $-c_0 \int_{\Omega} \ln \vartheta(t) + \int_0^t \int_{\Omega} |\nabla \ln \vartheta|^2 + \int_0^t \int_{\Omega} \partial_t \underbrace{\{\dots\dots\}}_{\text{good} \rightarrow \text{rhs}}$

RHS: **negative** (since $R \geq 0$) !!!

- $(1') \times 1 \longrightarrow$ LHS: $\int_{\Omega} w(t) + 0 \geq \delta_* \int_{\Omega} \vartheta(t)$

RHS: $\frac{3}{2} \int_0^t \int_{\Omega} |\partial_t \chi|^2$ **bad** !!! Therefore...

15 - The basic first a priori estimate (cont'd)

$\frac{3}{2} \int_0^t \int_{\Omega} |\partial_t \chi|^2$ on the RHS Therefore.....

Recall

$$(3) \quad \partial_t \chi - \Delta \chi + \vartheta_c F'(\chi) + \alpha(\vartheta) G'(\chi) - \sigma \cdot e \gamma'(\chi) = 0$$

$$\begin{aligned} \bullet \quad (3) \times \downarrow 2 \partial_t \chi &\rightarrow \downarrow 2 \int_0^t \int_{\Omega} |\partial_t \chi|^2 + 2 \int_0^t \int_{\Omega} \overbrace{\alpha(\vartheta) G'(\chi)}^{\text{bounded}} \partial_t \chi \\ &\quad \int_0^t \int_{\Omega} |\nabla \chi|^2 + 2 \vartheta_c \int_{\Omega} F(\chi(t)) \\ &\quad - \underbrace{2 \int_0^t \int_{\Omega} \sigma \cdot e \gamma'(\chi) \partial_t \chi}_{\uparrow} \quad ??? \end{aligned}$$

- However, the bad term $\frac{3}{2} \int_0^t \int_{\Omega} |\partial_t \chi|^2$ has been compensated

16 - The basic first a priori estimate (cont'd)

We had $-\underset{\uparrow}{2} \int_0^t \int_{\Omega} \boldsymbol{\sigma} \cdot \mathbf{e} \gamma'(\chi) \partial_t \chi$ on the LHS

Recall

$$(2_1) \quad \partial_t^2 u - \operatorname{div} \boldsymbol{\sigma} = B$$

$$(2_2) \quad \boldsymbol{\sigma} = \nabla u - \gamma(\chi) \mathbf{e}$$

$$\text{whence} \quad \nabla \partial_t u = \partial_t \boldsymbol{\sigma} + \gamma'(\chi) \partial_t \chi \mathbf{e}$$

$$\bullet \quad (2_1) \times \underset{\downarrow}{2} \partial_t u \longrightarrow$$

$$2 \int_0^t \int_{\Omega} \partial_t^2 u \partial_t u + 2 \int_0^t \int_{\Omega} \boldsymbol{\sigma} \cdot (\partial_t \boldsymbol{\sigma} + \gamma'(\chi) \partial_t \chi \mathbf{e}) = \dots$$

$$\int_{\Omega} |\partial_t u(t)|^2 + \int_{\Omega} |\boldsymbol{\sigma}(t)|^2 + \underset{\uparrow}{2} \int_0^t \int_{\Omega} \boldsymbol{\sigma} \cdot \mathbf{e} \gamma'(\chi) \partial_t \chi$$

and the last term cancels out the bad red term on the top !!!

17 - The basic first a priori estimate (cont'd)

Eventually, we obtain

$$\|w_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq c \quad \text{and} \quad \|\vartheta_\varepsilon - \ln \vartheta_\varepsilon\|_{L^\infty(0,T;L^1(\Omega))} \leq c$$

$$\|\ln \vartheta_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \leq c$$

$$\|u_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \leq c \quad \text{and} \quad \|\partial_t u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))} \leq c$$

$$(\|\sigma_\varepsilon\|_{L^2(0,T;(L^2(\Omega))^3)} \leq c)$$

$$\|\chi_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))} \quad \text{and} \quad \|\partial_t \chi_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq c$$

- Compactness $\rightarrow$ weak type convergence (proper topologies)

$$\ln \vartheta_\varepsilon \rightarrow \ell, \quad u_\varepsilon \rightarrow u, \quad \partial_t u_\varepsilon \rightarrow \partial_t u \quad \text{and} \quad \partial_t \chi_\varepsilon \rightarrow \partial_t \chi$$

(and, e.g., $\alpha_\varepsilon(\vartheta_\varepsilon) \xrightarrow{*} a$ in $L^\infty(Q)$) and strong convergence

$$\chi_\varepsilon \rightarrow \chi \quad \text{in } C^0([0,T];L^p(\Omega)), \quad p < 6$$

- **Nothing** for $w_\varepsilon, \vartheta_\varepsilon, F'(\chi_\varepsilon)$!!! $\rightarrow$ **big identification problems**

18 - First trouble

- One would like to test (3) by $F'(\chi)$ to estimate it in L^2 : impossible, since the **true** approximating equation $(3)_\varepsilon$ is

$$(3)_\varepsilon \quad \partial_t \chi - \Delta \chi - \varepsilon \partial_t \Delta \chi + \vartheta_c F'_\varepsilon(\chi) + \alpha_\varepsilon(\vartheta) G'_\varepsilon(\chi) - \sigma \cdot \mathbf{e} \gamma'_\varepsilon(\chi) = 0$$

- one can test by $-\Delta \chi$ and get $\|\Delta \chi\|_{L^2(0,T;L^2(\Omega))} \leq c$ (**good** !!!)
- on the contrary, one **cannot** test by $F'_\varepsilon(\chi)$!!! However
- $\varepsilon \partial_t \Delta \chi$ (even $\varepsilon^{1/2} \partial_t \Delta \chi$) bdd in $L^2(0,T; V^*)$, $V = H^1(\Omega)$:

$$\int_0^t \int_\Omega (-\varepsilon^{1/2} \partial_t \Delta \chi) v \leq \|\varepsilon^{1/2} \partial_t \nabla \chi\|_2 \|\nabla v\|_2 \leq c \|\nabla v\|_2$$

$(3)_\varepsilon \times \partial_t \chi$ **true** 1st estimate

- by comparison in $(3)_\varepsilon$: $F'_\varepsilon(\chi_\varepsilon)$ bounded in $L^2(0,T; V^*)$
- Good with a non-trivial complication:**
 monotonicity argument for (the monotone part of) F'_ε
 in the duality $L^2(0,T; V) - L^2(0,T; V^*)$ + use of **BCGG**

19 - Second trouble

- **No convergence** for ϑ , nor for w !!!
(just boundedness in spaces involving L^1)

Recall equation (1') for w

$$(1') \quad \partial_t w - \Delta \vartheta = \overbrace{R + \alpha(\vartheta)G'(\chi)}^{bdd \text{ in } L^2(0,T;L^2(\Omega))} \partial_t \chi + |\partial_t \chi|^2$$
$$w := c_0 \vartheta + (\alpha(\vartheta) - \vartheta \alpha'(\vartheta)) G(\chi), \quad \text{say, } w \approx \vartheta$$

- Estimates for sol's of parabolic eq's with $RHS \in L^1$ needed !!!
- **Boccardo-Gallouët-1989 argument** adapted to our situation

$$\|w_\varepsilon\|_{L^q(0,T;W^{1,q}(\Omega))} \leq c_q \quad \text{and} \quad \|\partial_t w_\varepsilon\|_{L^1(0,T;W^{1,q'}(\Omega)^*)} \leq c_q$$

for $q < 5/4$, whence $w_\varepsilon \rightarrow w$ **strongly** in L^1 and a.e.

- This is sufficient to identify the nonlinear terms involving ϑ

20 - Third trouble

- **Just boundedness** in L^1 for $|\partial_t \chi|^2$!!!

Recall equation (1) for ϑ_ε and (3) for χ_ε

$$(1) \quad (\dots\dots)\partial_t \vartheta_\varepsilon \dots\dots = R + |\partial_t \chi_\varepsilon|^2$$

$$(3) \quad \partial_t \chi_\varepsilon \dots\dots - \sigma_\varepsilon \cdot \mathbf{e} \gamma'(\chi_\varepsilon) = 0$$

- For (1), we need $|\partial_t \chi_\varepsilon|^2 \rightarrow |\partial_t \chi|^2$ in some topology
- **Strong convergence** in L^2 for $\partial_t \chi_\varepsilon$ needed !!!
- Due to weak convergence, a sufficient condition is

$$\limsup_{\varepsilon \searrow 0} \int_0^T \int_\Omega |\partial_t \chi_\varepsilon|^2 \leq \int_0^T \int_\Omega |\partial_t \chi|^2$$

but this needs $(3) \times \partial_t \chi_\varepsilon$, thus it involves $\sigma_\varepsilon \cdot \mathbf{e} \gamma'(\chi_\varepsilon) \partial_t \chi_\varepsilon$
i.e., a trouble due to **weak-weak** convergence !!!

21 - Energy identity

It is formally obtained by testing eq'n (2) for u by $\partial_t u$.

Rigorous proof both for the ε -sol'n and the final candidate (u, σ, χ)

$$\begin{aligned} & \frac{1}{2} \int_{\Omega} |\partial_t u(t)|^2 - \frac{1}{2} \int_{\Omega} |u'_0|^2 + \frac{1}{2} \int_{\Omega} |\sigma(t)|^2 - \frac{1}{2} \int_{\Omega} |\sigma(0)|^2 \\ & + \int_0^t \int_{\Omega} \sigma \cdot \mathbf{e} \gamma'(\chi) \partial_t \chi \quad \leftarrow \quad \text{the } \begin{matrix} B & A & D \\ \text{weak} & & \text{weak} \end{matrix} \text{ term} \\ & = \int_0^t \int_{\Omega} B_{\Omega} \partial_t u + \langle B_{\Gamma}(t), u(t) \rangle - \langle B_{\Gamma}(0), u_0 \rangle \\ & \quad - \int_0^t \langle \partial_t B_{\Gamma}(s), u(s) \rangle ds \quad \text{for every } t \in [0, T]. \end{aligned}$$

Write the bad term using the better ones of the identity in the ε -problem, account for semicontinuity, and finally use the identity for (u, σ, χ) . The inequality goes in the right direction !!!

- So, we arrived at the end of the proof !!!

Thank you for your attention

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Coordinates & info's: Google → gianni gilardi → 1st result is
<http://www-dimat.unipv.it/gilardi>

