

Quasistatic Evolution as Limit of Dynamic Evolutions: the Case of Perfect Plasticity

Gianni Dal Maso and Riccardo Scala
SISSA, Trieste, Italy



Levico

Diffuse Interface Models

September 13, 2013

Dynamic and viscous approximation

The quasistatic evolution of **rate independent** systems has been often obtained as the limit case of **viscosity driven evolutions**. For instance, **Suquet (1981)** obtained the quasistatic evolution in **perfect plasticity** as limit of the solutions of **visco-plastic** problems, as the **viscosity parameter tends to zero**.

The general philosophy underlying this kind of approximation is that **rate independent systems** are **highly idealized models**, obtained by neglecting all viscous effects, and that the true system is actually governed by a **viscous dynamics**, with small viscosity. The mathematical statements about the **convergence of the viscous solutions** to the quasistatic solutions provide a **justification** of the physical relevance of the quasistatic model as well as a **tool to prove** the **existence** of a **quasistatic solution**.

On the other hand, even the **viscosity driven models** are approximate models, obtained by neglecting all **inertial effects**. The purpose of this talk is to present a **case study** on the approximation of a quasistatic evolution by **dynamic evolutions**, which take into account all **inertial effects**.

Dynamic and viscous approximation

The quasistatic evolution of **rate independent** systems has been often obtained as the limit case of **viscosity driven evolutions**. For instance, **Suquet (1981)** obtained the quasistatic evolution in **perfect plasticity** as limit of the solutions of **visco-plastic** problems, as the **viscosity parameter tends to zero**.

The general philosophy underlying this kind of approximation is that **rate independent systems** are **highly idealized models**, obtained by neglecting all viscous effects, and that the true system is actually governed by a **viscous dynamics**, with small viscosity. The mathematical statements about the **convergence of the viscous solutions** to the quasistatic solutions provide a **justification** of the physical relevance of the quasistatic model as well as a **tool to prove** the **existence** of a **quasistatic solution**.

On the other hand, even the **viscosity driven models** are approximate models, obtained by neglecting all **inertial effects**. The purpose of this talk is to present a **case study** on the approximation of a quasistatic evolution by **dynamic evolutions**, which take into account all **inertial effects**.

Dynamic and viscous approximation

The quasistatic evolution of **rate independent** systems has been often obtained as the limit case of **viscosity driven evolutions**. For instance, **Suquet (1981)** obtained the quasistatic evolution in **perfect plasticity** as limit of the solutions of **visco-plastic** problems, as the **viscosity parameter tends to zero**.

The general philosophy underlying this kind of approximation is that **rate independent systems** are **highly idealized models**, obtained by neglecting all viscous effects, and that the true system is actually governed by a **viscous dynamics**, with small viscosity. The mathematical statements about the **convergence of the viscous solutions** to the quasistatic solutions provide a **justification** of the physical relevance of the quasistatic model as well as a **tool to prove the existence** of a **quasistatic solution**.

On the other hand, even the **viscosity driven models** are approximate models, obtained by neglecting all **inertial effects**. The purpose of this talk is to present a **case study** on the approximation of a quasistatic evolution by **dynamic evolutions**, which take into account all **inertial effects**.

Dynamic and viscous approximation

The quasistatic evolution of **rate independent** systems has been often obtained as the limit case of **viscosity driven evolutions**. For instance, **Suquet (1981)** obtained the quasistatic evolution in **perfect plasticity** as limit of the solutions of **visco-plastic** problems, as the **viscosity parameter tends to zero**.

The general philosophy underlying this kind of approximation is that **rate independent systems** are **highly idealized models**, obtained by neglecting all viscous effects, and that the true system is actually governed by a **viscous dynamics**, with small viscosity. The mathematical statements about the **convergence of the viscous solutions** to the quasistatic solutions provide a **justification** of the physical relevance of the quasistatic model as well as a **tool to prove** the **existence** of a **quasistatic solution**.

On the other hand, even the **viscosity driven models** are approximate models, obtained by neglecting all **inertial effects**. The purpose of this talk is to present a **case study** on the approximation of a quasistatic evolution by **dynamic evolutions**, which take into account all **inertial effects**.

The quasistatic evolution of **rate independent** systems has been often obtained as the limit case of **viscosity driven evolutions**. For instance, **Suquet (1981)** obtained the quasistatic evolution in **perfect plasticity** as limit of the solutions of **visco-plastic** problems, as the **viscosity parameter tends to zero**.

The general philosophy underlying this kind of approximation is that **rate independent systems** are **highly idealized models**, obtained by neglecting all viscous effects, and that the true system is actually governed by a **viscous dynamics**, with small viscosity. The mathematical statements about the **convergence of the viscous solutions** to the quasistatic solutions provide a **justification** of the physical relevance of the quasistatic model as well as a **tool to prove** the **existence** of a **quasistatic solution**.

On the other hand, even the **viscosity driven models** are approximate models, obtained by neglecting all **inertial effects**. The purpose of this talk is to present a **case study** on the approximation of a quasistatic evolution by **dynamic evolutions**, which take into account all **inertial effects**.

A finite dimensional analogue

For a **finite number of degrees of freedom** the state $\mathbf{u}(t) \in \mathbb{R}^n$ satisfies

$$M\ddot{\mathbf{u}}(t) + K\dot{\mathbf{u}}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}(t)) = \mathbf{f}(t), \quad t \in [0, T],$$

where dots denote time derivatives, M and K are positive definite matrices, related to **mass** and **friction**, $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a **potential**, and $\mathbf{f}(t)$ is a time dependent **external force**. To study the solutions when the data vary slowly, we consider the rescaled force $\mathbf{f}_{\varepsilon}(t) := \mathbf{f}(\varepsilon t)$, defined for $t \in [0, T/\varepsilon]$. On this interval we consider the solution $\mathbf{u}_{\varepsilon}$ of the equation with $\mathbf{f}_{\varepsilon}$ instead of $\mathbf{f}$. To study the limit behaviour of $\mathbf{u}_{\varepsilon}$ on the whole interval $[0, T/\varepsilon]$ it is convenient to consider the rescaled functions $\mathbf{u}^{\varepsilon}(t) := \mathbf{u}_{\varepsilon}(t/\varepsilon)$, which are defined for $t \in [0, T]$. By an easy change of variables we see that they satisfy

$$\varepsilon^2 M\ddot{\mathbf{u}}^{\varepsilon}(t) + \varepsilon K\dot{\mathbf{u}}^{\varepsilon}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}^{\varepsilon}(t)) = \mathbf{f}(t), \quad t \in [0, T].$$

If $\mathbf{u}^{\varepsilon}$ converges to $\mathbf{u}^0$, we expect that $\nabla_{\mathbf{x}}V(\mathbf{u}^0(t)) = \mathbf{f}(t), \quad t \in [0, T]$.

A finite dimensional analogue

For a **finite number of degrees of freedom** the state $\mathbf{u}(t) \in \mathbb{R}^n$ satisfies

$$M\ddot{\mathbf{u}}(t) + K\dot{\mathbf{u}}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}(t)) = \mathbf{f}(t), \quad t \in [0, T],$$

where dots denote time derivatives, M and K are positive definite matrices, related to **mass** and **friction**, $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a **potential**, and $\mathbf{f}(t)$ is a time dependent **external force**. To study the solutions when the data vary slowly, we consider the rescaled force $\mathbf{f}_{\varepsilon}(t) := \mathbf{f}(\varepsilon t)$, defined for $t \in [0, T/\varepsilon]$. On this interval we consider the solution $\mathbf{u}_{\varepsilon}$ of the equation with $\mathbf{f}_{\varepsilon}$ instead of $\mathbf{f}$. To study the limit behaviour of $\mathbf{u}_{\varepsilon}$ on the whole interval $[0, T/\varepsilon]$ it is convenient to consider the rescaled functions $\mathbf{u}^{\varepsilon}(t) := \mathbf{u}_{\varepsilon}(t/\varepsilon)$, which are defined for $t \in [0, T]$. By an easy change of variables we see that they satisfy

$$\varepsilon^2 M\ddot{\mathbf{u}}^{\varepsilon}(t) + \varepsilon K\dot{\mathbf{u}}^{\varepsilon}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}^{\varepsilon}(t)) = \mathbf{f}(t), \quad t \in [0, T].$$

If $\mathbf{u}^{\varepsilon}$ converges to $\mathbf{u}^0$, we expect that $\nabla_{\mathbf{x}}V(\mathbf{u}^0(t)) = \mathbf{f}(t), \quad t \in [0, T]$.

A finite dimensional analogue

For a **finite number of degrees of freedom** the state $\mathbf{u}(t) \in \mathbb{R}^n$ satisfies

$$M\ddot{\mathbf{u}}(t) + K\dot{\mathbf{u}}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}(t)) = \mathbf{f}(t), \quad t \in [0, T],$$

where dots denote time derivatives, M and K are positive definite matrices, related to **mass** and **friction**, $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a **potential**, and $\mathbf{f}(t)$ is a time dependent **external force**. To study the solutions when the data vary slowly, we consider the rescaled force $\mathbf{f}_{\varepsilon}(t) := \mathbf{f}(\varepsilon t)$, defined for $t \in [0, T/\varepsilon]$. On this interval we consider the solution $\mathbf{u}_{\varepsilon}$ of the equation with $\mathbf{f}_{\varepsilon}$ instead of $\mathbf{f}$. To study the limit behaviour of $\mathbf{u}_{\varepsilon}$ on the whole interval $[0, T/\varepsilon]$ it is convenient to consider the rescaled functions $\mathbf{u}^{\varepsilon}(t) := \mathbf{u}_{\varepsilon}(t/\varepsilon)$, which are defined for $t \in [0, T]$. By an easy change of variables we see that they satisfy

$$\varepsilon^2 M\ddot{\mathbf{u}}^{\varepsilon}(t) + \varepsilon K\dot{\mathbf{u}}^{\varepsilon}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}^{\varepsilon}(t)) = \mathbf{f}(t), \quad t \in [0, T].$$

If $\mathbf{u}^{\varepsilon}$ converges to $\mathbf{u}^0$, we expect that $\nabla_{\mathbf{x}}V(\mathbf{u}^0(t)) = \mathbf{f}(t), \quad t \in [0, T]$.

A finite dimensional analogue

For a **finite number of degrees of freedom** the state $\mathbf{u}(t) \in \mathbb{R}^n$ satisfies

$$M\ddot{\mathbf{u}}(t) + K\dot{\mathbf{u}}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}(t)) = \mathbf{f}(t), \quad t \in [0, T],$$

where dots denote time derivatives, M and K are positive definite matrices, related to **mass** and **friction**, $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a **potential**, and $\mathbf{f}(t)$ is a time dependent **external force**. To study the solutions when the data vary slowly, we consider the rescaled force $\mathbf{f}_{\varepsilon}(t) := \mathbf{f}(\varepsilon t)$, defined for $t \in [0, T/\varepsilon]$. On this interval we consider the solution $\mathbf{u}_{\varepsilon}$ of the equation with $\mathbf{f}_{\varepsilon}$ instead of $\mathbf{f}$. To study the limit behaviour of $\mathbf{u}_{\varepsilon}$ on the whole interval $[0, T/\varepsilon]$ it is convenient to consider the rescaled functions $\mathbf{u}^{\varepsilon}(t) := \mathbf{u}_{\varepsilon}(t/\varepsilon)$, which are defined for $t \in [0, T]$. By an easy change of variables we see that they satisfy

$$\varepsilon^2 M\ddot{\mathbf{u}}^{\varepsilon}(t) + \varepsilon K\dot{\mathbf{u}}^{\varepsilon}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}^{\varepsilon}(t)) = \mathbf{f}(t), \quad t \in [0, T].$$

If $\mathbf{u}^{\varepsilon}$ converges to $\mathbf{u}^0$, we expect that $\nabla_{\mathbf{x}}V(\mathbf{u}^0(t)) = \mathbf{f}(t), \quad t \in [0, T]$.

A finite dimensional analogue

For a **finite number of degrees of freedom** the state $\mathbf{u}(t) \in \mathbb{R}^n$ satisfies

$$M\ddot{\mathbf{u}}(t) + K\dot{\mathbf{u}}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}(t)) = \mathbf{f}(t), \quad t \in [0, T],$$

where dots denote time derivatives, M and K are positive definite matrices, related to **mass** and **friction**, $V: \mathbb{R}^n \rightarrow \mathbb{R}$ is a **potential**, and $\mathbf{f}(t)$ is a time dependent **external force**. To study the solutions when the data vary slowly, we consider the rescaled force $\mathbf{f}_{\varepsilon}(t) := \mathbf{f}(\varepsilon t)$, defined for $t \in [0, T/\varepsilon]$. On this interval we consider the solution $\mathbf{u}_{\varepsilon}$ of the equation with $\mathbf{f}_{\varepsilon}$ instead of $\mathbf{f}$. To study the limit behaviour of $\mathbf{u}_{\varepsilon}$ on the whole interval $[0, T/\varepsilon]$ it is convenient to consider the rescaled functions $\mathbf{u}^{\varepsilon}(t) := \mathbf{u}_{\varepsilon}(t/\varepsilon)$, which are defined for $t \in [0, T]$. By an easy change of variables we see that they satisfy

$$\varepsilon^2 M\ddot{\mathbf{u}}^{\varepsilon}(t) + \varepsilon K\dot{\mathbf{u}}^{\varepsilon}(t) + \nabla_{\mathbf{x}}V(\mathbf{u}^{\varepsilon}(t)) = \mathbf{f}(t), \quad t \in [0, T].$$

If $\mathbf{u}^{\varepsilon}$ converges to $\mathbf{u}^0$, we expect that $\nabla_{\mathbf{x}}V(\mathbf{u}^0(t)) = \mathbf{f}(t), \quad t \in [0, T].$

A case study: perfect plasticity

The case study we consider here is the quasistatic evolution in **perfect plasticity**. It was approximated by **Suquet (1981)** using **visco-plasticity** and neglecting all inertial effects.

We want to prove that it can be obtained also as limit of a **damped dynamic evolution**, which follows the law of **visco-elasticity** when the stress is in the elastic region, and of **visco-plasticity** when the stress is out of the elastic region.

The aim is to develop the mathematical tools to approximate a quasistatic evolution by damped dynamic evolutions. The final objective, not yet reached, is to introduce a **quasistatic evolution** for **cracks** as the limit case of suitable **dynamic evolutions** in **fracture mechanics**.

Of course, the problem of perfect plasticity considered here is easier, because only **convex** terms appear in its **energy formulation**.

A case study: perfect plasticity

The case study we consider here is the quasistatic evolution in **perfect plasticity**. It was approximated by **Suquet (1981)** using **visco-plasticity** and neglecting all inertial effects.

We want to prove that it can be obtained also as limit of a **damped dynamic evolution**, which follows the law of **visco-elasticity** when the stress is in the elastic region, and of **visco-plasticity** when the stress is out of the elastic region.

The aim is to develop the mathematical tools to approximate a quasistatic evolution by damped dynamic evolutions. The final objective, not yet reached, is to introduce a **quasistatic evolution** for **cracks** as the limit case of suitable **dynamic evolutions** in **fracture mechanics**.

Of course, the problem of perfect plasticity considered here is easier, because only **convex** terms appear in its **energy formulation**.

A case study: perfect plasticity

The case study we consider here is the quasistatic evolution in **perfect plasticity**. It was approximated by **Suquet (1981)** using **visco-plasticity** and neglecting all inertial effects.

We want to prove that it can be obtained also as limit of a **damped dynamic evolution**, which follows the law of **visco-elasticity** when the stress is in the elastic region, and of **visco-plasticity** when the stress is out of the elastic region.

The aim is to develop the mathematical tools to approximate a quasistatic evolution by damped dynamic evolutions. The final objective, not yet reached, is to introduce a **quasistatic evolution** for **cracks** as the limit case of suitable **dynamic evolutions** in **fracture mechanics**.

Of course, the problem of perfect plasticity considered here is easier, because only **convex** terms appear in its **energy formulation**.

A case study: perfect plasticity

The case study we consider here is the quasistatic evolution in **perfect plasticity**. It was approximated by **Suquet (1981)** using **visco-plasticity** and neglecting all inertial effects.

We want to prove that it can be obtained also as limit of a **damped dynamic evolution**, which follows the law of **visco-elasticity** when the stress is in the elastic region, and of **visco-plasticity** when the stress is out of the elastic region.

The aim is to develop the mathematical tools to approximate a quasistatic evolution by damped dynamic evolutions. The final objective, not yet reached, is to introduce a **quasistatic evolution** for **cracks** as the limit case of suitable **dynamic evolutions** in **fracture mechanics**.

Of course, the problem of perfect plasticity considered here is easier, because only **convex** terms appear in its **energy formulation**.

Dynamic visco-elasto-plastic evolution

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbb{E}u := \frac{1}{2}(\mathbb{D}u + \mathbb{D}u^T)$, where $u: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbb{E}u = e + p$, where e is the **elastic** part and p is the **plastic** part.
- **Stress:** $\sigma = A^0 e + A^1 \dot{e}$, where $A^0 e$ is the **elastic** part and $A^1 \dot{e}$ is the **viscous** part (A^0 is the **elasticity tensor**, A^1 is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{u} - \operatorname{div} \sigma = f$, where f is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{p} = \sigma_D - \pi_K \sigma_D$, where σ_D is the **deviatoric** part of σ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing 0 in its interior (K is interpreted as the **domain of visco-elasticity**)

Dynamic visco-elasto-plastic evolution

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbf{Eu} := \frac{1}{2}(\mathbf{Du} + \mathbf{Du}^T)$, where $\mathbf{u}: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbf{Eu} = \mathbf{e} + \mathbf{p}$, where $\mathbf{e}$ is the **elastic** part and $\mathbf{p}$ is the **plastic** part.
- **Stress:** $\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}}$, where $\mathbf{A}^0 \mathbf{e}$ is the **elastic** part and $\mathbf{A}^1 \dot{\mathbf{e}}$ is the **viscous** part ($\mathbf{A}^0$ is the **elasticity tensor**, $\mathbf{A}^1$ is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$, where $\mathbf{f}$ is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D$, where $\boldsymbol{\sigma}_D$ is the **deviatoric** part of $\boldsymbol{\sigma}$ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing $\mathbf{0}$ in its interior (K is interpreted as the **domain of visco-elasticity**)

Dynamic visco-elasto-plastic evolution

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbf{Eu} := \frac{1}{2}(\mathbf{Du} + \mathbf{Du}^T)$, where $\mathbf{u}: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbf{Eu} = \mathbf{e} + \mathbf{p}$, where $\mathbf{e}$ is the **elastic** part and $\mathbf{p}$ is the **plastic** part.
- **Stress:** $\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}}$, where $\mathbf{A}^0 \mathbf{e}$ is the **elastic** part and $\mathbf{A}^1 \dot{\mathbf{e}}$ is the **viscous** part ($\mathbf{A}^0$ is the **elasticity tensor**, $\mathbf{A}^1$ is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$, where $\mathbf{f}$ is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D$, where $\boldsymbol{\sigma}_D$ is the **deviatoric** part of $\boldsymbol{\sigma}$ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing $\mathbf{0}$ in its interior (K is interpreted as the **domain of visco-elasticity**)

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbf{Eu} := \frac{1}{2}(\mathbf{Du} + \mathbf{Du}^T)$, where $\mathbf{u}: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbf{Eu} = \mathbf{e} + \mathbf{p}$, where $\mathbf{e}$ is the **elastic** part and $\mathbf{p}$ is the **plastic** part.
- **Stress:** $\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}}$, where $\mathbf{A}^0 \mathbf{e}$ is the **elastic** part and $\mathbf{A}^1 \dot{\mathbf{e}}$ is the **viscous** part ($\mathbf{A}^0$ is the **elasticity tensor**, $\mathbf{A}^1$ is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$, where $\mathbf{f}$ is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D$, where $\boldsymbol{\sigma}_D$ is the **deviatoric** part of $\boldsymbol{\sigma}$ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing $\mathbf{0}$ in its interior (K is interpreted as the **domain of visco-elasticity**)

Dynamic visco-elasto-plastic evolution

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbf{Eu} := \frac{1}{2}(\mathbf{Du} + \mathbf{Du}^T)$, where $\mathbf{u}: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbf{Eu} = \mathbf{e} + \mathbf{p}$, where $\mathbf{e}$ is the **elastic** part and $\mathbf{p}$ is the **plastic** part.
- **Stress:** $\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}}$, where $\mathbf{A}^0 \mathbf{e}$ is the **elastic** part and $\mathbf{A}^1 \dot{\mathbf{e}}$ is the **viscous** part ($\mathbf{A}^0$ is the **elasticity tensor**, $\mathbf{A}^1$ is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$, where $\mathbf{f}$ is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D$, where $\boldsymbol{\sigma}_D$ is the **deviatoric** part of $\boldsymbol{\sigma}$ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing $\mathbf{0}$ in its interior (K is interpreted as the **domain of visco-elasticity**)

- **Reference configuration:** $\Omega \subset \mathbb{R}^n$, bounded open set, $\partial\Omega$ smooth.
- **Small strain** regime: we use the linearized strain $\mathbf{Eu} := \frac{1}{2}(\mathbf{Du} + \mathbf{Du}^T)$, where $\mathbf{u}: \Omega \rightarrow \mathbb{R}^n$ is the **displacement**.
- **Additive decomposition:** $\mathbf{Eu} = \mathbf{e} + \mathbf{p}$, where $\mathbf{e}$ is the **elastic** part and $\mathbf{p}$ is the **plastic** part.
- **Stress:** $\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}}$, where $\mathbf{A}^0 \mathbf{e}$ is the **elastic** part and $\mathbf{A}^1 \dot{\mathbf{e}}$ is the **viscous** part ($\mathbf{A}^0$ is the **elasticity tensor**, $\mathbf{A}^1$ is the **viscosity tensor**, both are **symmetric** and **positive definite**).
- **Balance of momentum:** $\ddot{\mathbf{u}} - \operatorname{div} \boldsymbol{\sigma} = \mathbf{f}$, where $\mathbf{f}$ is the **volume force** (we suppose, for simplicity, that the mass density is identically equal to 1).
- **Flow rule:** $\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D$, where $\boldsymbol{\sigma}_D$ is the **deviatoric** part of $\boldsymbol{\sigma}$ (i.e., its projection onto the space $\mathbb{M}_D^{n \times n}$ of **trace free matrices**) and π_K is the projection onto a prescribed bounded convex set K of $\mathbb{M}_D^{n \times n}$, containing $\mathbf{0}$ in its interior (K is interpreted as the **domain of visco-elasticity**)

The complete system of equations

To simplify the exposition, we will assume $\mathbf{f} = \mathbf{0}$.

The complete system of equations on $\Omega \times (0, T)$ is then

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}},$$

$$\ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma},$$

$$\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D.$$

Initial conditions: $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{p}(0) = \mathbf{p}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$.

Boundary conditions: $\mathbf{u}(t) = \mathbf{w}(t)$ on $\partial\Omega$ (to simplify the exposition).

Given $\mathbf{w} \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ is **kinematically admissible**, and write $(\mathbf{u}, \mathbf{e}, \mathbf{p}) \in A(\mathbf{w})$, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n)$, $\mathbf{e} \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\mathbf{p} \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p}$ on Ω , and $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega}$ on $\partial\Omega$.

The complete system of equations

To simplify the exposition, we will assume $\mathbf{f} = 0$.

The complete system of equations on $\Omega \times (0, T)$ is then

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}},$$

$$\ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma},$$

$$\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D.$$

Initial conditions: $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{p}(0) = \mathbf{p}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$.

Boundary conditions: $\mathbf{u}(t) = \mathbf{w}(t)$ on $\partial\Omega$ (to simplify the exposition).

Given $\mathbf{w} \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ is kinematically admissible, and write $(\mathbf{u}, \mathbf{e}, \mathbf{p}) \in A(\mathbf{w})$, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n)$, $\mathbf{e} \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\mathbf{p} \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p}$ on Ω , and $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega}$ on $\partial\Omega$.

The complete system of equations

To simplify the exposition, we will assume $\mathbf{f} = 0$.

The complete **system of equations** on $\Omega \times (0, T)$ is then

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}},$$

$$\ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma},$$

$$\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D.$$

Initial conditions: $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{p}(0) = \mathbf{p}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$.

Boundary conditions: $\mathbf{u}(t) = \mathbf{w}(t)$ on $\partial\Omega$ (to simplify the exposition).

Given $\mathbf{w} \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ is **kinematically admissible**, and write $(\mathbf{u}, \mathbf{e}, \mathbf{p}) \in A(\mathbf{w})$, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n)$, $\mathbf{e} \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\mathbf{p} \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p}$ on Ω , and $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega}$ on $\partial\Omega$.

The complete system of equations

To simplify the exposition, we will assume $\mathbf{f} = 0$.

The complete **system of equations** on $\Omega \times (0, T)$ is then

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}},$$

$$\ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma},$$

$$\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D.$$

Initial conditions: $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{p}(0) = \mathbf{p}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$.

Boundary conditions: $\mathbf{u}(t) = \mathbf{w}(t)$ on $\partial\Omega$ (to simplify the exposition).

Given $\mathbf{w} \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ is **kinematically admissible**, and write $(\mathbf{u}, \mathbf{e}, \mathbf{p}) \in A(\mathbf{w})$, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n)$, $\mathbf{e} \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\mathbf{p} \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p}$ on Ω , and $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega}$ on $\partial\Omega$.

The complete system of equations

To simplify the exposition, we will assume $\mathbf{f} = \mathbf{0}$.

The complete system of equations on $\Omega \times (0, T)$ is then

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e} + \mathbf{A}^1 \dot{\mathbf{e}},$$

$$\ddot{\mathbf{u}} = \operatorname{div} \boldsymbol{\sigma},$$

$$\dot{\mathbf{p}} = \boldsymbol{\sigma}_D - \pi_K \boldsymbol{\sigma}_D.$$

Initial conditions: $\mathbf{u}(0) = \mathbf{u}_0$, $\mathbf{e}(0) = \mathbf{e}_0$, $\mathbf{p}(0) = \mathbf{p}_0$, $\dot{\mathbf{u}}(0) = \mathbf{v}_0$.

Boundary conditions: $\mathbf{u}(t) = \mathbf{w}(t)$ on $\partial\Omega$ (to simplify the exposition).

Given $\mathbf{w} \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ is kinematically admissible, and write $(\mathbf{u}, \mathbf{e}, \mathbf{p}) \in A(\mathbf{w})$, if $\mathbf{u} \in H^1(\Omega; \mathbb{R}^n)$, $\mathbf{e} \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, $\mathbf{p} \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, $\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p}$ on Ω , and $\mathbf{u}|_{\partial\Omega} = \mathbf{w}|_{\partial\Omega}$ on $\partial\Omega$.

Theorem (DM-Scala 2013)

Assume that $w \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$,
 $\dot{w} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\ddot{w} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$,
 $(u_0, e_0, p_0) \in A(w(0))$, and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. Then there exists a *unique quadruple* (u, e, p, σ) , with $u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\dot{u} \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\dot{u} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\ddot{u} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$, $e \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$,
 $p \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\dot{e} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$
 $\dot{p} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\sigma \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, such that for
a.e. $t \in [0, T]$ we have

$$\begin{aligned} Eu(t) &= e(t) + p(t), & \sigma(t) &= A^0 e(t) + A^1 \dot{e}(t), \\ \ddot{u}(t) &= \operatorname{div} \sigma(t), & \dot{p}(t) &= \sigma_D(t) - \pi_K \sigma_D(t), \end{aligned}$$

and $u(t) = w(t)$ on $\partial\Omega$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $\dot{u}(0) = v_0$.

Theorem (DM-Scala 2013)

Assume that $w \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$,
 $\dot{w} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\ddot{w} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$,
 $(u_0, e_0, p_0) \in A(w(0))$, and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. Then there exists a **unique quadruple** (u, e, p, σ) , with $u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\dot{u} \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\dot{u} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\ddot{u} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$, $e \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$,
 $p \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\dot{e} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$
 $\dot{p} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\sigma \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, such that for
a.e. $t \in [0, T]$ we have

$$\begin{aligned} Eu(t) &= e(t) + p(t), & \sigma(t) &= A^0 e(t) + A^1 \dot{e}(t), \\ \ddot{u}(t) &= \operatorname{div} \sigma(t), & \dot{p}(t) &= \sigma_D(t) - \pi_K \sigma_D(t), \end{aligned}$$

and $u(t) = w(t)$ on $\partial\Omega$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $\dot{u}(0) = v_0$.

Theorem (DM-Scala 2013)

Assume that $w \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$,
 $\dot{w} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\ddot{w} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$,
 $(u_0, e_0, p_0) \in A(w(0))$, and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. Then there exists a **unique quadruple** (u, e, p, σ) , with $u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\dot{u} \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\dot{u} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\ddot{u} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$, $e \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$,
 $p \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\dot{e} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$
 $\dot{p} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\sigma \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, such that for
a.e. $t \in [0, T]$ we have

$$\begin{aligned} Eu(t) &= e(t) + p(t), & \sigma(t) &= A^0 e(t) + A^1 \dot{e}(t), \\ \ddot{u}(t) &= \operatorname{div} \sigma(t), & \dot{p}(t) &= \sigma_D(t) - \pi_K \sigma_D(t), \end{aligned}$$

and $u(t) = w(t)$ on $\partial\Omega$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $\dot{u}(0) = v_0$.

Theorem (DM-Scala 2013)

Assume that $w \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$,
 $\dot{w} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\ddot{w} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$,
 $(u_0, e_0, p_0) \in A(w(0))$, and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. Then there exists a **unique quadruple** (u, e, p, σ) , with $u \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\dot{u} \in L^\infty([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\ddot{u} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$,
 $\ddot{u} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$, $e \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$,
 $p \in L^\infty([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\dot{e} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$
 $\dot{p} \in L^2([0, T]; L^2(\Omega; \mathbb{M}_D^{n \times n}))$, $\sigma \in L^2([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$, such that for
a.e. $t \in [0, T]$ we have

$$\begin{aligned} Eu(t) &= e(t) + p(t), & \sigma(t) &= A^0 e(t) + A^1 \dot{e}(t), \\ \ddot{u}(t) &= \operatorname{div} \sigma(t), & \dot{p}(t) &= \sigma_D(t) - \pi_K \sigma_D(t), \end{aligned}$$

and $u(t) = w(t)$ on $\partial\Omega$, $u(0) = u_0$, $e(0) = e_0$, $p(0) = p_0$, $\dot{u}(0) = v_0$.

To express the energy balance it is useful to introduce the quadratic forms

$$\mathcal{Q}_0(e) := \frac{1}{2} \int_{\Omega} A^0 e(x) \cdot e(x) dx \quad \text{and} \quad \mathcal{Q}_1(e) = \int_{\Omega} A^1 e(x) \cdot e(x) dx,$$

defined for $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$.

We also need the positively one-homogeneous functional

$$\mathcal{H}(p) = \int_{\Omega} H(p(x)) dx,$$

defined for $p \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, where $H: \mathbb{M}_D^{n \times n} \rightarrow [0, +\infty[$ is the **support function** of K , i.e.,

$$H(\xi) = \sup_{\zeta \in K} \zeta \cdot \xi.$$

To express the energy balance it is useful to introduce the quadratic forms

$$\mathcal{Q}_0(e) := \frac{1}{2} \int_{\Omega} A^0 e(x) \cdot e(x) dx \quad \text{and} \quad \mathcal{Q}_1(e) = \int_{\Omega} A^1 e(x) \cdot e(x) dx,$$

defined for $e \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$.

We also need the positively one-homogeneous functional

$$\mathcal{H}(p) = \int_{\Omega} H(p(x)) dx,$$

defined for $p \in L^2(\Omega; \mathbb{M}_D^{n \times n})$, where $H: \mathbb{M}_D^{n \times n} \rightarrow [0, +\infty[$ is the **support function** of K , i.e.,

$$H(\xi) = \sup_{\zeta \in K} \zeta \cdot \xi.$$

Under the regularity assumptions of the existence theorem, assume that (u, e, p, σ) satisfies $\mathbf{E}u(t) = e(t) + p(t)$ and $\sigma(t) = A^0 e(t) + A^1 \dot{e}(t)$ for a.e. $t \in [0, T]$, as well as the **initial** and **boundary conditions**. Then (u, e, p, σ) satisfies $\ddot{u}(t) = \operatorname{div} \sigma(t)$ and $\dot{p}(t) = \sigma_D(t) - \pi_K \sigma_D(t)$ for a.e. $t \in [0, T]$ if and only if both the following conditions hold:

- **Equilibrium condition:** for a.e. $t \in [0, T]$ we have

$$-\mathcal{H}(q) \leq \langle A^0 e(t), \eta \rangle + \langle A^1 \dot{e}(t), \eta \rangle + \langle \dot{p}(t), q \rangle + \langle \ddot{u}(t), \varphi \rangle \leq \mathcal{H}(-q)$$

for every $(\varphi, \eta, q) \in A(0)$;

- **Energy-dissipation balance:** for every $t \in [0, T]$ we have

$$\begin{aligned} \mathcal{Q}_0(e(t)) + \frac{1}{2} \|\dot{u}(t) - \dot{w}(t)\|_{L^2}^2 + \int_0^t \mathcal{Q}_1(\dot{e}) ds + \int_0^t \|\dot{p}\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot{p}) ds \\ = \int_0^t \langle \sigma, E \dot{w} \rangle ds - \int_0^t \langle \ddot{w}, \dot{u} - \dot{w} \rangle ds + \mathcal{Q}_0(e_0) + \frac{1}{2} \|v_0 - \dot{w}(0)\|_{L^2}^2. \end{aligned}$$

Under the regularity assumptions of the existence theorem, assume that (u, e, p, σ) satisfies $\mathbf{E}u(t) = e(t) + p(t)$ and $\sigma(t) = A^0 e(t) + A^1 \dot{e}(t)$ for a.e. $t \in [0, T]$, as well as the **initial** and **boundary conditions**. Then (u, e, p, σ) satisfies $\ddot{u}(t) = \operatorname{div} \sigma(t)$ and $\dot{p}(t) = \sigma_D(t) - \pi_K \sigma_D(t)$ for a.e. $t \in [0, T]$ if and only if both the following conditions hold:

- **Equilibrium condition**: for a.e. $t \in [0, T]$ we have

$$-\mathcal{H}(q) \leq \langle A^0 e(t), \eta \rangle + \langle A^1 \dot{e}(t), \eta \rangle + \langle \dot{p}(t), q \rangle + \langle \ddot{u}(t), \varphi \rangle \leq \mathcal{H}(-q)$$

for every $(\varphi, \eta, q) \in A(0)$;

- **Energy-dissipation balance**: for every $t \in [0, T]$ we have

$$\begin{aligned} \mathcal{Q}_0(e(t)) + \frac{1}{2} \|\dot{u}(t) - \dot{w}(t)\|_{L^2}^2 + \int_0^t \mathcal{Q}_1(\dot{e}) ds + \int_0^t \|\dot{p}\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot{p}) ds \\ = \int_0^t \langle \sigma, E \dot{w} \rangle ds - \int_0^t \langle \ddot{w}, \dot{u} - \dot{w} \rangle ds + \mathcal{Q}_0(e_0) + \frac{1}{2} \|v_0 - \dot{w}(0)\|_{L^2}^2. \end{aligned}$$

Under the regularity assumptions of the existence theorem, assume that (u, e, p, σ) satisfies $E u(t) = e(t) + p(t)$ and $\sigma(t) = A^0 e(t) + A^1 \dot{e}(t)$ for a.e. $t \in [0, T]$, as well as the **initial** and **boundary conditions**. Then (u, e, p, σ) satisfies $\ddot{u}(t) = \operatorname{div} \sigma(t)$ and $\dot{p}(t) = \sigma_D(t) - \pi_K \sigma_D(t)$ for a.e. $t \in [0, T]$ if and only if both the following conditions hold:

- **Equilibrium condition:** for a.e. $t \in [0, T]$ we have

$$-\mathcal{H}(q) \leq \langle A^0 e(t), \eta \rangle + \langle A^1 \dot{e}(t), \eta \rangle + \langle \dot{p}(t), q \rangle + \langle \ddot{u}(t), \varphi \rangle \leq \mathcal{H}(-q)$$

for every $(\varphi, \eta, q) \in A(0)$;

- **Energy-dissipation balance:** for every $t \in [0, T]$ we have

$$\begin{aligned} \mathcal{Q}_0(e(t)) + \frac{1}{2} \|\dot{u}(t) - \dot{w}(t)\|_{L^2}^2 + \int_0^t \mathcal{Q}_1(\dot{e}) ds + \int_0^t \|\dot{p}\|_{L^2}^2 ds + \int_0^t \mathcal{H}(\dot{p}) ds \\ = \int_0^t \langle \sigma, E \dot{w} \rangle ds - \int_0^t \langle \ddot{w}, \dot{u} - \dot{w} \rangle ds + \mathcal{Q}_0(e_0) + \frac{1}{2} \|v_0 - \dot{w}(0)\|_{L^2}^2. \end{aligned}$$

Proof of the existence theorem

The proof of the existence theorem uses a **discrete time approximation** of the energy formulation.

Given an integer $N \geq 1$, let $\tau := T/N$ and $t_i = i\tau$. We set $u_{-1} = u_0 - \tau v_0$, $w_{-1} = w_0 - \tau \dot{w}(0)$, and $w_i = w(t_i)$. We construct the sequence (u_i, e_i, p_i) with $i = 0, 1, \dots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data. Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \dots, i$. Then $(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the **unique minimizer** on $A(w_{i+1})$ of the functional

$$\frac{1}{2} \langle A^0 e, e \rangle + \frac{1}{2\tau} \langle A^1 (e - e_i), e - e_i \rangle + \frac{1}{2\tau} \|p - p_i\|_{L^2}^2 + \mathcal{H}(p - p_i) + \frac{1}{2} \left\| \frac{u - u_i}{\tau} - \frac{u_i - u_{i-1}}{\tau} \right\|_{L^2}^2.$$

Then we consider continuous time **interpolations** of these functions (and of their discrete time derivatives), and we prove that a suitable subsequence converges, as $N \rightarrow \infty$, to a triple (u, e, p) such that, if we define σ in the natural way, the quadruple (u, e, p, σ) is a solution of our system.

Proof of the existence theorem

The proof of the existence theorem uses a **discrete time approximation** of the energy formulation.

Given an integer $N \geq 1$, let $\tau := T/N$ and $t_i = i\tau$. We set $u_{-1} = u_0 - \tau v_0$, $w_{-1} = w_0 - \tau \dot{w}(0)$, and $w_i = w(t_i)$. We construct the sequence (u_i, e_i, p_i) with $i = 0, 1, \dots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data. Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \dots, i$. Then $(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the **unique minimizer** on $A(w_{i+1})$ of the functional

$$\begin{aligned} & \frac{1}{2} \langle A^0 e, e \rangle + \frac{1}{2\tau} \langle A^1 (e - e_i), e - e_i \rangle + \frac{1}{2\tau} \|p - p_i\|_{L^2}^2 \\ & + \mathcal{H}(p - p_i) + \frac{1}{2} \left\| \frac{u - u_i}{\tau} - \frac{u_i - u_{i-1}}{\tau} \right\|_{L^2}^2. \end{aligned}$$

Then we consider continuous time **interpolations** of these functions (and of their discrete time derivatives), and we prove that a suitable subsequence converges, as $N \rightarrow \infty$, to a triple (u, e, p) such that, if we define σ in the natural way, the quadruple (u, e, p, σ) is a solution of our system.

Proof of the existence theorem

The proof of the existence theorem uses a **discrete time approximation** of the energy formulation.

Given an integer $N \geq 1$, let $\tau := T/N$ and $t_i = i\tau$. We set $u_{-1} = u_0 - \tau v_0$, $w_{-1} = w_0 - \tau \dot{w}(0)$, and $w_i = w(t_i)$. We construct the sequence (u_i, e_i, p_i) with $i = 0, 1, \dots, N$ by induction. First (u_0, e_0, p_0) coincides with the initial data. Let us fix i and let us suppose $(u_j, e_j, p_j) \in A(w_j)$ to have been defined for $j = 0, \dots, i$. Then $(u_{i+1}, e_{i+1}, p_{i+1})$ is defined as the **unique minimizer** on $A(w_{i+1})$ of the functional

$$\frac{1}{2} \langle A^0 e, e \rangle + \frac{1}{2\tau} \langle A^1 (e - e_i), e - e_i \rangle + \frac{1}{2\tau} \|p - p_i\|_{L^2}^2 \\ + \mathcal{H}(p - p_i) + \frac{1}{2} \left\| \frac{u - u_i}{\tau} - \frac{u_i - u_{i-1}}{\tau} \right\|_{L^2}^2.$$

Then we consider continuous time **interpolations** of these functions (and of their discrete time derivatives), and we prove that a suitable subsequence converges, as $N \rightarrow \infty$, to a triple (u, e, p) such that, if we define σ in the natural way, the quadruple (u, e, p, σ) is a solution of our system.

Proof of the existence theorem

The proof of the existence theorem uses a **discrete time approximation** of the energy formulation.

Given an integer $N \geq 1$, let $\tau := T/N$ and $t_i = i\tau$. We set $\mathbf{u}_{-1} = \mathbf{u}_0 - \tau \mathbf{v}_0$, $\mathbf{w}_{-1} = \mathbf{w}_0 - \tau \dot{\mathbf{w}}(0)$, and $\mathbf{w}_i = \mathbf{w}(t_i)$. We construct the sequence $(\mathbf{u}_i, \mathbf{e}_i, \mathbf{p}_i)$ with $i = 0, 1, \dots, N$ by induction. First $(\mathbf{u}_0, \mathbf{e}_0, \mathbf{p}_0)$ coincides with the initial data. Let us fix i and let us suppose $(\mathbf{u}_j, \mathbf{e}_j, \mathbf{p}_j) \in A(\mathbf{w}_j)$ to have been defined for $j = 0, \dots, i$. Then $(\mathbf{u}_{i+1}, \mathbf{e}_{i+1}, \mathbf{p}_{i+1})$ is defined as the **unique minimizer** on $A(\mathbf{w}_{i+1})$ of the functional

$$\begin{aligned} & \frac{1}{2} \langle A^0 \mathbf{e}, \mathbf{e} \rangle + \frac{1}{2\tau} \langle A^1(\mathbf{e} - \mathbf{e}_i), \mathbf{e} - \mathbf{e}_i \rangle + \frac{1}{2\tau} \|\mathbf{p} - \mathbf{p}_i\|_{L^2}^2 \\ & + \mathcal{H}(\mathbf{p} - \mathbf{p}_i) + \frac{1}{2} \left\| \frac{\mathbf{u} - \mathbf{u}_i}{\tau} - \frac{\mathbf{u}_i - \mathbf{u}_{i-1}}{\tau} \right\|_{L^2}^2. \end{aligned}$$

Then we consider continuous time **interpolations** of these functions (and of their discrete time derivatives), and we prove that a suitable subsequence converges, as $N \rightarrow \infty$, to a triple $(\mathbf{u}, \mathbf{e}, \mathbf{p})$ such that, if we define σ in the natural way, the quadruple $(\mathbf{u}, \mathbf{e}, \mathbf{p}, \sigma)$ is a solution of our system.

To describe the behaviour of the solutions as the time dependent data w becomes slower and slower, we perform to the **change of variables** discussed at the beginning of the talk and we are lead to the study of the systems

$$\begin{aligned} \mathbf{E} \mathbf{u}^\varepsilon &= \mathbf{e}^\varepsilon + \mathbf{p}^\varepsilon, \\ \sigma^\varepsilon &= \mathbf{A}^0 \mathbf{e}^\varepsilon + \varepsilon \mathbf{A}^1 \dot{\mathbf{e}}^\varepsilon, \\ \varepsilon^2 \ddot{\mathbf{u}}^\varepsilon &= \operatorname{div} \sigma^\varepsilon, \\ \varepsilon \dot{\mathbf{p}}^\varepsilon &= \sigma_D^\varepsilon - \pi_K \sigma_D^\varepsilon, \end{aligned}$$

as $\varepsilon \rightarrow 0+$.

To simplify the exposition we assume that the boundary condition satisfies $w \in L^\infty(H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0(L^2(\Omega; \mathbb{R}^n)) \cap L^2(H^1(\Omega; \mathbb{R}^n))$, and $\ddot{w} \in L^2(H^{-1}(\Omega; \mathbb{R}^n))$. Moreover we assume that $(u_0, e_0, p_0) \in A(w(0))$ and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. More general results can be obtained when the boundary conditions and the initial conditions depend on ε .

To describe the behaviour of the solutions as the time dependent data w becomes slower and slower, we perform to the **change of variables** discussed at the beginning of the talk and we are lead to the study of the systems

$$\begin{aligned} \mathbb{E}u^\varepsilon &= e^\varepsilon + p^\varepsilon, \\ \sigma^\varepsilon &= A^0 e^\varepsilon + \varepsilon A^1 \dot{e}^\varepsilon, \\ \varepsilon^2 \ddot{u}^\varepsilon &= \operatorname{div} \sigma^\varepsilon, \\ \varepsilon \dot{p}^\varepsilon &= \sigma_D^\varepsilon - \pi_K \sigma_D^\varepsilon, \end{aligned}$$

as $\varepsilon \rightarrow 0+$.

To simplify the exposition we assume that the boundary condition satisfies $w \in L^\infty(H^1(\Omega; \mathbb{R}^n))$, $\dot{w} \in C^0(L^2(\Omega; \mathbb{R}^n)) \cap L^2(H^1(\Omega; \mathbb{R}^n))$, and $\ddot{w} \in L^2(H^{-1}(\Omega; \mathbb{R}^n))$. Moreover we assume that $(u_0, e_0, p_0) \in A(w(0))$ and $v_0 \in L^2(\Omega; \mathbb{R}^n)$. More general results can be obtained when the boundary conditions and the initial conditions depend on ε .

We shall prove that $(u^\varepsilon, e^\varepsilon, p^\varepsilon, \sigma^\varepsilon)$ converges, up to a subsequence, to a solution of the **quasistatic evolution** problem in **perfect plasticity**, formally written as

$$Eu = e + p,$$

$$\sigma = A^0 e,$$

$$\operatorname{div} \sigma = 0,$$

$$\sigma_D \in K \text{ and } \dot{p} \in N_K(\sigma_D),$$

where $N_K(\sigma_D)$ denotes the **normal cone** to K at σ_D .

The precise formulation of the notion of solution requires the space $BD(\Omega)$ of functions of **bounded deformation**, defined as

$$BD(\Omega) := \{u \in L^1(\Omega; \mathbb{R}^n) : Eu \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})\}.$$

Here and henceforth $\mathcal{M}_b(X; Y)$ is the space of **bounded Radon measures** on the locally compact space X with values in the finite dimensional vector space Y .

We shall prove that $(\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{p}^\varepsilon, \boldsymbol{\sigma}^\varepsilon)$ converges, up to a subsequence, to a solution of the **quasistatic evolution** problem in **perfect plasticity**, formally written as

$$\mathbf{E}\mathbf{u} = \mathbf{e} + \mathbf{p},$$

$$\boldsymbol{\sigma} = \mathbf{A}^0 \mathbf{e},$$

$$\operatorname{div} \boldsymbol{\sigma} = 0,$$

$$\boldsymbol{\sigma}_D \in \mathbf{K} \quad \text{and} \quad \dot{\mathbf{p}} \in \mathbf{N}_K(\boldsymbol{\sigma}_D),$$

where $\mathbf{N}_K(\boldsymbol{\sigma}_D)$ denotes the **normal cone** to $\mathbf{K}$ at $\boldsymbol{\sigma}_D$.

The precise formulation of the notion of solution requires the space $\mathbf{BD}(\Omega)$ of functions of **bounded deformation**, defined as

$$\mathbf{BD}(\Omega) := \{\mathbf{u} \in L^1(\Omega; \mathbb{R}^n) : \mathbf{E}\mathbf{u} \in \mathcal{M}_b(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})\}.$$

Here and henceforth $\mathcal{M}_b(X; Y)$ is the space of **bounded Radon measures** on the locally compact space X with values in the finite dimensional vector space Y .

To study perfect plasticity, we have to modify the notion of kinematic admissibility. Given $w \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple (u, e, p) is **BD-kinematically admissible**, and write $(u, e, p) \in A_{BD}(w)$, if $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, $p \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$, $Eu = e + p$ on Ω , and $p|_{\partial\Omega} = (w|_{\partial\Omega} - u|_{\partial\Omega}) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1}|_{\partial\Omega}$ on $\partial\Omega$, where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$, $\odot$ is the symmetrized tensor product, and $\mathcal{H}^{n-1}$ is the $(n-1)$ -dimensional Hausdorff measure.

Since now $p \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$, we also need to redefine $\mathcal{H}(p)$ by setting

$$\mathcal{H}(p) = \int_{\Omega} H\left(\frac{dp}{d|p|}\right) d|p|,$$

where $|p|$ is the total variation of the measure p and $\frac{dp}{d|p|}$ denotes the Radon-Nikodym derivative.

To study perfect plasticity, we have to modify the notion of kinematic admissibility. Given $w \in H^1(\Omega, \mathbb{R}^n)$, we say that a triple (u, e, p) is **BD-kinematically admissible**, and write $(u, e, p) \in A_{BD}(w)$, if $u \in BD(\Omega)$, $e \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, $p \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$, $Eu = e + p$ on Ω , and $p|_{\partial\Omega} = (w|_{\partial\Omega} - u|_{\partial\Omega}) \odot \nu_{\partial\Omega} \mathcal{H}^{n-1}|_{\partial\Omega}$ on $\partial\Omega$, where $\nu_{\partial\Omega}$ is the outer unit normal to $\partial\Omega$, $\odot$ is the symmetrized tensor product, and $\mathcal{H}^{n-1}$ is the $(n-1)$ -dimensional Hausdorff measure.

Since now $p \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$, we also need to redefine $\mathcal{H}(p)$ by setting

$$\mathcal{H}(p) = \int_{\Omega} H\left(\frac{dp}{d|p|}\right) d|p|,$$

where $|p|$ is the total variation of the measure p and $\frac{dp}{d|p|}$ denotes the Radon-Nikodym derivative.

The term $\int_0^t \mathcal{H}(\dot{p}) ds$ must also be replaced, because now $t \mapsto p(t)$ is not assumed to be regular in the energy formulation of the quasistatic evolution. This term will be replaced by the **plastic dissipation**, defined by

$$\mathcal{D}_H(0, t; p) := \sup \sum_{i=1}^N \mathcal{H}(p(t_i) - p(t_{i-1})), \quad (1)$$

where the supremum is taken over all the possible choices of the integer $N > 0$ and of the real numbers $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = t$. One can prove that, if $p: [0, T] \rightarrow \mathcal{M}_b(\bar{\Omega}; \mathbb{M}_D^{n \times n})$ is absolutely continuous, then

$$\mathcal{D}_H(0, t; p) = \int_0^t \mathcal{H}(\dot{p}(s)) ds,$$

where $\dot{p}$ is the derivative of p defined by

$$\dot{p}(t) := w^* - \lim_{s \rightarrow t} \frac{p(s) - p(t)}{s - t}.$$

The term $\int_0^t \mathcal{H}(\dot{p}) ds$ must also be replaced, because now $t \mapsto p(t)$ is not assumed to be regular in the energy formulation of the quasistatic evolution. This term will be replaced by the **plastic dissipation**, defined by

$$\mathcal{D}_H(0, t; p) := \sup \sum_{i=1}^N \mathcal{H}(p(t_i) - p(t_{i-1})), \quad (1)$$

where the supremum is taken over all the possible choices of the integer $N > 0$ and of the real numbers $0 = t_0 < t_1 < \dots < t_{N-1} < t_N = t$. One can prove that, if $p: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is absolutely continuous, then

$$\mathcal{D}_H(0, t; p) = \int_0^t \mathcal{H}(\dot{p}(s)) ds,$$

where $\dot{p}$ is the derivative of p defined by

$$\dot{p}(t) := w^* - \lim_{s \rightarrow t} \frac{p(s) - p(t)}{s - t}.$$

Energy formulation of the quasistatic evolution

Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, and $p_0 \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$. A quasistatic evolution in perfect plasticity with initial conditions u_0 , e_0 , p_0 , and boundary condition w is a function (u, e, p, σ) from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, with

$$\begin{aligned} u(0) &= u_0, \quad e(0) = e_0, \quad p(0) = p_0, \\ \sigma(t) &= A^0 e(t) \quad \text{for every } t \in [0, T], \end{aligned}$$

such that $t \mapsto p(t)$ has bounded variation and the following two conditions are satisfied for every $t \in [0, T]$:

- Minimality: $(u(t), e(t), p(t)) \in \Lambda_{BD}(w(t))$ and $Q_0(e(t)) \leq Q_0(\eta) + \mathcal{H}(q - p(t))$ for every $(\varphi, \eta, q) \in \Lambda_{BD}(w(t))$;
- Energy-dissipation balance: $Q_0(e(t)) + \mathcal{D}_H(p; 0, t) = Q_0(e_0) + \int_0^t \langle \sigma, Ew \rangle ds$.

Energy formulation of the quasistatic evolution

Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, and $p_0 \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$. A **quasistatic evolution in perfect plasticity** with initial conditions u_0 , e_0 , p_0 , and boundary condition w is a function (u, e, p, σ) from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, with

$$\begin{aligned} u(0) &= u_0, \quad e(0) = e_0, \quad p(0) = p_0, \\ \sigma(t) &= A^0 e(t) \quad \text{for every } t \in [0, T], \end{aligned}$$

such that $t \mapsto p(t)$ has **bounded variation** and the following two conditions are satisfied for every $t \in [0, T]$:

- **Minimality:** $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and

$$\mathcal{Q}_0(e(t)) \leq \mathcal{Q}_0(\eta) + \mathcal{H}(q - p(t)) \quad \text{for every } (\varphi, \eta, q) \in A_{BD}(w(t));$$

- **Energy-dissipation balance:**

$$\mathcal{Q}_0(e(t)) + \mathcal{D}_H(p; 0, t) = \mathcal{Q}_0(e_0) + \int_0^t \langle \sigma, E\dot{w} \rangle ds.$$

Energy formulation of the quasistatic evolution

Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, and $p_0 \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$. A **quasistatic evolution in perfect plasticity** with initial conditions u_0 , e_0 , p_0 , and boundary condition w is a function (u, e, p, σ) from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \times \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \times L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n})$, with

$$\begin{aligned} u(0) &= u_0, \quad e(0) = e_0, \quad p(0) = p_0, \\ \sigma(t) &= A^0 e(t) \quad \text{for every } t \in [0, T], \end{aligned}$$

such that $t \mapsto p(t)$ has **bounded variation** and the following two conditions are satisfied for every $t \in [0, T]$:

- **Minimality:** $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and

$$Q_0(e(t)) \leq Q_0(\eta) + \mathcal{H}(q - p(t)) \quad \text{for every } (\varphi, \eta, q) \in A_{BD}(w(t));$$

- **Energy-dissipation balance:**

$$Q_0(e(t)) + \mathcal{D}_H(p; 0, t) = Q_0(e_0) + \int_0^t \langle \sigma, E\dot{w} \rangle ds.$$

Energy formulation of the quasistatic evolution

Let $u_0 \in BD(\Omega)$, $e_0 \in L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, and $p_0 \in \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$. A **quasistatic evolution in perfect plasticity** with initial conditions u_0 , e_0 , p_0 , and boundary condition w is a function (u, e, p, σ) from $[0, T]$ into $BD(\Omega) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n}) \times \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \times L^2(\Omega; \mathbb{M}_{sym}^{n \times n})$, with

$$\begin{aligned} u(0) &= u_0, \quad e(0) = e_0, \quad p(0) = p_0, \\ \sigma(t) &= A^0 e(t) \quad \text{for every } t \in [0, T], \end{aligned}$$

such that $t \mapsto p(t)$ has **bounded variation** and the following two conditions are satisfied for every $t \in [0, T]$:

- **Minimality:** $(u(t), e(t), p(t)) \in A_{BD}(w(t))$ and

$$Q_0(e(t)) \leq Q_0(\eta) + \mathcal{H}(q - p(t)) \quad \text{for every } (\varphi, \eta, q) \in A_{BD}(w(t));$$

- **Energy-dissipation balance:**

$$Q_0(e(t)) + \mathcal{D}_H(p; 0, t) = Q_0(e_0) + \int_0^t \langle \sigma, E\dot{w} \rangle ds.$$

Theorem (DM-Scala 2013)

Let $(\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{p}^\varepsilon, \sigma^\varepsilon)$ be the solution of the ε -problems with initial data $(\mathbf{u}_0, \mathbf{e}_0, \mathbf{p}_0) \in A(\mathbf{w}(0))$, $\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^n)$, and boundary condition $\mathbf{w}$ satisfying $\mathbf{w} \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{\mathbf{w}} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\dot{\mathbf{w}} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, and $\ddot{\mathbf{w}} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$. Then there exist a quasistatic evolution in perfect plasticity $(\mathbf{u}, \mathbf{e}, \mathbf{p}, \sigma)$, with the same initial and boundary conditions, and a subsequence of $(\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{p}^\varepsilon, \sigma^\varepsilon)$, not relabelled, such that

$$\begin{aligned} \mathbf{u}^\varepsilon(t) &\rightharpoonup \mathbf{u}(t) \quad \text{weakly* in } BD(\Omega) \text{ for a.e. } t \in [0, T], \\ \mathbf{e}^\varepsilon(t) &\rightarrow \mathbf{e}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ for a.e. } t \in [0, T], \\ \mathbf{p}^\varepsilon(t) &\rightharpoonup \mathbf{p}(t) \quad \text{weakly* in } \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \text{ for every } t \in [0, T]. \end{aligned}$$

Theorem (DM-Scala 2013)

Let $(\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{p}^\varepsilon, \sigma^\varepsilon)$ be the solution of the ε -problems with initial data $(\mathbf{u}_0, \mathbf{e}_0, \mathbf{p}_0) \in A(\mathbf{w}(0))$, $\mathbf{v}_0 \in L^2(\Omega; \mathbb{R}^n)$, and boundary condition $\mathbf{w}$ satisfying $\mathbf{w} \in L^\infty([0, T]; H^1(\Omega; \mathbb{R}^n))$, $\dot{\mathbf{w}} \in C^0([0, T]; L^2(\Omega; \mathbb{R}^n))$, $\dot{\mathbf{w}} \in L^2([0, T]; H^1(\Omega; \mathbb{R}^n))$, and $\ddot{\mathbf{w}} \in L^2([0, T]; H^{-1}(\Omega; \mathbb{R}^n))$. Then there exist a *quasistatic evolution in perfect plasticity* $(\mathbf{u}, \mathbf{e}, \mathbf{p}, \sigma)$, with the same initial and boundary conditions, and a *subsequence* of $(\mathbf{u}^\varepsilon, \mathbf{e}^\varepsilon, \mathbf{p}^\varepsilon, \sigma^\varepsilon)$, not relabelled, such that

$$\mathbf{u}^\varepsilon(t) \rightharpoonup \mathbf{u}(t) \quad \text{weakly* in } BD(\Omega) \text{ for a.e. } t \in [0, T],$$

$$\mathbf{e}^\varepsilon(t) \rightarrow \mathbf{e}(t) \quad \text{strongly in } L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}) \text{ for a.e. } t \in [0, T],$$

$$\mathbf{p}^\varepsilon(t) \rightharpoonup \mathbf{p}(t) \quad \text{weakly* in } \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n}) \text{ for every } t \in [0, T].$$

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ weakly* in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ weakly* in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $Eu^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ weakly* in $L^\infty([0, T]; BD(\Omega))$, hence weakly* in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ **weakly*** in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ **weakly*** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $Eu^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ **weakly*** in $L^\infty([0, T]; BD(\Omega))$, hence **weakly*** in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ **weakly*** in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ **weakly*** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $\text{Eu}^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ **weakly*** in $L^\infty([0, T]; \text{BD}(\Omega))$, hence **weakly*** in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ **weakly*** in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ **weakly*** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $Eu^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ **weakly*** in $L^\infty([0, T]; BD(\Omega))$, hence **weakly*** in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ **weakly*** in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ **weakly*** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $\text{Eu}^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ **weakly*** in $L^\infty([0, T]; \text{BD}(\Omega))$, hence **weakly*** in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

Sketch of the proof

Using the energy formulation for the ε -problem we prove that e^ε is **bounded** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$ and $\dot{p}^\varepsilon$ is **bounded** in $L^1([0, T]; L^1(\Omega; \mathbb{M}_D^{n \times n}))$. This implies that $p^\varepsilon: [0, T] \rightarrow \mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ have **uniformly bounded variation**. Since $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ is the dual of a separable Banach space, a variant of **Helly Theorem** implies that there exists a subsequence, still denoted by p^ε , such that $p^\varepsilon(t) \rightharpoonup p(t)$ **weakly*** in $\mathcal{M}_b(\overline{\Omega}; \mathbb{M}_D^{n \times n})$ for every $t \in [0, T]$. Moreover, there exists a further subsequence, not relabelled, such that $e^\varepsilon \rightharpoonup e$ **weakly*** in $L^\infty([0, T]; L^2(\Omega; \mathbb{M}_{\text{sym}}^{n \times n}))$. Since $\text{Eu}^\varepsilon = e^\varepsilon + p^\varepsilon$ in Ω and $u^\varepsilon = w$ on $\partial\Omega$, we conclude that $u^\varepsilon \rightharpoonup u$ **weakly*** in $L^\infty([0, T]; \text{BD}(\Omega))$, hence **weakly*** in $L^\infty([0, T]; L^{n/(n-1)}(\Omega; \mathbb{R}^n))$. Then we pass to the limit in the **equilibrium condition** and in the **energy-dissipation balance** for the ε -problem and we obtain the **minimality condition** and in the **energy-dissipation balance** for the quasistatic evolution. The **strong** convergence of e^ε is obtained from the energy-dissipation balance, which implies the convergence of the norms.

THANK YOU!