Esercitazioni di ANALISI MATEMATICA II

A.a. 2021/22

$$\begin{cases} u'(t) = \cosh^2 u(t) \\ u(0) = 1 \end{cases}$$

$$u'(t) = f(t, u(t)) = a(t)\, b(u(t))$$

$$t \in I = [t_0, T]$$

$$b(y) \neq 0 \quad \forall y \in \mathbb{R}$$

$$\frac{u'(t)}{b(u(t))} = a(t)$$

$$\int_{t_0}^{t} \frac{u'(s)}{b(u(s))}\, ds = \int_{t_0}^{t} a(s)\, ds$$

$$y = u(t)$$

$$dy = u'(t)\, dt$$

$$\int_{u(t_0) = u_0}^{u(t)} \frac{dy}{b(y)} = \int_{t_0}^{t} a(s)\, ds$$

$$B(y) = \int_{t_0}^{t} a(s)\, ds$$

$$y = B^{-1}\left(\int_{t_0}^{t} e(s)\,ds\right)$$

$$\mu(t) = e\, B^{-1}\left(\int_{t_0}^{t} e(\gamma)\,d\gamma\right)$$

$$\begin{cases} \mu'(t) = \cosh^2 \mu(t) \\ \mu(0) = 1 \end{cases}$$

$$e(t) = 1 \quad \forall\, t \in [0,T]$$

$$b(y) = \cosh^{2,}(y)$$

$$\frac{\mu'(t)}{\cosh^2 \mu(t)} = 1$$

$$\int_0^t \frac{\mu'(s)}{\cosh^2(\mu(s))}\,ds = \int_0^t 1\,ds$$

$$y = \mu(t) \quad dy = \mu'(t)\,dt$$

$$\int_{\mu(0)}^{\mu(t)} \frac{dy}{\cosh^2(y)} = t$$

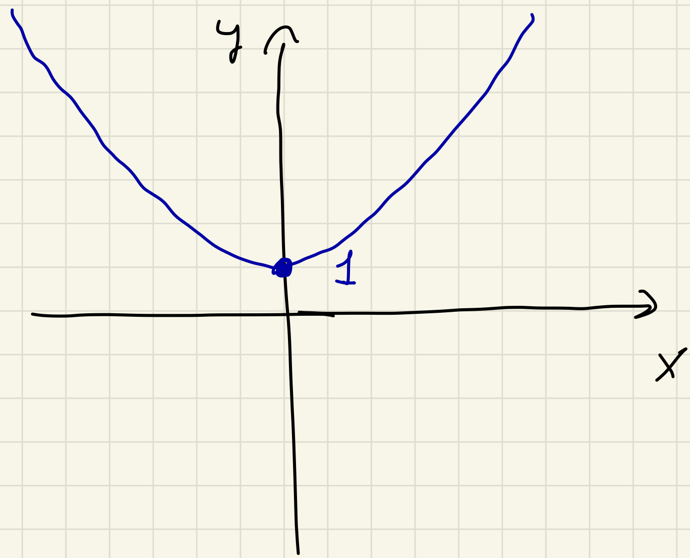
$$\int_{u(0)}^{u(t)} \frac{dy}{\cosh^2(y)} = t$$

$$\tanh(y)\Big|_{1}^{u(t)} = t$$

$$\tanh(u(t)) - \tanh(1) = t$$

$$\tanh(u(t)) = t + \tanh(1)$$

$$u(t) = \tanh^{-1}(t + \tanh(1))$$



$$\begin{cases} u'(t) = \exp(t - u(t)) \\ u(0) = 0 \end{cases}$$

$$f(t,y) = \exp(t-y) = \exp(t)\exp(-y)$$

$$a(t) = \exp(t) \qquad b(y) = \exp(-y)$$

$$b(y) \neq 0 \quad \forall y \in \mathbb{R}$$

$$\int_{u(0)}^{u(t)} \frac{dy}{\exp(-y)} = \int_0^t \exp(s)\,ds$$

$$\int_{u(0)}^{u(t)} \exp(y)\,dy = \int_0^t \exp(s)\,ds$$

$$\exp(y)\Big|_{u(0)}^{u(t)} = \exp(t) - 1$$

$$\exp(u(t)) - \overbrace{\exp(u_0)}^{\exp(0)=1} = \exp(t) - 1$$

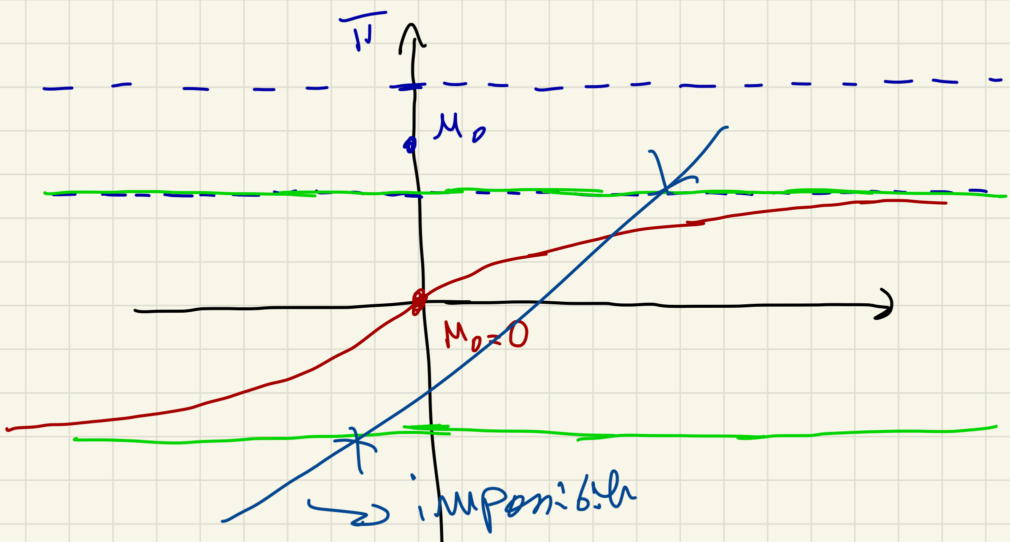
$$\exp(u(t)) \cancel{-1} = \exp(t) \cancel{-1}$$

$$\exp(u(t)) = \exp(t)$$

$$\exp(u(t)) = \exp(t)$$

$$u(t) = t \quad \forall t \in [0,T] \quad (\forall t \in [0,+\infty))$$

$$\begin{cases} u'(t) = \cos^2(u(t)) \\ u(0) = u_0 \end{cases} \qquad u_0 \in (-\pi/2, \pi/2)$$



$$f(t,y) = \cos^2(y) \qquad a(t) = 1 \quad \forall t \in [0,T]$$

$$b(y) = \cos^2(y)$$

$$\cos(y) = 0 \qquad y = \pi/2$$

$$u_0 = \pi/2 \Rightarrow \cos^2(u_0) = 0$$

$u(t) = u_0$ è soluzione del problema di CAUCHY completo quando $u_0 = \pi/2$

$$\begin{cases} u'(t) = \cos^2(u(t)) \\ u_0 = \pi/2 \end{cases}$$

$u(t) = \pi/2 \quad \forall t \in [0, \pi]$

$\Rightarrow \quad u'(t) = 0$

EQUAZIONI DIFFERENZIALI LINEARI DEL PRIMO ORDINE (COEFF. VARIABILI)

$$\begin{cases} u'(t) = a(t)u(t) + b(t) & t \in I \\ u(t_0) = u_0 & t_0 \in I \end{cases}$$

$u(t_0)$

$a, b \in C^0(I)$

$$A(t) := \int_{t_0}^{t} a(s)\,ds$$

$$\frac{d}{dt}\left(u(t) \exp(-A(t)) = u'(t) \exp(-A(t)) - \right.$$
$$+ u(t)\, a(t) \exp(-A(t)) = \cdot$$

$$= \exp(-A(t))\,(\underbrace{u'(t) - a(t)\,u(t)}_{=\,b(t)}) =$$

$$= \exp(-A(t))\, b(t)$$

$$\frac{d}{dt}\left(u(t)\exp(-A(t))\right) = \exp(-A(t))\,b(t)$$

$$u(t)\exp(-A(t)) = c + \int_{t_0}^{t} b(s)\exp(-A(s))\,ds$$

$$u(t) = c\exp(+A(t)) + \exp(+A(t))\int_{t_0}^{t} b(s)\exp(-A(s))\,ds$$

$$u(t_0) = u_0$$

Valuto in t_0:

$$u_0 = c\exp(A(t_0)) \rightarrow c = u_0\exp(-A(t_0))$$

$$= u_0$$

$$u(t) = u_0\exp(A(t)) + \exp(A(t))\int_{t_0}^{t} b(s)\exp(A(s))\,ds$$

$\boxed{ES.}$ $\mu'(t) - 2\mu(t) = t \qquad \mu(0) = 0$

$$\mu'(t) = 2\mu(t) + t$$

$$\mu(t) = c\exp(+A(t)) + \exp(+A(t))\int_{t_0}^{t} b(s)\exp(-A(s))ds$$

$$A(t) = \int_0^t 2ds = 2t$$

$$\mu(t) = c\exp(2t) + \exp(2t)\int_0^t s\exp(-2s)ds$$

$$\int_0^t s\exp(-2s)ds = s\left(-\frac{1}{2}\exp(-2s)\right)\Big|_0^t =$$

$$-\int_0^t \left(-\tfrac{1}{2}\exp(-2s)\right)ds =$$

$$= -\frac{1}{2}t\exp(-2t) - \frac{1}{4}\exp(-2s)\Big|_0^t =$$

$$= -\frac{1}{2}t\exp(-2t) - \tfrac{1}{4}\exp(-2t) + \tfrac{1}{4}$$

$$\rightarrow \mu(t) = c\exp(2t) + \exp(2t)\left(-\tfrac{1}{2}t\exp(-2t) + \tfrac{1}{4} - \tfrac{1}{4}\exp(-2t)\right) =$$

$\mu(t) = c\exp(2t) + \exp(2t)\left(-\frac{1}{2}t\exp(-2t) + \frac{1}{4}\right.$
$\left. - \frac{1}{4}\exp(-2t)\right) = c\exp(2t) - \frac{1}{2}t + \frac{1}{4}\exp(2t)$
$- \frac{1}{4}$

$$\mu(t) = c\exp(2t) - \frac{1}{2}t + \frac{1}{4}\exp(2t) - \frac{1}{4}$$

$$0 = c + \cancel{\frac{1}{4}} - \cancel{\frac{1}{4}}$$

$$\mu(t) = -\frac{1}{2}t + \frac{1}{4}\exp(2t) - \frac{1}{4}$$

$$\begin{cases} \mu'(t) = 3\,|\mu(t)|^{2/3} & t \in [0,+\infty) \\ \mu(0) = 0 \end{cases}$$

$\mu(t) = 0$ è soluzione costante dell'equazione

$\mu(t) > 0$

$$\mu'(t) = 3\,\mu^{2/3}(t)$$

$$\frac{\mu'(t)}{\mu^{2/3}(t)} = 3$$

$$\frac{\mu'(t)}{\mu^{2/3}(t)} = 3$$

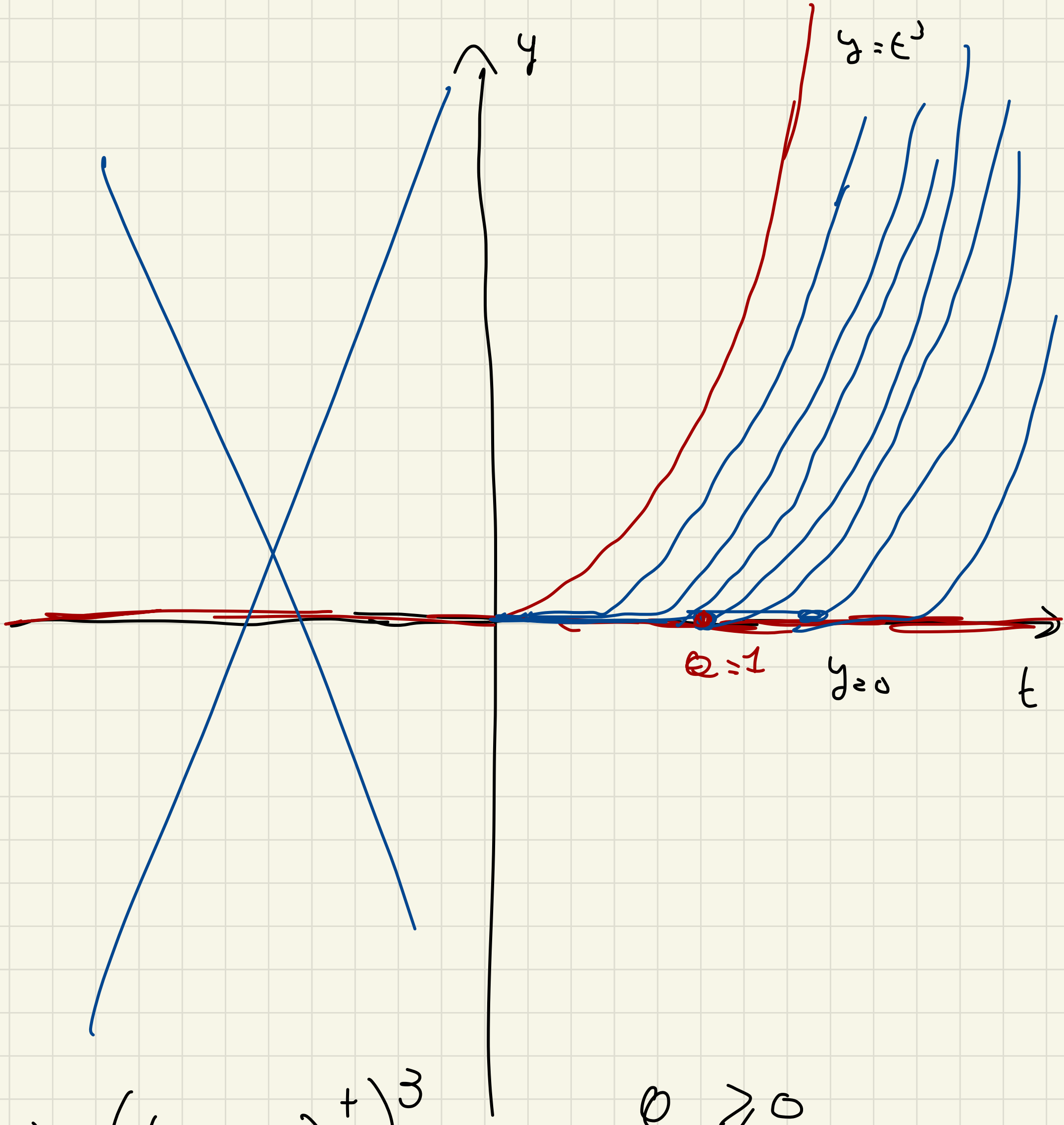
$$\int_0^{\mu(t)} y^{-2/3}\, dy = 3t$$

$$\left.\frac{y^{1/3}}{1/3}\right|_0^{\mu(t)} = 3t$$

$$\cancel{3}\, y^{1/3}\Big|_0^{\mu(t)} = \cancel{3}t$$

$$\mu^{1/3}(t) = t$$

$$\mu(t) = t^3$$



$$u(t) = \left((t-\theta)^+ \right)^3 \qquad \theta \geq 0$$

PENNELLO DI PEANO!

EQUAZIONI DIFFERENZIALI LINEARI DEL 2° ORDINE

COEF. COSTANTI . $a, b, c \in \mathbb{R}$

$$a\, u''(t) + b\, u'(t) + c\, u(t) = f(t)$$

f:
- costanti (quando $f = 0$ problema omogeneo)
- polinomi
- exp
- sin / cos

$$u(t) \in S + \tilde{u}$$

S = spazio delle soluzioni del sistema omogeneo $a\, u''(t) + b\, u'(t) + c\, u(t) = 0$

$$a\, \tilde{u}''(t) + b\, \tilde{u}'(t) + c\, \tilde{u}(t) = f(t)$$

$$u^0 \in S \iff u^0(t) = c_1 u^0_1(t) + c_2 u^0_2(t)$$

$c_1, c_2 \in \mathbb{R}$.

ES.

$$\begin{cases} u''(t) - u'(t) = 2e^{-t} \\ u(0) = 0 \\ u'(0) = -1 \end{cases}$$

SOLUZIONE OMOGENEO:

$$(u^0)''(t) - (u^0)'(t) = 0$$

$$\lambda^2 - \lambda = 0$$

$$\lambda(\lambda - 1) = 0 \iff \lambda = 0 \cup \lambda = 1$$

$$u_1^0(t) = e^{0 \cdot t} = 1$$

$$u_2^0(t) = e^{1 \cdot t} = e^t$$

$$u^0(t) = c_1 + c_2 e^t$$

SOLUZIONE PARTICOLARE:

$\tilde{u}(t) = a e^{-t}$ $\quad a \in \mathbb{R}$ da determinare

$\tilde{u}'(t) = -a e^{-t}$

$\tilde{u}''(t) = a e^{-t}$

$a e^{-t} - (-a e^{-t}) = 2 e^{-t}$

$\cancel{2} a \cancel{e^{-t}} = \cancel{2} \cancel{e^{-t}}$

$a = 1$

$u(t) = c_1 + c_2 e^{t} + e^{-t}$

$u'(t) = c_2 e^{t} - e^{-t}$

$u(0) = 0 = c_1 + c_2 + 1$

$u'(0) = -1 = c_2 - 1$

$$\begin{cases} c_1 + c_2 + 1 = 0 \longrightarrow c_1 = -1 \\ \cancel{-1} = c_2 \cancel{-1} \longrightarrow c_2 = 0 \end{cases}$$

$$\begin{cases} c_1 + c_2 + 1 = 0 \longrightarrow c_1 = -1 \\ \cancel{-1} = c_2 \cancel{-1} \longrightarrow c_2 = 0 \end{cases}$$

$$\boxed{u(t) = -1 + e^{-t}} \longrightarrow$$ Soluzione del problema di Cauchy iniziale

Es.

$$u''(t) + 2u'(t) + 2u(t) = \sin(t) + \cos(t)$$

SOLUZIONE OMOGENEO:

$$(u^0)''(t) + 2(u^0)'(t) + 2u^0(t) = 0$$

$$\lambda^2 + 2\lambda + 2 = 0$$

$$\lambda_{1,2} = -1 \pm \sqrt{1-2} = -1 \pm i \quad \begin{cases} \lambda_1 = -1 + i \\ \lambda_2 = -1 - i \end{cases}$$

$$u_1^0(t) = \exp(\mathrm{Re}(\lambda_1)) \cos(|\mathrm{Im}(\lambda_1)| t)$$

$$u_2^0(t) = \exp(\mathrm{Re}(\lambda_2)) \cos(|\mathrm{Im}(\lambda_2)| t)$$

$$u^0(t) = c_1 \exp(-t) \cos(t) + c_2 \exp(-t) \sin(t)$$

SOLUZIONE PARTICOLARE:

$A, B \in \mathbb{R}$ da determinare

$$\tilde{\mu}(t) = A\sin(t) + B\cos(t)$$

$$\tilde{\mu}'(t) = A\cos(t) - B\sin(t)$$

$$\tilde{\mu}''(t) = -A\sin(t) - B\cos(t)$$

$$-A\sin(t) - B\cos(t) + 2\left(A\cos(t) - B\sin(t)\right) + 2\left(A\sin(t) + B\cos(t)\right) =$$
$$= \sin(t) + \cos(t)$$

$$-A\sin(t) - B\cos(t) + 2A\cos(t) - 2B\sin(t) + 2A\sin(t) + 2B\cos(t) =$$
$$= \sin(t) + \cos(t)$$

$$\sin(t)\left[A - 2B\right] + \cos(t)\left[B + 2A\right] = \sin(t) + \cos(t)$$

$$\begin{cases} A - 2B = 1 \rightarrow A = 1 + 2B \\ B + 2A = 1 \rightarrow B + 2 + 4B = 1 \end{cases}$$

$$5B = -1$$

$$B = -1/5 \rightarrow A = 3/5$$

$$\tilde{u}(t) = \frac{3}{5}\sin(t) - \frac{1}{5}\cos(t)$$

$$u(t) = c_1 e^{-t}\cos(t) + c_2 e^{-t}\sin(t) + \frac{3}{5}\sin(t) - \frac{1}{5}\cos(t)$$

soluzione completa.

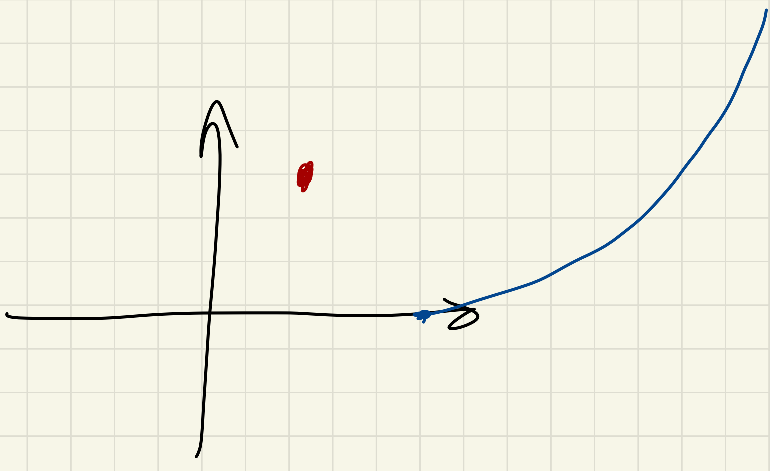
LIMITI E CONTINUITÀ IN PIÙ VARIABILI

$A \subseteq \mathbb{R}^n \qquad f: A \to \mathbb{R}^m \qquad n, m \in \mathbb{N}$

$x_0 \in A$ dico che f è continua in x_0

se $\forall J$ intorno di $f(x_0)$ $\exists I$ int. x_0

tale che $\forall x \in I \cap A$ risulti $f(x) \in J$



1) $f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} x & \text{se } (x,y) \neq (0,1) \\ 2 & \text{se } (x,y) = (0,1) \end{cases}$$

$\forall \{e_n\}_{n \in \mathbb{N}} \subseteq A$ t.c. $e_n \xrightarrow{n \to \infty} x_0$ allora

$f(e_n) \xrightarrow{n \to \infty} f(x_0) \iff f$ è continua.

$$e_n = \left(\frac{1}{n}, 1\right) \longrightarrow (0,1) = \underline{x}_0$$

$$f(e_n) = f\left(\frac{1}{n}, 1\right) = \frac{1}{n} \xrightarrow{n \to \infty} 0$$

$f(\underline{x}_0) = f(0,1) = 2 \implies f$ NON PUÒ ESSERE CONTINUA IN $(0,1)$.

$g(x,y) = x \quad \forall (x,y) \in \mathbb{R}$ è continua perché funzione elementare ed estende f e una funzione continua.

$$f(x,y) = \begin{cases} \dfrac{y}{x^2+y^2} & x\ y > 0 \\ 0 & x\ y \leq 0 \end{cases}$$

$$(x,y) = (0,0)$$

$$\lim_{(x,y)\to(0,0)} \frac{y}{x^2+y^2} \overset{?}{=} f(0,0) = 0$$

$$\{(x,y) \in \mathbb{R}^2 : x = y, y > 0\} := \mathcal{C}$$

$$\lim_{x\to0^+} f(x,x) = \lim_{x\to0^+} \frac{x}{2x^2} = \lim_{x\to0^+} \frac{1}{2x} = +\infty$$

$$f(x,y) = \frac{xy}{x^2+y^2}$$

$\mathcal{C} = \{ (x,y) \in \mathbb{R}^2 : \exists\, m \in \mathbb{R}$ che verifica $y = mx \}$

$$\lim_{x\to0} f(x, mx)$$

$$\lim_{x \to 0} f(x, mx) = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2 x^2} =$$

$$= \lim_{x \to 0} \frac{m \cancel{x^2}}{(m^2+1)\cancel{x^2}} = \frac{m}{m^2+1}$$

Allora $\forall m \in \mathbb{R}$ si ha un valore diverso del limite lungo tale restrizione (Ad. es). se $m = 1 \longrightarrow$ il limite fa $1/2$

se $m = 0$ il limite fa 0) $\Rightarrow$ tale limite $\nexists$ e la funzione è quindi DISCONTINUA in $(0;0)$.