Global solution to a singular integrodifferential system related to the entropy balance

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Abstract. This paper is devoted to the mathematical analysis of a thermodynamic model describing phase transitions with thermal memory in terms of an entropy equation and a momentum balance for the microforces. The initial and boundary value problem is addressed for the related integro-differential system of PDE's. Existence and uniqueness, continuous dependence on the data, and regularity results are proved for the global solution, in a finite time interval.

Key words: entropy equation, thermal memory, phase field model, nonlinear partial differential equations, existence and uniqueness, regularity of solutions.

2000 Mathematics Subject Classification: 35K60, 35Q99, 80A22.

1. Introduction

This paper deals with a singular integro-differential system describing a phase transition phenomenon with thermal memory. The system is written in terms of a rescaled balance of energy and a balance law for the microforces that govern the phase transition. The related thermodynamic model is somehow new, and presents interesting features on which we would like to suitably comment. From the analytical point of view, we mainly aim to investigate existence and uniqueness of the solutions, continuous dependence on the data, regularity, and long-time behaviour. Therefore, since the complete treatment of modelling and analytical aspects would lead to a long and thorough manuscript, we prefer to split our investigation into two parts. In particular, this paper will focus only on analytical aspects, while modelling derivation and long-time behaviour of solutions will be the subject of the twin paper [4]. Nonetheless, in order to make the presentation clear from the beginning, let us briefly introduce here the main ingredients of the resulting PDE's system and give some comments on their physical meaning.

We consider a two-phase system located in a smooth bounded domain $\Omega \subset \mathbb{R}^N$, with $N \leq 3$, and let $T > 0$ denote a final time. The unknowns of the problem are the absolute temperature ϑ, and a phase parameter χ which may represent the local proportion of one of the two phases. To ensure thermomechanical consistence, suitable physical constraints on χ are introduced: if it is assumed, e.g., that the two phases may coexist at each point with different proportions, it turns out to be reasonable to require that

$$\chi \in [0,1] \tag{1.1}$$

letting $1-\chi$ denote the proportion of the second phase. In particular, in (1.1) the values $\chi = 0, 1$ correspond to the pure phases, while χ is between 0 and 1 in the cases when both phases are present (i.e., in the so called "mushy region"). Clearly, the model should provide an evolution for χ that complies with the physical constraint (1.1). A way to force χ to attain values in the interval $[0,1]$ consists in including, in the energy functional, the indicator function of the interval $[0,1]$, which is defined for $\xi \in \mathbb{R}$ by $I_{[0,1]}(\xi) := 0$ if $\xi \in [0,1]$ and $I_{[0,1]}(\xi) = +\infty$ otherwise. The presence in the energy functional of a non-smooth convex function, like the indicator function, leads to an evolution equation for χ which involves a maximal monotone graph. Our analysis works in a fairly general framework that allows constraints on the phase parameter, which arise in physical applications and are possibly more general than the subdifferential of the indicator function.

Now, let us state precisely the system of PDE's as well as initial and boundary conditions. The two equations governing the evolution of ϑ and χ are recovered as balance laws. The first equation is an equation for the entropy and is obtained as a reduction of the energy balance equation divided by the absolute temperature ϑ (see [4, formulas (2.33–35)]). In previous contributions this kind of equation has been termed "entropy balance" (cf. the titles of references [5, 6] and our title) to point out that it describes the evolution of the entropy in place of the more usual internal energy. The second equation accounts for the phase dynamics and is deduced from a balance law for the microscopic forces that are responsible for the phase transition process. Concerning the heat flux law and the thermal properties of the system, we allow the material to exhibit some thermal memory effect, i.e., the heat flux (and consequently the entropy flux) is assumed to involve both the present value at time t of the gradient of the temperature and the summed past history of it. Hence, the so-called entropy balance can be written in $\Omega \times (0,T)$ as follows

$$\partial_t (c_s \log \vartheta - \lambda(\chi)) + \operatorname{div}\left(-k_0 \nabla\vartheta - \int_0^\infty k(s) \nabla\vartheta(t-s)\, ds \right) = R \tag{1.2}$$

where $c_s > 0$ represents the specific heat of the system, $k_0 > 0$ is a thermal coefficient for the entropy flux, k is a sufficiently regular thermal memory kernel, the factor $\lambda'(\chi)$ in $\partial_t(\lambda(\chi)) = \lambda'(\chi)\chi_t$ plays as latent heat, and R stands for an external entropy source. We aim to point out that in (1.2) one finds the entropy flux $\mathbf{Q}$, related to the heat flux vector $\mathbf{q}$ by $\mathbf{Q} = \mathbf{q}/\vartheta$, and specified by

$$\mathbf{Q}(t) = -k_0\nabla\vartheta(t) - \int_0^\infty k(s)\nabla\vartheta(t-s)\,ds. \tag{1.3}$$

Let us comment at once on the position (1.3). A general form for the heat conductor is given by

$$\mathbf{q} = -\kappa(\vartheta)\nabla\vartheta$$

where the conductivity $\kappa(\vartheta)$ is a function of the temperature. For many dielectrics, as ice and water, $\kappa(\vartheta)$ can be supposed (see, e.g., [23]) to be a linear function of ϑ. In such a case

$$\mathbf{q} = -k_0\vartheta\nabla\vartheta$$

where $k_0 > 0$. In our paper, it turns out that we actually generalize this constitutive equation to dielectric materials with fading memory on the gradient of temperature (however, cf. [5, 20] as well as our extended comments below). Therefore, we obtain the equation

$$\mathbf{q} = -k_0\vartheta\Big(\nabla\vartheta + \int_0^\infty h(s)\nabla\vartheta(t-s)ds\Big)$$

where $h(t) = k(t)/k_0$ for all $t > 0$. After an integration by parts in time, (1.2) reduces to

$$\partial_t(c_s\log\vartheta - \lambda(\chi)) - \Delta(k_0\vartheta + k * \vartheta) = R \tag{1.4}$$

the convolution product being defined by $(a * b)(t) := \int_0^t a(t-s)b(s)ds$. Further, by abuse of notation, R now denotes an entropy source accounting also for the past history contribution

$$\operatorname{div}\int_{-\infty}^0 k(t-s)\nabla\vartheta(s)ds$$

which in our approach is assumed to be known. Next, according to the Gurtin terminology [19]), the microforce balance reads

$$\chi_t - \nu\Delta\chi + \beta(\chi) + \sigma'(\chi) \ni -\lambda'(\chi)\vartheta \tag{1.5}$$

where ν is a small positive parameter, σ is a smooth real function, and $\beta := \partial j$ denotes the subdifferential of a proper convex lower semicontinuous function $j : \mathbb{R} \to \mathbb{R}\cup\{+\infty\}$ which accounts for physical constraint on χ (e.g., (1.1) if $j(\chi) = I_{[0,1]}(\chi)$). In particular, β turns out to be a maximal monotone graph in $\mathbb{R}\times\mathbb{R}$. The setting of λ, β, and σ depends on the phenomenology we are considering. Even if it is fairly usual that (1.1) is ensured if the effective domain of j is a subset of $[0, 1]$, different choices of σ, β, and λ have been introduced in the literature (see, e.g., [2, 30, 14]). In the case of a solid-liquid phase transition, λ and σ may be chosen in such a way that one of the two minima $\chi = 0$ and $\chi = 1$ is always preferred for equilibrium, depending on whether $\vartheta < \vartheta_c$ or $\vartheta > \vartheta_c$, where ϑ_c denotes the critical phase change temperature. A second interesting situation, included in our modelling approach by a suitable choice of λ and σ , is concerned with the Ising model of ferromagnetism. In this case, the energy functional may assume either two absolute minima with the same value if $\vartheta < \vartheta_c$ (in the presence of two symmetric phase variants) or just one absolute minimum in the midpoint if $\vartheta > \vartheta_c$. Here, ϑ_c represents the so-called Curie temperature. For further details, comments, and examples we refer to our following paper [4] (the reader may also see [13]).

Before proceeding, let us briefly recall some related results in the literature. A key reference is the paper [5], where a first model using the entropy balance to describe phase transitions with thermal memory has been investigated. In [5] the authors deal with an entropy flux law leading to an evolution equation different from (1.4) and reading

$$\partial_t(c_s \log\vartheta - \lambda(\chi)) - \Delta(k_0 \log\vartheta + k * \vartheta) = R. \tag{1.6}$$

Note that here the part involving the present value of the temperature in the entropy flux yields a $\log\vartheta$ (in place of a ϑ) term as an argument of the Laplacian. However, both the approach of [5] and ours offer an important advantage for the related models. Indeed, by the presence of the logarithm of the temperature (at least, under the time derivative) in the entropy equation (1.4) or (1.6), the positivity of the variable representing the absolute temperature follows directly from solving the problem, i.e., from finding a solution component ϑ to which the logarithm applies. This is important since we can avoid the use of other methods, like e.g. the maximum principle, to determine whether ϑ is positive in the space-time domain; and it is known that the maximum principle is not always applicable, especially in the case of integro-differential equations. On the other hand, let us note at once that in (1.4) the global operator mapping ϑ into $-\Delta(k_0\vartheta + k * \vartheta)$ is now linear, which is not the case for the corresponding operator in (1.6). In the context of phase change models with thermal memory, linearity yields a substantial help to recover good analytical results (see, e.g., [7, 10, 11, 18] and references therein). In particular, for our PDE's system we can prove here uniqueness and regularity results (which was not possible in the framework of [5]) and discuss the large time behaviour of the solution in the related paper [4]. To complete the review of the related contributions, two other papers can be mentioned. In [6] a first simplified version of the entropy system was introduced, but neglecting any thermal memory effect as well as local interactions for the phases (i.e., $\nu = 0$ in (1.5)). Then, the results of [5] have been extended in [3] to the case of some thermal anisotropy in the material.

Anyway, we emphasize that both in [5] and in this paper we are assuming thermal memory for the system, i.e., the heat flux vector $\mathbf{q} = \mathbf{q}_p + \mathbf{q}_h$ is the sum of a present value term $\mathbf{q}_p$, depending on ϑ at time t , and an history contribution $\mathbf{q}_h$ involving the summed past history of the temperature. Consequently, the entropy flux $\mathbf{Q} = \mathbf{q}/\vartheta$ is $\mathbf{Q} = \mathbf{Q}_p + \mathbf{Q}_h = (\mathbf{q}_p + \mathbf{q}_h)/\vartheta$, and in our new approach (i.e., (1.4)) the heat flux part $\mathbf{q}_p$ is not provided by the usual Fourier law

$$\mathbf{q}_p(t) = -k_0 \nabla\vartheta(t), \quad k_0 > 0$$

which would yield $\mathbf{Q}_p = -k_0 \nabla \log \vartheta$ as in (1.6), but $\mathbf{q}_p(t)$ is supposed to be proportional to the gradient of the temperature through a non costant thermal coefficient depending on the temperature itself. In particular, we let this coefficient be equal to $-k_0\vartheta$, that is,

$$\mathbf{q}_p(t) = -k_0 \vartheta(t) \nabla \vartheta(t)$$

so that at $\vartheta = 0$ no heat flux is allowed and, in turn, $\mathbf{Q}_p = -k_0 \nabla \vartheta$. This position is in agreement with thermodynamics since it postulates that when the absolute temperature approches zero the heat flux coefficient degenerates. Moreover, it is coherent with the law for the past history contribution of the heat flux $\mathbf{q}_h$, which also in [5] is assumed to depend on a thermal coefficient involving the temperature ϑ and vanishing as $\vartheta \searrow 0$. Thanks to this new diffusion term, which is linear with respect to ϑ , (1.4) presents some mathematical advantages with respect to (1.6): indeed, for our modified model we are able to prove uniqueness of the solution and investigate its long-time behaviour, while in [5] the existence of a solution was proved on a finite time interval $(0, T)$ and the uniqueness result was given only for some regularized version of the system. As you can easily agree, uniqueness and long-time stability are also relevant as physical properties.

About our thermodynamical approach, it turned out that we actually deal with the theory of thermal memory materials in agreement with the position of [9], as we include a present contribution $\mathbf{q}_p(t)$ in the definition of $\mathbf{q}$. However, for a general introduction to the theory of thermal memory, we invite the reader to see the other famous reference [20] and possibly the more recent contributions [15, 17, 21, 22, 31]. Besides, concerning the idea of recovering (1.5) or analogous equations as a balance law for microforces, we mainly refer to two single approaches by Gurtin [19] and Frémond [16] (cf. also [12, 26, 27, 28, 29] for related analytical results and complementary remarks).

The full problem investigated in this paper consists of equations (1.4) and (1.5) coupled with suitable initial and boundary conditions. In particular, we consider a Dirichlet boundary condition for ϑ (absolute temperature known on the boundary Γ of the body Ω) and a Neumann homogeneous boundary condition for χ , which is fairly usual in this kind of problems, as it corresponds to prescribe that no surface forces are applied on the boundary. Then, we set on $\Gamma \times (0, T)$

$$\vartheta = \vartheta_\Gamma, \quad \partial_n \chi = 0 \tag{1.7}$$

where ∂_n denotes the outward normal derivative on Γ . In addition, we prescribe Cauchy conditions for ϑ and χ , i.e.,

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0. \tag{1.8}$$

The resulting system is highly nonlinear, and the main difficulties lie in the coupling of the nonlinear evolutive term in the entropy balance with the convolution product involving the temperature and the nonlinearities in the variational inclusion governing the dynamics of the phase parameter. Nonetheless, for the related initial boundary value problem we can prove the existence and uniqueness of a global solution, as well as some regularity properties, by deducing uniform estimates and using monotonicity and compacteness methods. The possibility of applying our results to other settings, such as different boundary conditions for the temperature (i.e., Neumann or third type boundary conditions), remains, for the moment, not clear as the techniques we are using carefully exploit the structure of the Dirichlet condition in (1.7). However, we will discuss the technical aspects more in detail in the next sections.

Here is the outline of the paper. We state precisely assumptions and main results in Section 2. The existence result (cf. Theorem 2.1) is proved in Section 4 by a priori estimates and passage to the limit, after implementing a double approximating procedure in Section 3. In particular, the inner approximation makes use of a delicate Faedo-Galerkin scheme. Hence, uniqueness and continuous dependence on the data, which are stated in Theorem 2.2, are shown in Section 5. Finally, some further regularity properties of the solution (cf. Theorem 2.3) are recovered in Section 6.

2. Statement of the problem

In this section, we take some care in describing the problem we are going to deal with. Moreover, we list our assumptions and state our results. We start with the assumption on the structure of the system.

We are given four functions $\widehat{\beta}$, λ, σ, k, and a constant k_0 satisfying the conditions listed below.

$$\widehat{\beta} : \mathbb{R} \to [0, +\infty] \text{ is convex, proper, lower semicontinuous, and } \widehat{\beta}(0) = 0 \tag{2.1}$$

$$\lim_{|r| \to +\infty} |r|^{-2} \, \widehat{\beta}(r) = +\infty \tag{2.2}$$

$$\lambda, \sigma \in C^1(\mathbb{R}) \text{ and } \lambda' \text{ and } \sigma' \text{ are Lipschitz continuous} \tag{2.3}$$

$$k \in W^{1,1}(0, +\infty) \quad \text{and} \quad k_0 > 0. \tag{2.4}$$

We define the graph β in $\mathbb{R} \times \mathbb{R}$ by

$$\beta := \partial \widehat{\beta} \tag{2.5}$$

and note that β is maximal monotone and that $\beta(0) \ni 0$. The same symbol β will be used for the maximal monotone operators induced on L^2 spaces.

Next, we list our assumptions on the data. To this aim, we introduce a notation. In the sequel, Ω is a bounded open set in $\mathbb{R}^3$ whose boundary Γ is assumed to be smooth. Given a final time T, we set

$$Q := \Omega \times (0, T) \quad \text{and} \quad \Sigma := \Gamma \times (0, T). \tag{2.6}$$

It is convenient to set also

$$H := L^2(\Omega), \qquad V := H^1(\Omega), \qquad V_0 := H_0^1(\Omega) \tag{2.7}$$

$$W := \big\{ v \in H^2(\Omega) : \ \partial_n v = 0 \big\} \tag{2.8}$$

where $\partial_n v$ is the normal derivative. We endowe H, V, and W with their usual scalar products and norms, and use a self-explaining notation, like $\|\cdot\|_V$. For the sake of simplicity, the same symbol will be used both for a space and for any power of it. We note that the norms $\|v\|_V$ and $\|\nabla v\|_H$ are equivalent for $v \in V_0$, thanks to the Poincaré inequality, and recall that V_0' coincides with the Sobolev space $H^{-1}(\Omega)$. Finally, for $p \in [1,\infty]$, the symbol $\|\cdot\|_p$ denotes the standard norm in $L^p(\Omega)$.

As far as the data of our problem are concerned, let the four functions R, ϑ_Γ, ϑ_0, and χ_0 satisfy

$$R \in L^2(0,T;H) \tag{2.9}$$

$$\vartheta_\Gamma \in C^0([0,T];H^{1/2}(\Gamma)) \cap W^{1,1}(0,T;L^\infty(\Gamma)) \cap H^1(0,T;H^{-1/2}(\Gamma)),$$
$$\vartheta_\Gamma > 0 \quad \text{a.e. on } \Sigma \quad \text{and} \quad 1/\vartheta_\Gamma \in L^\infty(\Sigma) \tag{2.10}$$

$$\vartheta_0 \in L^\infty(\Omega), \quad \vartheta_0 > 0 \quad \text{a.e. in } \Omega, \quad \text{and} \quad 1/\vartheta_0 \in L^\infty(\Omega) \tag{2.11}$$

$$\chi_0 \in V \quad \text{and} \quad \widehat{\beta}(\chi_0) \in L^1(\Omega). \tag{2.12}$$

Due to (2.10–11), there exist two positive constants ϑ_* and ϑ^* such that

$$\vartheta_* \le \vartheta_\Gamma \le \vartheta^* \quad \text{a.e. on } \Sigma \quad \text{and} \quad \vartheta_* \le \vartheta_0 \le \vartheta^* \quad \text{a.e. in } \Omega. \tag{2.13}$$

The function ϑ_Γ is the boundary datum for the temperature and we would like to consider a function $u := \vartheta - \vartheta_{\mathcal{H}}$ vanishing on the boundary as associated unknown function. Hence, a natural choice of $\vartheta_{\mathcal{H}}$ is the harmonic extension of ϑ_Γ, so that $\Delta u = \Delta\vartheta$. Therefore, we define $\vartheta_{\mathcal{H}} : Q \to \mathbb{R}$ as follows

$$\vartheta_{\mathcal{H}}(t) \in V, \quad \Delta\vartheta_{\mathcal{H}}(t) = 0, \quad \text{and} \quad \vartheta_{\mathcal{H}}(t)|_\Gamma = \vartheta_\Gamma(t) \quad \text{for a.a. } t \in (0,T) \tag{2.14}$$

and note that assumptions (2.10) yield

$$\vartheta_{\mathcal{H}} \in C^0([0,T];V) \cap W^{1,1}(0,T;L^\infty(\Omega)) \cap H^1(0,T;H).$$

More precisely, owing to the theory of harmonic functions, in particular to the maximum principle, we have

$$\begin{aligned}
&\|\vartheta_{\mathcal{H}}\|_{L^2(0,T;V)} \le M_\Omega \|\vartheta_\Gamma\|_{L^2(0,T;H^{1/2}(\Gamma))}, \\
&\|\vartheta_{\mathcal{H}}\|_{L^1(0,T;H)} \le M_\Omega \|\vartheta_\Gamma\|_{L^1(0,T;H^{-1/2}(\Gamma))}, \\
&\|\partial_t\vartheta_{\mathcal{H}}\|_{L^2(0,T;H)} \le M_\Omega \|\partial_t\vartheta_\Gamma\|_{L^2(0,T;H^{-1/2}(\Gamma))}, \\
&\vartheta_* \le \vartheta_{\mathcal{H}} \le \vartheta^* \quad \text{a.e. in } Q, \\
&\|\partial_t\vartheta_{\mathcal{H}}\|_{L^1(0,T;L^\infty(\Omega))} = \|\partial_t\vartheta_\Gamma\|_{L^1(0,T;L^\infty(\Gamma))}
\end{aligned} \tag{2.15}$$

where M_Ω is a constant depending on Ω, only. So, our aim is finding a triplet (ϑ,χ,ξ) satisfying the regularity conditions

$$\vartheta \in L^2(0,T;V) \quad \text{and} \quad u := \vartheta - \vartheta_{\mathcal{H}} \in L^2(0,T;V_0) \tag{2.16}$$

$$\vartheta > 0 \quad \text{a.e. in } Q \quad \text{and} \quad \ln\vartheta \in L^\infty(0,T;H) \cap H^1(0,T;V_0') \tag{2.17}$$

$$\chi \in L^2(0,T;W) \cap H^1(0,T;H) \tag{2.18}$$

$$\xi \in L^2(Q) \tag{2.19}$$

and fulfilling the following equations

$$\partial_t\big(\ln\vartheta(t) - \lambda(\chi(t))\big) - \Delta(k_0 u + k * u)(t) = R(t) \quad \text{in } V_0', \text{ for a.a. } t \in (0,T) \tag{2.20}$$

$$\partial_t\chi - \Delta\chi + \xi + \sigma'(\chi) = -\lambda'(\chi)\vartheta \qquad \text{a.e. in } Q \tag{2.21}$$

$$\xi \in \beta(\chi) \quad \text{a.e. in } Q \tag{2.22}$$

$$(\ln\vartheta)(0) = \ln\vartheta_0 \quad \text{and} \quad \chi(0) = \chi_0. \tag{2.23}$$

We remark that (2.16) and (2.4) imply that $k_0\vartheta + k*\vartheta \in L^2(0,T;V)$, so that each term of (2.20) belongs to $L^2(0,T;V_0')$. Moreover, we note that (2.16) and (2.18) contain the Dirichlet condition for ϑ and the homogeneous Neumann condition for χ, respectively (see (2.7–8)). Finally, we point out that $\ln\vartheta \in C^0([0,T];V_0')$ and $\chi \in C^0([0,T];V)$ due to (2.16–18), whence both conditions (2.23) are meaningful. Now, we present our results.

Theorem 2.1. *Assume* (2.1–5) *and* (2.9–13) *with the notation* (2.6–8). *Then there exists a unique triplet* (ϑ,χ,ξ) *satisfying* (2.16–19) *and solving problem* (2.20–23). *Moreover, if* M *is a constant and all the norms related to* (2.9–12) *are bounded by the constant* M, *then the solution* (ϑ,χ,ξ) *satisfies the estimate*

$$\begin{aligned} &\|\vartheta\|_{L^2(0,T;V)} + \|\ln\vartheta\|_{L^\infty(0,T;H)\cap H^1(0,T;V_0')} \\ &\quad + \|\chi\|_{L^2(0,T;W)\cap H^1(0,T;H)} + \|\xi\|_{L^2(0,T;H)} \le M' \end{aligned} \tag{2.24}$$

where M' *depends on* Ω, T, *on the constants and functions listed in* (2.1–5), (2.13), *and on the constant* M, *only.* ∎

Theorem 2.2. *In the same framework of Theorem 2.1, the solution* (ϑ,χ,ξ) *continuously depends on the data in the following sense. If* $(R_i,\vartheta_{i\Gamma},\vartheta_{0i},\chi_{0i})$, $i=1,2$, *are two sets of data whose norms related to assumptions* (2.9–12) *are bounded by a constant* M, *then the corresponding solutions* $(\vartheta_i,\chi_i,\xi_i)$ *fulfil the estimate*

$$\begin{aligned} &\int_Q (\ln\vartheta_1 - \ln\vartheta_2)(\vartheta_1-\vartheta_2) + \int_\Omega |(1*\nabla(u_1-u_2))(t)|^2 \\ &\quad + \int_\Omega |\chi_1(t)-\chi_2(t)|^2 + \int_Q |\nabla(\chi_1-\chi_2)|^2 + \int_Q (\xi_1-\xi_2)(\chi_1-\chi_2) \\ &\le M''\Big\{\|\vartheta_{1\Gamma}-\vartheta_{2\Gamma}\|_{L^1(0,T;H^{-1/2}(\Gamma))} + \|R_1-R_2\|^2_{L^1(0,T;H)} + \|\eta_{01}-\eta_{02}\|_H^2\Big\} \end{aligned} \tag{2.25}$$

where $\eta_{0i} := \ln\vartheta_{0i} - \lambda(\chi_{0i})$ *for* $i=1,2$ *and* M'' *has the same dependences as* M' *does in Theorem 2.1.* ∎

Theorem 2.3. *In the same framework of Theorem 2.1, assume that*

$$\chi_0 \in L^\infty(\Omega) \quad \textit{and} \quad \vartheta_\Gamma \in L^2(0,T;H^{3/2}(\Gamma)) \tag{2.26}$$

in addition. Then, the solution (ϑ, χ, ξ) *enjoys the following further properties*

$$|\chi|^r \in L^2(0,T;V) \quad \textit{and} \quad \chi \in L^\infty(0,T;L^r(\Omega)) \tag{2.27}$$
$$\vartheta^p \in L^2(0,T;V) \quad \textit{and} \quad \vartheta \in L^\infty(0,T;L^{2p-1}(\Omega)) \tag{2.28}$$

for all $1 \le r < \infty$ *and* $1 \le p < 2$. ■

The rest of the paper is organized as follows. Next section is devoted to approximating problem (2.20–23). The other sections deal with the proof of our results. More precisely, we first prove the existence and stability parts of Theorem 2.1. Then we complete the proof showing uniqueness and prove Theorem 2.2. Finally, in the last section, we detail the proof of Theorem 2.3 and give some ideas on further regularity of the solution. In the sequel, we widely use the notation

$$Q_t := \Omega \times (0,t) \quad \text{for } 0 < t \le T$$

and the elementary Young inequality

$$ab \le \delta a^p + c_{\delta,p} b^{p'} \qquad \forall\, a,b \ge 0 \qquad \forall\, \delta > 0 \tag{2.29}$$

where $p, p' > 1$ satisfy $(1/p) + (1/p') = 1$ and $c_{\delta,p}$ is a positive constant, which depends on δ and p .

As far as constants are concerned, we use a general rule. In the whole paper, the symbol c stands for different constants which depend only on Ω , on the final time T , and on the constants and the norms of the functions involved in the assumptions of our statements. A notation like c_δ allows the constant to depend on the positive parameter δ , in addition. Hence, the meaning of c and c_δ might change from line to line and even in the same chain of inequalities. On the contrary, we use different symbols to denote precise constants which we could refer to.

As far as convolutions are concerned, we recall the identities (which hold whenever they make sense)

$$a * b = a(0) * b + (\partial_t a) * 1 * b \qquad \text{and} \qquad \partial_t (a * b) = a(0) b + (\partial_t a) * b \tag{2.30}$$

and the Young theorem

$$\|a * b\|_{L^r(0,T;X)} \le \|a\|_{L^p(0,T)} \|b\|_{L^q(0,T;X)} \tag{2.31}$$

where X is a Banach space and $p, q, r \in [1, \infty]$ satisfy $1/r = (1/p) + (1/q) - 1$. Moreover, we recall that $V \subset L^p(\Omega)$ for $1 \le p \le 6$, the embedding being compact if $p < 6$, and often use the corresponding Sobolev inequality

$$\|v\|_p \le M_\Omega \|v\|_V \quad \text{for } 1 \le p \le 6 \tag{2.32}$$

which holds for every $v \in H^1(\Omega)$ and for some constant M_Ω (depending on Ω , only), since Ω is a bounded smooth domain in $\mathbb{R}^3$.

3. Approximating problems

In order to prove our existence, stability, and regularity results, we introduce a family of approximating problems depending on the positive parameter ε. In such problems, some terms are regularized. In the next sections, we let ε tend to 0 and deal with the original problem.

We consider the Yosida regularizations β_ε and $\ln_\varepsilon$ of the maximal monotone graphs β and $\ln$, respectively (see, e.g., [8, p. 28]), and define $\widehat{\beta}_\varepsilon$, $\mathrm{Ln}_\varepsilon : \mathbb{R} \to \mathbb{R}$ by

$$\widehat{\beta}_\varepsilon(r) := \int_0^r \beta_\varepsilon(s)\,ds \quad \text{and} \quad \mathrm{Ln}_\varepsilon(r) := \varepsilon r + \ln_\varepsilon(r). \tag{3.1}$$

We note that both β_ε and Ln_ε are monotone and Lipschitz continuous(see, e.g., [8, p. 28]). Moreover, one can easily check that Ln_ε is even C^∞ and satisfies $\mathrm{Ln}'_\varepsilon(r) \geq \varepsilon$ for every $r \in \mathbb{R}$. Next, we replace λ with a smoother $\lambda_\varepsilon \in C^1(\mathbb{R})$ satisfying

$$\lambda_\varepsilon \text{ and } \lambda'_\varepsilon \text{ are Lipschitz continuous and } |\lambda_\varepsilon(0)| + |\lambda'_\varepsilon(0)| + \|\lambda''_\varepsilon\|_{L^\infty(\mathbb{R})} \leq M_\lambda \tag{3.2}$$

$$\lambda_\varepsilon(r) \to \lambda(r) \quad \text{and} \quad \lambda'_\varepsilon(r) \to \lambda'(r) \quad \text{as } \varepsilon \searrow 0 \text{ for every } r \in \mathbb{R}. \tag{3.3}$$

Finally, we introduce regularized data $\vartheta_{0\varepsilon}$ fulfilling the following conditions

$$\vartheta_{0\varepsilon} \in V \quad \text{and} \quad \vartheta_* \leq \vartheta_{0\varepsilon} \leq \vartheta^* \quad \text{a.e. in } \Omega \quad \text{for any } \varepsilon > 0 \tag{3.4}$$

$$\vartheta_{0\varepsilon} \to \vartheta_0 \quad \text{in } H \text{ and a.e. in } \Omega \quad \text{as } \varepsilon \searrow 0. \tag{3.5}$$

Hence, the approximating problem consists in finding a pair $(\vartheta_\varepsilon, \chi_\varepsilon)$ satisfying the regularity conditions

$$\vartheta_\varepsilon \in L^2(0,T;V) \cap H^1(0,T;H) \quad \text{and} \quad u_\varepsilon := \vartheta_\varepsilon - \vartheta_{\mathcal{H}} \in L^2(0,T;V_0) \tag{3.6}$$

$$\chi_\varepsilon \in L^2(0,T;W) \cap H^1(0,T;H) \tag{3.7}$$

and fulfilling the following equations

$$\partial_t\big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big) - \Delta(k_0 u_\varepsilon + k * u_\varepsilon) = R \qquad \text{a.e. in } Q \tag{3.8}$$

$$\partial_t \chi_\varepsilon - \Delta\chi_\varepsilon + \beta_\varepsilon(\chi_\varepsilon) + \sigma'(\chi_\varepsilon) = -\lambda'_\varepsilon(\chi_\varepsilon)\vartheta_\varepsilon \qquad \text{a.e. in } Q \tag{3.9}$$

$$\vartheta_\varepsilon(0) = \vartheta_{0\varepsilon} \quad \text{and} \quad \chi_\varepsilon(0) = \chi_0. \tag{3.10}$$

Even though the above problem looks better than the original one, it is not obvious that it has a solutions. Therefore, we discretize it by means of a Galerkin procedure. More precisely, we prefer to approximate u_ε rather than ϑ_ε.

To this aim, we introduce two increasing sequences $\{V^n\}$ and $\{V_0^n\}$ of finite dimensional subspaces of V and V_0 such that the corresponding unions are dense in V and V_0, respectively. We can choose such subspaces in order that

$$V^n \subset W \quad \text{and} \quad V_0^n \subset H^2(\Omega). \tag{3.11}$$

Moreover, we approximate the data of the approximating problem by sequences $\{R_n\}$, $\{u_{0n}\}$, and $\{\chi_{0n}\}$ satisfying for every n

$$R_n \in C^0([0,T];H), \quad u_{0n} \in V_0^n, \quad \text{and} \quad \chi_{0n} \in V^n \tag{3.12}$$

and converging as $n \nearrow \infty$ in the following sense

$$\begin{aligned} &R_n \to R \quad \text{in } L^2(0,T;H) \\ &u_{0n} \to \vartheta_{0\varepsilon} - \vartheta_{\mathcal{H}}(0) \quad \text{and} \quad \chi_{0n} \to \chi_0 \quad \text{in } V. \end{aligned} \tag{3.13}$$

Then, the discrete problem consists in finding a pair (u_n, χ_n) satisfying

$$u_n \in C^1([0,T];V_0^n) \quad \text{and} \quad \chi_n \in C^1([0,T];V^n) \tag{3.14}$$

and solving the following variational equations

$$\begin{aligned} &\big(\partial_t(\mathrm{Ln}_\varepsilon(\vartheta_n(t)) - \lambda_\varepsilon(\chi_n(t))), v\big)_H + \big(k_0 \nabla u_n(t) + (k * \nabla u_n)(t), \nabla v\big)_H \\ &\quad = \big(R_n(t), v\big)_H \quad \text{for any } t \in [0,T] \text{ and } v \in V_0^n \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\big(\partial_t \chi_n(t), v\big)_H + \big(\nabla \chi_n(t), \nabla v\big)_H + \big(\beta_\varepsilon(\chi_n(t)) + \sigma'(\chi_n(t)), v\big)_H \\ &\quad = -\big(\lambda_\varepsilon'(\chi_n(t))\, \vartheta_n(t), v\big)_H \quad \text{for any } t \in [0,T] \text{ and } v \in V^n \end{aligned} \tag{3.16}$$

$$u_n(0) = u_{0n} \quad \text{and} \quad \chi_n(0) = \chi_{0n} \tag{3.17}$$

where we have set

$$\vartheta_n := u_n + \vartheta_{\mathcal{H}}\,. \tag{3.18}$$

It turns out that the above problem has a global solution. Then, we let n tend to infinity in order to solve the approximating problem (3.8–10). However, as nothing is obvious, we state precise existence results. Although the solutions to problems (3.15–17) and (3.8–10) should be unique, we do not care about uniqueness of such approximating solutions.

Theorem 3.1. *The discrete problem* (3.15–17) *has a global solution.* ∎

Proof. Many of the constants and the functions we introduce depend on both n and ε. However, as such parameters are fixed at the moment, we often do not stress such a dependence in the notation. Let n' and n'_0 be the dimensions of the subspaces V^n and V_0^n and introduce two bases $\mathcal{B} = (w_j)$ and $\mathcal{B}_0 = (w_{0j})$ for V_0^n and V^n, respectively. Here and in the rest of the proof, the index j runs from 1 to either n' or n'_0, according to the basis which it refers to. Clearly, the true unknowns are the coefficients u_j and y_j of the representations of u_n and χ_n with respect to the bases we have fixed. If $\mathbf{u}$ and $\mathbf{y}$ are the vectors of such coefficients, sistem (3.15–16) can be written in the form of a system of integrodifferential ordinary equations, namely

$$\mathbf{E}(t, \mathbf{u}(t), \mathbf{y}(t), \mathbf{u}'(t), \mathbf{y}'(t), (k * \mathbf{u})(t)) = 0 \tag{3.19}$$

where $\mathbf{E}=(\mathbf{F},\mathbf{G})$ and the components F_i and G_i of $\mathbf{F}$ and $\mathbf{G}$ are defined by

$$\begin{aligned}
&F_i(t,\mathbf{u},\mathbf{y},\mathbf{u}',\mathbf{y}',\mathbf{z})\\
&= \int_\Omega \Bigl(\mathrm{Ln}'_\varepsilon\Bigl(\textstyle\sum_j u_j w_{0j} + \vartheta_{\mathcal{H}}(t)\Bigr)\Bigl(\textstyle\sum_j u'_j w_{0j} + \partial_t\vartheta_{\mathcal{H}}(t)\Bigr) - \lambda'_\varepsilon\Bigl(\textstyle\sum_j y_j w_j\Bigr)\textstyle\sum_j y'_j w_j\Bigr) w_{0i}\\
&\quad + \textstyle\sum_j (k_0 u_j + z_j)\displaystyle\int_\Omega \nabla w_{0j}\cdot\nabla w_{0i} - \int_\Omega R_n(t) w_{0i}\\
&G_i(t,\mathbf{u},\mathbf{y},\mathbf{u}',\mathbf{y}',\mathbf{z})\\
&= \textstyle\sum_j y'_j \displaystyle\int_\Omega w_j w_i + \textstyle\sum_j y_j \displaystyle\int_\Omega \nabla w_j\cdot\nabla w_i\\
&\quad + \int_\Omega (\beta_\varepsilon + \sigma')\Bigl(\textstyle\sum_j y_j w_j\Bigr) w_i + \displaystyle\int_\Omega \lambda'_\varepsilon\Bigl(\textstyle\sum_j y_j w_j\Bigr)\Bigl(\textstyle\sum_j u_j w_{0j} + \vartheta_{\mathcal{H}}(t)\Bigr) w_i
\end{aligned}$$

for $i=1,\dots,n'_0$ and $i=1,\dots,n'$, respectively. In the above formulas, $\mathbf{u}'$, $\mathbf{y}'$, and $\mathbf{z}$ are understood to be independent variables as well as $\mathbf{u}$ and $\mathbf{y}$. Clearly, the first thing to do is putting system (3.19) it its normal form, and this corresponds to solve the equation

$$\mathbf{E}(t,\mathbf{u},\mathbf{y},\mathbf{u}',\mathbf{y}',\mathbf{z}) = 0$$

with respect to the variable $(\mathbf{u}',\mathbf{y}')$. To this aim, we want to use the implicite function theorem and thus have to verify a number of assumption. First of all, $\mathbf{E}$ is a continuous function and has continuous derivatives with respect to the variables u'_j and y'_j. Next, we have to study its jacobian matrix with respect to the variables just mentioned. Clearly, one thinks of four blocks, namely, the derivatives of $\mathbf{F}$ and $\mathbf{G}$ with respect to the variables $\mathbf{u}'$ and $\mathbf{y}'$. As $\mathbf{G}$ does not depend on $\mathbf{u}'$ explicitely, the determinant we have to compute is given by

$$\det\frac{\partial(\mathbf{F},\mathbf{G})}{\partial(\mathbf{u}',\mathbf{y}')} = \det\frac{\partial\mathbf{F}}{\partial\mathbf{u}'}\cdot\det\frac{\partial\mathbf{G}}{\partial\mathbf{y}'}\,.$$

We note that all partial derivatives involved are scalar products, namely,

$$\frac{\partial F_i}{\partial u'_j} = \bigl(w_{0j}, w_{0i}\bigr)_{t,\mathbf{u}} \quad\text{and}\quad \frac{\partial G_i}{\partial y'_j} = \bigl(w_j, w_i\bigr)_H$$

where we have set

$$\bigl(v_1,v_2\bigr)_{t,\mathbf{u}} := \int_\Omega \mathrm{Ln}'_\varepsilon\Bigl(\textstyle\sum_j u_j w_{0j} + \vartheta_{\mathcal{H}}(t)\Bigr) v_1 v_2 \quad \text{for } v_1,v_2\in H\,.$$

More precisely, $(\,\cdot\,,\,\cdot\,)_{t,\mathbf{u}}$ is an equivalent scalar product in H since $\varepsilon \le \mathrm{Ln}'_\varepsilon(r) \le \varepsilon + 1/\varepsilon$ for every $r\in\mathbb{R}$. Therefore, the above matrices are positive definite. Finally, we have to choose the point $(t_*,\mathbf{u}_*,\mathbf{y}_*,\mathbf{u}'_*,\mathbf{y}'_*,\mathbf{z}_*)$ to be used to apply the implicite function theorem, and this clearly has to be related to the initial conditions (3.17). Hence, we take $t_*=0$ and $\mathbf{z}_*=0$. Next, we consider the representations of u_{0n} and χ_{0n} with respect to the bases $\mathcal{B}_0$ and $\mathcal{B}$, respectively, and choose $\mathbf{u}_*$ and $\mathbf{y}_*$

to be the vectors of the coefficients. It remains to select vectors $\mathbf{u}'_*$ and $\mathbf{y}'_*$ such that $\mathbf{E}(t_*, \mathbf{u}_*, \mathbf{y}_*, \mathbf{u}'_*, \mathbf{y}'_*, \mathbf{z}_*) = 0$. To this aim, we define $\chi^\bullet_{0n}$ by means of the equation

$$\chi^\bullet_{0n} - \Delta\chi_{0n} + (\beta_\varepsilon + \sigma')(\chi_{0n}) = -\lambda'_\varepsilon(\chi_{0n})(u_{0n} + \vartheta_{\mathcal{H}}(0)).$$

Noting that $\chi^\bullet_{0n}$ belongs to H thanks to (3.11), we take its projection χ^*_{0n} on V^n with respect to the standard scalar product of H and consider the vector $\mathbf{y}'_*$ of the coefficients of the representation of χ^*_{0n} in terms of the the basis $\mathcal{B}$. Hence, we have $\mathbf{G}(t_*, \mathbf{u}_*, \mathbf{y}_*, \mathbf{u}', \mathbf{y}'_*, \mathbf{z}_*) = 0$ for every $\mathbf{u}'$ since $\mathbf{G}$ does not depend on $\mathbf{u}'$ explicitely. Next, we define $u^\bullet_{0n}$ by means of the equation

$$\mathrm{Ln}'_\varepsilon(u_{0n} + \vartheta_{\mathcal{H}}(0))\big(u^\bullet_{0n} + \partial_t\vartheta_{\mathcal{H}}(0)\big) - \lambda'_\varepsilon(\chi_0)\chi^*_{0n} - k_0\Delta u_{0n} = R_n(0).$$

As $u^\bullet_{0n} \in H$ due to the inequality $\mathrm{Ln}'_\varepsilon \geq \varepsilon$ and to our regularity assumptions, we can take its projection u^*_{0n} on V^n_0 with respect to the scalar product $(\,\cdot\,,\,\cdot\,)_{t_*,\mathbf{u}_*}$ and consider the vector $\mathbf{u}'_*$ of the coefficients of the representation of u^*_{0n} in terms of the basis $\mathcal{B}_0$. Hence, we have $\mathbf{F}(t_*, \mathbf{u}_*, \mathbf{y}_*, \mathbf{u}'_*, \mathbf{y}'_*, \mathbf{z}_*) = 0$. Therefore, we are in position to apply the implicit function theorem and conclude that system (3.19) is (locally) equivalent to an equation of the form

$$(\mathbf{u}'(t), \mathbf{y}'(t)) = \mathcal{E}(t, \mathbf{u}(t), \mathbf{y}(t), (k * \mathbf{u})(t)) \tag{3.20}$$

where $\mathcal{E}$ is a continuous function that is Lipschitz with respect to the variables $\mathbf{u}$, $\mathbf{y}$ and $\mathbf{z}$. Hence, the corresponding Cauchy problem (3.17) has a unique local solution, i.e., a solution defined in a possibly smaller interval $[0,\tau)$: indeed, one can reduce (3.20) to a fixed point problem (integral equation) by integration with respect to t and then suitably apply the Contraction Principle.

Our next goal is showing that every maximal solution is actually global, and this can be done in a quite standard way, i.e., noting that the initial regularity is preserved and proving some a priori estimates. This would give a final value $(u_n(\tau), \chi_n(\tau))$ and thus the possibility of extending the solution, against maximality if $\tau < T$. However, the a priori estimate we scketch in a moment already would do the job, whence we avoid any further detail on this point.

Theorem 3.2. *Assume* (2.1–5), (2.9–12), (3.2), *and* (3.4). *Then there exists a pair* $(\vartheta_\varepsilon, \chi_\varepsilon)$ *satisfying* (3.6–7) *and solving problem* (3.8–10). ■

Proof. We argue on the solutions to the discrete problems (3.15–17) and let n tend to infinity. This should be done by performing a number of a priori estimates. In order to obtain the basic one, one writes (3.15–16) at time $t = s$, tests such equations by $v = u_n(s)$ and $v = \partial_t\chi_n(s)$, respectively, integrates over $(0,t)$, and sums the obtained equalities to each others. The $\lambda_\varepsilon -$ terms cancel out, the convolution can be treated as a perturbation, and one arrives at

$$\|u_n\|_{L^2(0,T;V_0)} + \|\chi_n\|_{H^1(0,T;H)} + \|\chi_n\|_{L^\infty(0,T;V)} \leq c_\varepsilon. \tag{3.21}$$

As a similar estimate is given with full detail in the proof of Theorem 2.1, let us skip the precise proof of (3.21).

Next, we derive an a priori bound for u_n' in $L^2(0,T;H)$, where "prime" stands for the time derivative till the end of the proof. In this case we give some detail by proceeding fast. We write (3.15) in the form

$$\begin{aligned} &\big(\partial_t \operatorname{Ln}_\varepsilon(\vartheta_n(s)), v\big)_H + \big(k_0 \nabla u_n(s), \nabla v\big)_H \\ &= -\big((k * \nabla u_n)(s), \nabla v\big)_H + \big(R_n(s) + \partial_t \lambda_\varepsilon(\chi_n(s)), v\big)_H \\ &\qquad \text{for any } s \in [0,T] \text{ and } v \in V_0^n. \end{aligned} \tag{3.22}$$

Then, we test such an equation by $v = u_n'(s)$ and integrate over $(0,t)$. The leading terms coming from the left hand side are

$$\int_{Q_t} \operatorname{Ln}_\varepsilon'(\vartheta_n) |\vartheta_n'|^2 + \frac{k_0}{2} \int_\Omega |\nabla u_n(t)|^2$$

and we recall that $\operatorname{Ln}_\varepsilon'(\vartheta_n) \geq \varepsilon$ a.e. in Q. Moreover, we note that the difference $\vartheta_n' - u_n' = \vartheta_{\mathcal{H}}'$ is bounded in $L^2(0,T;H)$, whence it is essentially equivalent to use either ϑ_n' or u_n' in treating each single term. The integral coming from the left hand side and containing $\vartheta_{\mathcal{H}}$ is moved to the right hand side and estimated this way

$$\int_{Q_t} \partial_t \operatorname{Ln}_\varepsilon(\vartheta_n)\, \vartheta_{\mathcal{H}}' \leq \frac{2}{\varepsilon} \int_{Q_t} \vartheta_n' \vartheta_{\mathcal{H}}' \leq \frac{\varepsilon}{2} \int_{Q_t} |\vartheta_n'|^2 + c_\varepsilon$$

with the help of the inequality $\operatorname{Ln}_\varepsilon'(r) \leq \varepsilon + 1/\varepsilon \leq 2/\varepsilon$, which holds for every $r \in \mathbb{R}$ and $\varepsilon \in (0,1)$, and of (2.15). The convolution term can be treated as follows

$$\begin{aligned} &- \int_{Q_t} (k * \nabla u_n) \cdot \nabla u_n' \\ &= - \int_\Omega (k * \nabla u_n)(t) \cdot \nabla u_n(t) + \int_{Q_t} (k(0) \nabla u_n + k' * \nabla u_n) \cdot \nabla u_n \\ &\leq \frac{k_0}{4} \int_\Omega |\nabla u_n(t)|^2 + c \|u_n\|^2_{L^2(0,T;V)} \leq \frac{k_0}{4} \int_\Omega |\nabla u_n(t)|^2 + c_\varepsilon. \end{aligned}$$

The last inequality uses the Young theorem (2.31) (see the next section for similar situations) and (3.21). As far as the last term on the right hand side is concerned, it suffices to observe that (3.21), (3.12), and (3.2) ensure that

$$\|R_n + \partial_t \lambda_\varepsilon(\chi_n)\|_{L^2(0,T;H)} \leq c_\varepsilon.$$

As the initial values $u_n(0)$ (coming from the second term of (3.22)) are bounded in V by (3.13), we get, in particular

$$\|\vartheta_n'\|_{L^2(0,T;H)} \leq c_\varepsilon. \tag{3.23}$$

At this point, it is straightforward to take the limit as $n \nearrow \infty$ (for a subsequence) and show that one obtains a solution to the $\varepsilon-$ problem (3.6–10) in the limit, namely

$$u_\varepsilon := \lim_{n\to\infty} u_n, \quad \vartheta_\varepsilon := \lim_{n\to\infty} \vartheta_n = \vartheta_{\mathcal{H}} + u_\varepsilon, \quad \text{and} \quad \chi_\varepsilon := \lim_{n\to\infty} \chi_n$$

weakly in the spaces mentioned in the above estimates. However, as similar arguments are used later on, we confine ourselves in sketching the proof here. The weak convergences we have obtained are already sufficient to ensure that the Cauchy conditions (3.10) and the property (3.6) hold in the limit. Next, due to the smoothness of all nonlinearities, all the nonlinear terms are weakly convergent as well, in the appropriate spaces. On the other hand, we deduce strong convergence in $L^p -$ type spaces for both $\{\vartheta_n\}$ and $\{\chi_n\}$, accounting for compact embeddings and the Aubin lemma (see, e.g., [25, p. 58]). This yields convergence a.e., whence we can identify the limits of the nonlinear terms. Now, we shortly show how to prove that $(\vartheta_\varepsilon, \chi_\varepsilon)$ actually solves the approximating problem. We fix m for a while. We write (3.15–16) at time $t = s$, test such equation with $v(s)$, where v is arbitrary in $L^2(0,T;V_0^m)$ and in $L^2(0,T;V^m)$, respectively, and integrate over $(0,T)$. This can be done assuming $n \ge m$ since the sequences of finite dimensional subspaces are increasing. At this point, we let n tend to infinity (for the convergent subsequence) and obtain

$$\int_Q \partial_t \big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon(t)) - \lambda_\varepsilon(\chi_\varepsilon(t))\big) v + \int_Q \nabla(k_0 u_\varepsilon + k * u_\varepsilon) \cdot \nabla v = \int_Q R v$$

$$\int_Q \partial_t \chi_\varepsilon \, v - \int_Q \nabla \chi_\varepsilon \cdot \nabla v + \int_Q (\beta_\varepsilon + \sigma')(\chi_\varepsilon)\, v = - \int_Q \lambda_\varepsilon'(\chi_\varepsilon) \vartheta_\varepsilon v$$

for any $v \in L^2(0,T;V_0^m)$ and $v \in L^2(0,T;V^m)$, respectively. As m is arbitrary, the same variational equations hold for any V_0 and V-valued functions, by density. So, we obtain a variational version of system (3.8–9) with the implicit Neumann condition for χ. At this point, looking at the regularity of the functions involved, we conclude that χ_ε satisfies (3.7) and that $(\vartheta_\varepsilon, \chi_\varepsilon)$ is a solution in the sense specified in the statements.

4. Existence and stability

In this section, we prove the existence and stability parts of Theorem 2.1. We start from an arbitrary solution $(\vartheta_\varepsilon, \chi_\varepsilon)$ to the approximating problem (3.8–10) and perform a number of a priori estimates in order to let $\varepsilon \searrow 0$ using compactness methods. The estimates we prove hold for ε small enough, in general. However, in order not to be too boring, let us avoid to recall every time the smallness conditions on ε already required in previous arguments, being understood that every upperbound for ε already needed is supposed to be respected in the sequel of the section. We remind the reader that the functions β_ε and $\ln_\varepsilon$ are the Yosida regularizations of β and $\ln$, respectively, and that the functions $\widehat{\beta}_\varepsilon$ and Ln_ε are defined in (3.1). We need one more function, namely,

$$\mathrm{I}_\varepsilon(r) := \int_0^r s \, \mathrm{Ln}_\varepsilon'(s)\, ds, \quad r \in \mathbb{R} \tag{4.1}$$

which is an approximation of the identity on $(0, +\infty)$. Before starting to prove our estimates, we collect a number of properties of the functions we have mentioned. Such properties will be used in the sequel. The first one is

$$\widehat{\beta}_\varepsilon(r) \le \widehat{\beta}(r) \quad \text{for every } r \in \mathbb{R} \tag{4.2}$$

and holds for any $\varepsilon > 0$ (see, e.g., [8, Prop. 2.11, p. 39]). Moreover, owing, e.g., to [8, p. 28], we easily deduce the chain of inequalities

$$0 \le \mathrm{Ln}_\varepsilon^-(r) \le \ln_\varepsilon^-(r) \le |\ln_\varepsilon(r)| \le |\ln r| \quad \text{for every } r > 0 \text{ and } \varepsilon > 0 . \tag{4.3}$$

Lemma 4.1. *For any $\delta > 0$ we have*

$$r^2 \le \delta \widehat{\beta}_\varepsilon(r) + c_\delta \quad \textit{for every } r \in \mathbb{R} \tag{4.4}$$

provided that $\varepsilon \in (0, \delta/3)$. ∎

Proof. Fix $\delta > 0$. Owing to (2.2), we find $r_\delta > 0$ such that $\widehat{\beta}(r)/r^2 \ge 1/\delta$ for $|r| \ge r_\delta$. Hence, we have

$$r^2 \le \delta \widehat{\beta}(r) + r_\delta^2 \quad \text{for every } r \in \mathbb{R}$$

whence also

$$\frac{1}{2\varepsilon/\delta}(s-r)^2 + s^2 \le \delta\left(\frac{1}{2\varepsilon}(s-r)^2 + \widehat{\beta}(s)\right) + r_\delta^2 \quad \text{for every } r, s \in \mathbb{R} .$$

For fixed r , taking the infimum with respect to s , we deduce that (see, e.g., [8, Prop. 2.11, p. 39])

$$\frac{r^2}{1 + 2\varepsilon/\delta} \le \delta \widehat{\beta}_\varepsilon(r) + r_\delta^2 .$$

Hence, assuming $\varepsilon \in (0, \delta)$, we infer that

$$r^2 \le 3\delta \widehat{\beta}_\varepsilon(r) + 3r_\delta^2 .$$

To conclude, it is sufficient to replace δ by $\delta/3$. ∎

Lemma 4.2. *There holds the inequality*

$$\ln_\varepsilon'(r) \le \frac{2}{r} \quad \textit{for every } r > 0 \tag{4.5}$$

provided that ε is small enough. ∎

Proof. By virtue of the definition of $\ln_\varepsilon$ (see, e.g., [8, p. 28]), we have

$$\ln_\varepsilon(r) = \frac{r - \rho_\varepsilon(r)}{\varepsilon} \quad \text{for } r \in \mathbb{R} \tag{4.6}$$

where $\rho_\varepsilon : \mathbb{R} \to \mathbb{R}$, the resolvent of $\ln$, is defined this way

$$\rho_\varepsilon(r) \text{ is the unique } \rho > 0 \text{ such that } \rho + \varepsilon \ln \rho = r . \tag{4.7}$$

Now we prove the lemma. Let $\varepsilon_* > 0$ be such that $\varepsilon_* \ln\rho \le \rho$ for every $\rho > 0$ and assume $\varepsilon \in (0, \varepsilon_*)$. Then, we have $\rho + \varepsilon\ln\rho \le 2\rho$ for every $\rho > 0$, whence, taking the the inverse functions, we deduce that

$$\rho_\varepsilon(r) \ge \frac{r}{2} \quad \text{for every } r \in \mathbb{R}. \tag{4.8}$$

On the other hand, (4.6) allows us to compute the derivative of $\ln_\varepsilon$ using ρ_ε. From the definition (4.7) of ρ_ε, we have

$$\rho_\varepsilon'(r) + \varepsilon\,\frac{\rho_\varepsilon'(r)}{\rho_\varepsilon(r)} = 1, \quad \text{whence} \quad \ln_\varepsilon'(r) = \frac{1}{\rho_\varepsilon(r) + \varepsilon}$$

and we immediately conclude. ∎

Lemma 4.3. *We have*

$$\mathrm{I}_\varepsilon(r) \le \frac{\varepsilon}{2}\, r^2 + 2r \quad \textit{for every } r > 0 \tag{4.9}$$

provided that ε is small enough. ∎

Proof. We have indeed $\mathrm{I}_\varepsilon(0) = 0$ and $\mathrm{I}_\varepsilon{}'(r) = \varepsilon r + r\ln_\varepsilon'(r) \le \varepsilon r + 2$ for every $r > 0$ and ε small enough, thanks to Lemma 4.2. ∎

Lemma 4.4. *For any $\delta > 0$ there holds the estimate*

$$\mathrm{Ln}_\varepsilon^+(r) \le \delta\,\mathrm{I}_\varepsilon(r) + c_\delta \quad \textit{for every } r \in \mathbb{R} \tag{4.10}$$

provided that ε is small enough. ∎

Proof. Fix $\delta > 0$ and let $r_\varepsilon \in (0,1)$ be the unique solution to the equation $\mathrm{Ln}_\varepsilon(r) = 0$. For $r \le r_\varepsilon$ we have $\mathrm{Ln}_\varepsilon(r) \le 0$ and $\mathrm{I}_\varepsilon(r) \ge 0$, whence (4.10) holds for any $c_\delta \ge 0$. To deal with the case $r > r_\varepsilon$, we prove that $r_\varepsilon > 1/2$ for ε small enough. We use the notation (4.6–7). We have $\mathrm{Ln}_\varepsilon(r) = \varepsilon r + (r - \rho_\varepsilon(r))/\varepsilon$, so that the definition of r_ε reads $(\varepsilon^2 + 1)r_\varepsilon = \rho_\varepsilon(r_\varepsilon)$, and consequently, by (4.7),

$$r_\varepsilon = (\varepsilon^2+1)r_\varepsilon + \varepsilon\ln((\varepsilon^2+1)r_\varepsilon), \quad \text{that is,} \quad \ln((\varepsilon^2+1)r_\varepsilon) + \varepsilon r_\varepsilon = 0.$$

As the function $r \mapsto \ln((\varepsilon^2+1)r) + \varepsilon r$ is positive for $r = 1$ and its value at $r = 1/2$ tends to $-\ln 2$ as $\varepsilon \searrow 0$, it is clear that $r_\varepsilon > 1/2$ for ε small enough. For such values of ε and for $r > r_\varepsilon$, owing to $r_\varepsilon \ge 1/2$ and to Lemma 4.2, we have

$$\begin{aligned}
\mathrm{Ln}_\varepsilon^+(r) = \mathrm{Ln}_\varepsilon(r) &= \int_{r_\varepsilon}^{r} \mathrm{Ln}_\varepsilon'(s)\,ds \le \int_{r_\varepsilon}^{r} \left(\delta s + c_\delta s^{-2}\right)\mathrm{Ln}_\varepsilon'(s)\,ds \\
&\le \delta\int_0^r s\,\mathrm{Ln}_\varepsilon'(s)\,ds + c_\delta\int_{1/2}^{+\infty} s^{-2}(\varepsilon + 2/s)\,ds \le \delta\,\mathrm{I}_\varepsilon(r) + c_\delta\,. \ \blacksquare
\end{aligned}$$

Now, we start proving our a priori estimates on the solutions $(\vartheta_\varepsilon, \chi_\varepsilon)$ to the approximating problems (3.8–10). We recall that the norms $\|v\|_V$ and $\|\nabla v\|_H$ are equivalent for $v \in V_0$, as already observed, and that the inequality

$$|\lambda_\varepsilon(r)| + |\sigma(r)| \le c(1 + r^2) \quad \text{for every } r \in \mathbb{R} \tag{4.11}$$

holds true, thanks to assumptions (2.3) and (3.2).

First a priori estimate. We multiply (3.8) and (3.9) by u_ε and $\partial_t \chi_\varepsilon$, respectively. Then, we sum the obtained equalities and integrate over Q_t. Finally, we add the same suitable integral to both sides. Noting that the $\lambda_\varepsilon -$ terms partially cancel and using the initial and boundary conditions, we obtain

$$\begin{aligned}
&\int_\Omega \mathrm{I}_\varepsilon(\vartheta_\varepsilon(t)) + \vartheta_* \int_\Omega \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon(t)) + k_0 \int_{Q_t} |\nabla u_\varepsilon|^2 \\
&\quad + \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_\varepsilon(t)|^2 + \int_\Omega (\widehat{\beta}_\varepsilon + \sigma)(\chi_\varepsilon(t)) \\
&= \int_\Omega \mathrm{I}_\varepsilon(\vartheta_{0\varepsilon}) + \frac{1}{2} \int_\Omega |\nabla \chi_0|^2 + \int_\Omega (\widehat{\beta}_\varepsilon + \sigma)(\chi_0) + \int_{Q_t} \partial_t \big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big) \vartheta_{\mathcal{H}} \\
&\quad - \int_{Q_t} (k * \nabla u_\varepsilon) \cdot \nabla u_\varepsilon + \int_{Q_t} R u_\varepsilon + \vartheta_* \int_\Omega \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon(t)).
\end{aligned} \tag{4.12}$$

We separately treat each term that needs some manipulation. The only term on the left hand side we have to handle is the last one. Due to (4.11) and to Lemma 4.1, we have

$$\int_\Omega (\widehat{\beta}_\varepsilon + \sigma)(\chi_\varepsilon(t)) \ge \int_\Omega |\chi_\varepsilon(t)|^2 + (1 - \delta) \int_\Omega \widehat{\beta}_\varepsilon(\chi_\varepsilon(t)) - c_\delta\,.$$

The corresponding term on the right hand side can be easily estimated using (4.2) and (2.12), while the term involving $\vartheta_{0\varepsilon}$ is seen to be uniformly bounded thanks to Lemma 4.3 and the inequalities in (2.15). Now, we deal with the first volume integral and integrate it by parts in time, accounting for the very last integral of (4.12) as well. Taking advantage of Lemma 4.4 and assuming ε to be small enough, we obtain

$$\begin{aligned}
&\int_{Q_t} \partial_t \big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big) \vartheta_{\mathcal{H}} + \vartheta_* \int_\Omega \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon(t)) \\
&= \int_\Omega \mathrm{Ln}_\varepsilon^+(\vartheta_\varepsilon(t)) \vartheta_{\mathcal{H}}(t) + \int_\Omega (\vartheta_* - \vartheta_{\mathcal{H}}(t))\, \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon(t)) - \int_\Omega \lambda_\varepsilon(\chi_\varepsilon(t)) \vartheta_{\mathcal{H}}(t) \\
&\quad - \int_\Omega \mathrm{Ln}_\varepsilon^+(\vartheta_{0\varepsilon}) \vartheta_{\mathcal{H}}(0) + \int_\Omega \vartheta_{\mathcal{H}}(0)\, \mathrm{Ln}_\varepsilon^-(\vartheta_{0\varepsilon}) + \int_\Omega \lambda_\varepsilon(\chi_0) \vartheta_{\mathcal{H}}(0) \\
&\quad - \int_{Q_t} \big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big) \partial_t \vartheta_{\mathcal{H}}.
\end{aligned} \tag{4.13}$$

We deal with each integral on the right hand side of (4.13) separately. Thanks to (2.13) and to Lemma 4.4, we have

$$\int_\Omega \mathrm{Ln}_\varepsilon^+(\vartheta_\varepsilon(t)) \vartheta_{\mathcal{H}}(t) \le \vartheta^* \int_\Omega \mathrm{Ln}_\varepsilon^+(\vartheta_\varepsilon(t)) \le \delta \int_\Omega \mathrm{I}_\varepsilon(\vartheta_\varepsilon(t)) + c_\delta.$$

The second term of (4.13) is nonpositive, while the third one is estimated using (2.15), (4.11), and Lemma 4.1 this way

$$-\int_\Omega \lambda_\varepsilon(\chi_\varepsilon(t))\vartheta_{\mathcal{H}}(t) \le c\int_\Omega |\chi_\varepsilon(t)|^2 + c \le \delta\int_\Omega \widehat{\beta}_\varepsilon(\chi(t)) + c_\delta\,.$$

Ignoring a nonpositive integral, we deal with the next one. We have

$$\int_\Omega \vartheta_{\mathcal{H}}(0)\,\mathrm{Ln}_\varepsilon^-(\vartheta_{0\varepsilon}) \le \vartheta^*\int_\Omega |\ln\vartheta_{0\varepsilon}| \le c$$

because of (2.15), (4.3), and (3.4). Finally, we estimate the last integral of (4.13). Using Lemmas 4.4 and 4.1 and estimates (4.3), (4.11), and (2.15), we obtain

$$\begin{aligned}
&-\int_{Q_t}\big(\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) - \lambda_\varepsilon(\chi_\varepsilon)\big)\partial_t\vartheta_{\mathcal{H}} \\
&\le c\int_{Q_t}|\partial_t\vartheta_{\mathcal{H}}|\big(\mathrm{Ln}_\varepsilon^+(\vartheta_\varepsilon) + \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon) + |\chi_\varepsilon|^2 + 1\big) \\
&\le c\int_0^t \|\partial_t\vartheta_{\mathcal{H}}(s)\|_{L^\infty(\Omega)}\int_\Omega\Big(\mathrm{I}_\varepsilon(\vartheta_\varepsilon(s)) + \mathrm{Ln}_\varepsilon^-(\vartheta_\varepsilon(s)) + \widehat{\beta}_\varepsilon(\chi(s)) + 1\Big)\,ds
\end{aligned}$$

and point out that the function $\|\partial_t\vartheta_{\mathcal{H}}(\,\cdot\,)\|_{L^\infty(\Omega)}$ belongs to $L^1(0,T)$, due to (2.15). As all integrals of (4.13) are estimated, we come back to (4.12) and deal with the convolution term. We use the regularity assumption (2.4) and the Young inequality (2.31) and get

$$\begin{aligned}
&-\int_{Q_t}(k*\nabla u_\varepsilon)\cdot\nabla u_\varepsilon = -\int_{Q_t}(k(0)*\nabla u_\varepsilon + k'*1*\nabla u_\varepsilon)\cdot\nabla u_\varepsilon \\
&\le \delta\int_{Q_t}|\nabla u_\varepsilon|^2 + c_\delta\Big(|k(0)|^2 + \|k'\|^2_{L^1(0,T)}\Big)\int_0^t\|\nabla u_\varepsilon\|^2_{L^2(0,s;H)}\,ds.
\end{aligned}$$

Finally, we immediately have

$$\int_{Q_t} Ru_\varepsilon \le \delta\int_{Q_t}|\nabla u_\varepsilon|^2 + c_\delta\,.$$

At this point, we choose δ small enough and then apply the Gronwall lemma. Thus, we obtain, in particular, the basic a priori estimate

$$\|\mathrm{I}_\varepsilon(\vartheta_\varepsilon)\|_{L^\infty(0,T;L^1(\Omega))} + \|u_\varepsilon\|_{L^2(0,T;V)} + \|\chi_\varepsilon\|_{L^\infty(0,T;V)\cap H^1(0,T;H)} \le c. \tag{4.14}$$

We note that we could have written more terms in (4.14). We have stressed just the estimates we use later on.

Consequences. Accounting for (2.15), it is straightforward to deduce that

$$\|\vartheta_\varepsilon\|_{L^2(0,T;V)} \le c. \tag{4.15}$$

Moreover, owing to the assumptions (3.3) on λ_ε and to the Sobolev and Hölder inequalities, we infer that

$$\|\partial_t \lambda_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;L^{3/2}(\Omega))} \leq c\|\chi_\varepsilon\|_{L^\infty(0,T;L^6(\Omega))}\|\partial_t \chi_\varepsilon\|_{L^2(0,T;L^2(\Omega))} \leq c. \tag{4.16}$$

In particular, the above derivative is bounded in $L^2(0,T;V_0')$, since the Sobolev inequality (2.32) implies that the continuous embedding $L^q(\Omega) \subset V_0'$ holds for any $q \geq 6/5$. As the boundedness property in $L^2(0,T;V_0')$ holds true for all the other terms of equation (3.8) but the first one, we deduce by comparison

$$\|\partial_t \operatorname{Ln}_\varepsilon(\vartheta_\varepsilon)\|_{L^2(0,T;V_0')} \leq c. \tag{4.17}$$

Second a priori estimate. We multiply (3.9) by $\beta_\varepsilon(\chi_\varepsilon)$ and integrate over Q_t. We obtain

$$\begin{aligned}
&\int_\Omega \widehat{\beta}_\varepsilon(\chi_\varepsilon(t)) + \int_{Q_t} \beta_\varepsilon'(\chi_\varepsilon)|\nabla\chi_\varepsilon|^2 + \int_{Q_t} |\beta_\varepsilon(\chi_\varepsilon)|^2 \\
&= \int_\Omega \widehat{\beta}_\varepsilon(\chi_0) - \int_{Q_t} \big(\sigma'(\chi_\varepsilon) + \lambda_\varepsilon'(\chi_\varepsilon)\vartheta_\varepsilon\big)\beta_\varepsilon(\chi_\varepsilon).
\end{aligned}$$

Now, the first integral on the right hand side is bounded thanks to (4.2) and (2.12), while the second one can be controlled by the left hand side. Indeed, the above estimates, our assumptions on λ_ε and σ, and the Sobolev and Hölder inequalities imply that

$$\|\sigma'(\chi_\varepsilon)\|_{L^\infty(0,T;V)} + \|\lambda_\varepsilon'(\chi_\varepsilon)\vartheta_\varepsilon\|_{L^2(0,T;L^3(\Omega))} \leq c. \tag{4.18}$$

Hence, we conclude that

$$\|\beta_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;H)} \leq c. \tag{4.19}$$

At this point, we infer a bound for the laplacian of χ_ε by comparison in (3.9), whence it follows that

$$\|\chi_\varepsilon\|_{L^2(0,T;W)} \leq c \tag{4.20}$$

because of (3.7) and the elliptic regularity theory.

Third a priori estimate. We integrate equation (3.8) in time and find

$$\begin{aligned}
&\operatorname{Ln}_\varepsilon(\vartheta_\varepsilon) - \operatorname{Ln}_\varepsilon(\vartheta_\mathcal{H}) - k_0 * \Delta u_\varepsilon \\
&= 1 * k * \Delta u_\varepsilon + \lambda_\varepsilon(\chi_\varepsilon) + 1 * R + \operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon}) - \lambda_\varepsilon(\chi_0) - \operatorname{Ln}_\varepsilon(\vartheta_\mathcal{H})
\end{aligned} \tag{4.21}$$

where we have added $-\operatorname{Ln}_\varepsilon(\vartheta_\mathcal{H})$ to both sides for convenience. Then, we multiply (4.21) by $-\Delta u_\varepsilon = -\Delta\vartheta_\varepsilon$ and integrate over Q_t. Finally, we integrate by parts using the Dirichlet boundary conditions. We obtain

$$\begin{aligned}
&\int_{Q_t} \nabla\big(\operatorname{Ln}_\varepsilon(\vartheta_\varepsilon) - \operatorname{Ln}_\varepsilon(\vartheta_\mathcal{H})\big) \cdot \nabla\vartheta_\varepsilon + \frac{k_0}{2}\int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 \\
&= -\int_{Q_t} (1 * k * \Delta u_\varepsilon)(\Delta u_\varepsilon) + \int_{Q_t} \lambda_\varepsilon(\chi_\varepsilon)(-\Delta u_\varepsilon) \\
&\quad + \int_{Q_t} (1 * R)(-\Delta u_\varepsilon) + \int_{Q_t} \big(\operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon}) - \lambda_\varepsilon(\chi_0)\big)(-\Delta u_\varepsilon) - \int_{Q_t} \operatorname{Ln}_\varepsilon(\vartheta_\mathcal{H})(-\Delta u_\varepsilon)
\end{aligned}$$

and deal with each term separately. The first integral on the left hand side gives two contributions. The first one is nonnegative. We move the other one to the right hand side and estimate it this way

$$\int_{Q_t} \nabla \operatorname{Ln}_\varepsilon(\vartheta_{\mathcal{H}}) \cdot \nabla \vartheta_\varepsilon \le \sup_{\vartheta_* \le r \le \vartheta^*} \operatorname{Ln}'_\varepsilon(r) \, \|\nabla \vartheta_{\mathcal{H}}\|_{L^2(Q)} \|\nabla \vartheta_\varepsilon\|_{L^2(Q)} \le c$$

where we have used Lemma 4.2 and estimates (2.15) and (4.15). The convolution term is treated integrating by parts. Accounting for the regularity of k (see (2.4)) and using the Young inequality (2.31), we have

$$\begin{aligned}
&-\int_{Q_t} (1 * k * \Delta u_\varepsilon)(\Delta u_\varepsilon) \\
&= -\int_\Omega (k * 1 * \Delta u_\varepsilon)(t)\,(1 * \Delta u_\varepsilon)(t) + \int_{Q_t} \big((k(0) + 1 * k') * \Delta u_\varepsilon\big)(1 * \Delta u_\varepsilon) \\
&\le \delta \int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 + c_\delta \|k\|^2_{L^2(0,T)} \|1 * \Delta u_\varepsilon\|^2_{L^2(0,t;H)} \\
&\quad + \big(|k(0)| + \|k'\|_{L^1(0,T)}\big) \int_{Q_t} |1 * \Delta u_\varepsilon|^2.
\end{aligned}$$

Now, we deal with the next term and integrate it by parts in time. Using the quadratic growth of λ_ε, the Sobolev inequality (2.32), and estimate (4.14), we have

$$\begin{aligned}
&\int_{Q_t} \lambda_\varepsilon(\chi_\varepsilon)(-\Delta u_\varepsilon) = -\int_\Omega \lambda_\varepsilon(\chi_\varepsilon(t))(1 * \Delta u_\varepsilon)(t) + \int_{Q_t} \partial_t \lambda_\varepsilon(\chi_\varepsilon)(1 * \Delta u_\varepsilon). \\
&\le \delta \int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 + c_\delta \Big(1 + \|\chi_\varepsilon\|^4_{L^\infty(0,T;L^4(\Omega))}\Big) \\
&\quad + \int_0^t \|\partial_t \lambda_\varepsilon(\chi_\varepsilon(s))\|_H \|(1 * \Delta u_\varepsilon)(s)\|_H \, ds \\
&\le \delta \int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 + c_\delta + \int_0^t \|\partial_t \lambda_\varepsilon(\chi_\varepsilon(s))\|_H \|(1 * \Delta u_\varepsilon)(s)\|_H \, ds.
\end{aligned}$$

We note at once that $\|\partial_t \lambda_\varepsilon(\chi_\varepsilon(\,\cdot\,))\|_H$ is bounded in $L^1(0,T)$. Indeed

$$\int_0^T \|\partial_t \lambda_\varepsilon(\chi_\varepsilon(s))\|_H \, ds \le c \|\lambda'_\varepsilon(\chi_\varepsilon)\|_{L^2(0,T;L^\infty(\Omega))} \|\partial_t \chi_\varepsilon\|_{L^2(0,T;H)} \le c$$

thanks to (3.2), (4.14), (4.20), and to the continuous embedding $W \subset L^\infty(\Omega)$. Finally, we treat the last three integrals together as follows

$$\begin{aligned}
&\int_{Q_t} \big(1 * R + \operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon}) - \lambda_\varepsilon(\chi_0) - \operatorname{Ln}_\varepsilon(\vartheta_{\mathcal{H}})\big)(-\Delta u_\varepsilon) \\
&= -\int_\Omega \big((1 * R)(t) + \operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon}) - \lambda_\varepsilon(\chi_0) - \operatorname{Ln}_\varepsilon(\vartheta_{\mathcal{H}}(t))\big)(1 * \Delta u_\varepsilon)(t)
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{Q_t} \bigl(R - \partial_t \operatorname{Ln}_\varepsilon(\vartheta_{\mathcal H})\bigr)(1 * \Delta u_\varepsilon) \\
&\le \delta \int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 + \int_{Q_t} |(1 * \Delta u_\varepsilon)|^2 \\
&\quad + c_\delta \Bigl(1 + \|R\|^2_{L^2(0,T;H)} + \|\operatorname{Ln}_\varepsilon(\vartheta_{\mathcal H})\|^2_{L^\infty(0,T;H)} + \|\partial_t \operatorname{Ln}_\varepsilon(\vartheta_{\mathcal H})\|^2_{L^2(0,T;H)}\Bigr) \\
&\le \delta \int_\Omega |(1 * \Delta u_\varepsilon)(t)|^2 + \int_{Q_t} |(1 * \Delta u_\varepsilon)|^2 + c_\delta \,.
\end{aligned}
$$

The last estimate is easily obtained by combining inequality (4.3), Lemmas 4.4, 4.3, and 4.2 with estimates (2.15) for $\vartheta_{\mathcal H}$. At this point, we choose δ small enough and apply and extended version of the Gronwall lemma (e.g., [8, Lemmas A.4 and A.5, pp. 156–157]). We obtain, in particular, the following estimate

$$\|1 * \Delta u_\varepsilon\|_{L^\infty(0,T;H)} \le c \tag{4.22}$$

whence we easily infer

$$\|\operatorname{Ln}_\varepsilon(\vartheta_\varepsilon)\|_{L^\infty(0,T;H)} \le c \tag{4.23}$$

by comparison in (4.21).

Conclusion. We collect all the obtained estimates and use well-known compactness results. It is understood that the convergences we are going to write hold for a suitable subsequence $\varepsilon_n \searrow 0$. By weak compactness, there exist five functions u, ϑ, χ, ξ, ℓ such that

$$u_\varepsilon \to u \quad \text{weakly in } L^2(0,T;V_0) \quad \text{and} \quad \vartheta_\varepsilon \to \vartheta \quad \text{weakly in } L^2(0,T;V) \tag{4.24}$$

$$\chi_\varepsilon \to \chi \quad \text{weakly in } L^2(0,T;W) \text{ and in } H^1(0,T;H) \tag{4.25}$$

$$\beta_\varepsilon(\chi_\varepsilon) \to \xi \quad \text{weakly in } L^2(0,T;H) \tag{4.26}$$

$$\operatorname{Ln}_\varepsilon(\vartheta_\varepsilon) \to \ell \quad \text{weakly star in } L^\infty(0,T;H) \text{ and in } H^1(0,T;V_0') \tag{4.27}$$

Note that the limit functions u and χ satisfy the regularity properties stated in (2.16) and (2.18), as well as the homogeneous Dirichlet and Neumann boundary conditions, respectively. Moreover, it is clear that $u = \vartheta - \vartheta_{\mathcal H}$ and that the Cauchy conditions $\ell(0) = \ln \vartheta_0$ and $\chi(0) = \chi_0$ are fulfilled (see (3.4–5)). Indeed, the weak convergence in $H^1(0,T;X)$ of a sequence implies the weak convergence in X of the corresponding initial values and one easily checks that $\operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon})$ tends to $\ln \vartheta_0$ a.e. in Ω, hence, e.g., in $L^2(\Omega)$. Next, $\lambda_\varepsilon(\chi_\varepsilon)$, $\lambda_\varepsilon'(\chi_\varepsilon)\vartheta_\varepsilon$, and $\sigma'(\chi_\varepsilon)$ are weakly convergent in the appropriate spaces as well (see (4.16) and (4.18)), so that we can pass to the limit in both equations (3.8) and (3.9). This yields (2.20) and (2.21) whenever we can identify the limits of all nonlinear terms. We start with the nonlinearities involving χ. The above estimates, well known strong compactness results, and the Aubin lemma (see, e.g., [25, p. 58]) yield

$$\chi_\varepsilon \to \chi \quad \text{strongly in } L^2(0,T;V) \,. \tag{4.28}$$

Such convergence and (4.24) enable us to identify the limits of $\sigma'(\chi_\varepsilon)$, $\lambda_\varepsilon(\chi_\varepsilon)$, and $\lambda'_\varepsilon(\chi_\varepsilon)\vartheta_\varepsilon$, with the help of (2.3) and (3.2–3). Indeed, we have that

$$\|\sigma'(\chi_\varepsilon) - \sigma'(\chi\|_{L^2(0,T;H)} \le c\|\chi_\varepsilon - \chi\|_{L^2(0,T;H)} .$$

Moreover, owing to the Taylor expansion

$$|\varphi(r+h) - \varphi(r) - h\varphi'(r)| \le ch^2 \quad \text{for every } r, h \in \mathbb{R} \tag{4.29}$$

which holds for any function φ with Lipschitz derivative φ', we deduce that

$$\begin{aligned} &|\lambda_\varepsilon(\chi_\varepsilon) - \lambda(\chi)| \le |\lambda_\varepsilon(\chi_\varepsilon) - \lambda_\varepsilon(\chi)| + |\lambda_\varepsilon(\chi) - \lambda(\chi)| \\ &\le |\lambda'_\varepsilon(\chi)(\chi_\varepsilon - \chi)| + c|\chi_\varepsilon - \chi|^2 + |\lambda_\varepsilon(\chi) - \lambda(\chi)| \\ &\le c(1 + |\chi|)|\chi_\varepsilon - \chi| + c|\chi_\varepsilon - \chi|^2 + |\lambda_\varepsilon(\chi) - \lambda(\chi)| \end{aligned}$$

whence $\lambda_\varepsilon(\chi_\varepsilon) \to \lambda(\chi)$ strongly, e.g., in $L^1(Q)$. Finally, noting that

$$|\lambda'_\varepsilon(\chi_\varepsilon) - \lambda'(\chi)| \le c|\chi_\varepsilon - \chi| + |\lambda'_\varepsilon(\chi) - \lambda'(\chi)|$$

it turns out that $\lambda'_\varepsilon(\chi_\varepsilon) \to \lambda'(\chi_\varepsilon)$ strongly in $L^2(0,T;H)$, and consequently that $\lambda'_\varepsilon(\chi_\varepsilon)\vartheta_\varepsilon \to \lambda'(\chi_\varepsilon)\vartheta$ weakly in $L^1(Q)$.

As far as the nonlinearity associated to the maximal monotone graph β is concerned, we can apply, e.g., [1, p. 42] and conclude that $\chi \in D(\beta)$ and $\xi \in \beta(\chi)$ a.e. in Q. Finally, we deal with the logarithmic term. Owing to the compact embedding $H \subset V_0'$ and to the Aubin lemma once more, we see that (4.27) implies

$$\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon) \to \ell \quad \text{weakly star in } L^\infty(0,T;H) \text{ and strongly in } L^2(0,T;V_0')$$

whence also

$$\ln_\varepsilon(\vartheta_\varepsilon) \to \ell \quad \text{weakly in } L^2(0,T;H) \text{ and strongly in } L^2(0,T;V_0') \tag{4.30}$$

since $\varepsilon\vartheta_\varepsilon \to 0$ strongly in $L^2(0,T;V)$. Hence, we infer that

$$\lim_{\varepsilon\searrow 0} \int_Q \vartheta_\varepsilon \ln_\varepsilon(\vartheta_\varepsilon) = \lim_{\varepsilon\searrow 0} \int_Q (u_\varepsilon + \vartheta_{\mathcal{H}}) \ln_\varepsilon(\vartheta_\varepsilon) = \langle \ell, u \rangle + \int_Q \vartheta_{\mathcal{H}}\ell = \int_Q \vartheta\ell \tag{4.31}$$

where $\langle\,\cdot\,,\,\cdot\,\rangle$ stands for the duality pairing between $L^2(0,T;V_0')$ and $L^2(0,T;V_0)$, and we can apply [1, p. 42] also in this case. We conclude that $\vartheta > 0$ and $\ell = \ln\vartheta$ a.e. in Q. This completes the proof.

Remark 4.5. The above proof clearly shows that we could have taken two different parameters ε' and ε'' in approximating ln and β. Moreover, we could have kept fixed either of them, say ε', and let ε'' tend to 0. This leads to an existence result for a half-regularized problem. Moreover, all the a priori estimates are conserved in the limit by the semicontinuity of the norms involved, so that it is possible to let ε' tend to 0. The same can be done by exchanging the parameters ε' and ε''.

Remark 4.6. It is clear that the solution we have obtain enjoys further regularity properties, due to the uniform estimates we have proved. In particular, we stress that

$$1 * \Delta u \in L^\infty(0,T;H) \tag{4.32}$$

thanks to (4.22).

5. Uniqueness and continuous dependence

In this section, we conclude the proof of Theorem 2.1, by showing that the solution is unique, and then prove Theorem 2.2. To this aim, we consider the integrated version of (2.20), namely the equality

$$\ln\vartheta - k_0 * \Delta u = 1 * k * \Delta u + \lambda(\chi) + 1 * R + \eta_0 \quad \text{where} \quad \eta_0 := \ln\vartheta_0 - \lambda(\chi_0) \tag{5.1}$$

which is a necessary condition for (ϑ,χ,ξ) to be a solution to problem (2.20–23), and couple it with (2.21) and the initial condition for χ. We pick two solutions $(\vartheta_i,\chi_i,\xi_i)$ to such a system corresponding to two sets of data $(R_i,\vartheta_{i\Gamma},\vartheta_{0i},\chi_{0i})$, $i=1,2$. Then, we write both (5.1) and (2.21) for such solutions and multiply the difference of the first equations by $u := u_1 - u_2$ and the difference of the second ones by $\chi := \chi_1 - \chi_2$. Finally, we sum the equalities we have obtained to each other and integrate over Q_t. We introduce a similar notation for all the differences involved (i.e., related either to solutions or to data), setting $\vartheta := \vartheta_1 - \vartheta_2$, ..., $\eta_0 := \eta_{01} - \eta_{02}$, and have

$$\begin{aligned}
&\int_{Q_t} (\ln\vartheta_1 - \ln\vartheta_2)\, u + \frac{k_0}{2}\int_\Omega |(1*\nabla u)(t)|^2 + \frac12\int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla\chi|^2 + \int_{Q_t} \xi\chi \\
&= -\int_{Q_t} (k * 1 * \nabla u)\cdot\nabla u + \int_{Q_t} (1*R+\eta_0)u + \frac12\int_\Omega |\chi_0|^2 - \int_{Q_t} \bigl(\sigma'(\chi_1)-\sigma'(\chi_2)\bigr)\chi \\
&\quad + \int_{Q_t} \bigl\{ \bigl(\lambda(\chi_1)-\lambda(\chi_2)\bigr)\vartheta - \bigl(\lambda'(\chi_1)\vartheta_1 - \lambda'(\chi_2)\vartheta_2\bigr)\chi \bigr\}.
\end{aligned}$$

We deal with each integral separately, as usual. On the left hand side, just the first term needs some treatment, since the other ones are nonnegative (in particular, the last term is nonnegative since β is monotone). We have

$$\int_{Q_t} (\ln\vartheta_1 - \ln\vartheta_2)\, u = \int_{Q_t} (\ln\vartheta_1 - \ln\vartheta_2)\,\vartheta - \int_{Q_t} (\ln\vartheta_1 - \ln\vartheta_2)\,\vartheta_{\mathcal H}\,.$$

The first contribution is nonnegative. We move the second integral to the right hand side and estimate it as follows

$$\begin{aligned}
&\int_{Q_t} (\ln\vartheta_1 - \ln\vartheta_2)\,\vartheta_{\mathcal H} \le \bigl(\|\ln\vartheta_1\|_{L^\infty(0,T;H)} + \|\ln\vartheta_2\|_{L^\infty(0,T;H)}\bigr)\,\|\vartheta_{\mathcal H}\|_{L^1(0,T;H)} \\
&\le c\,\bigl(\|\ln\vartheta_1\|_{L^\infty(0,T;H)} + \|\ln\vartheta_2\|_{L^\infty(0,T;H)}\bigr)\,\|\vartheta_\Gamma\|_{L^1(0,T;H^{-1/2}(\Gamma))}
\end{aligned}$$

thanks to (2.15) and (2.24). Let us consider the convolution term on the right hand side. We have

$$\begin{aligned}
&-\int_{Q_t} (k*1*\nabla u)\cdot\nabla u \\
&= -\int_\Omega (k*1*\nabla u)(t)\cdot(1*\nabla u)(t) + \int_{Q_t} \partial_t(k*1*\nabla u)\cdot(1*\nabla u) \\
&\le \delta\int_\Omega |(1*\nabla u)(t)|^2 + c_\delta\int_{Q_t} |1*\nabla u|^2
\end{aligned}$$

thanks to the regularity (2.4) of k and the Young inequality, as in the previous section. We treat the next term integrating it by parts in time and using the Poincaré inequality and (2.31). Then, we infer

$$
\begin{aligned}
&\int_{Q_t}(1*R+\eta_0)u = \int_\Omega \big((1*R)(t)+\eta_0\big)(1*u)(t) - \int_{Q_t} R(1*u)\\
&\le \delta\int_\Omega |(1*\nabla u)(t)|^2 + c_\delta \int_\Omega |(1*R)(t)+\eta_0|^2 + c\int_0^t \|R(s)\|_H\,\|(1*\nabla u)(s)\|_H\,ds\\
&\le \delta\int_\Omega |(1*\nabla u)(t)|^2 + c_\delta \int_0^t \|R(s)\|_H\,\|(1*\nabla u)(s)\|_H\,ds + c_\delta\|R\|^2_{L^1(0,T;H)} + c_\delta\|\eta_0\|^2_H.
\end{aligned}
$$

The subsequent term is dealt with using assumption (2.3) on σ'. We have

$$
-\int_{Q_t}\big(\sigma'(\chi_1)-\sigma'(\chi_2)\big)\chi \le c\int_{Q_t}|\chi|^2
$$

while the last integral needs some treatment. To this aim, we recall (4.29) and, after rearranging, we use the above estimate and the Hölder and Sobolev inequalities and obtain

$$
\begin{aligned}
&\int_{Q_t}\big\{\big(\lambda(\chi_1)-\lambda(\chi_2)\big)\vartheta - \big(\lambda'(\chi_1)\vartheta_1-\lambda'(\chi_2)\vartheta_2\big)\chi\big\}\\
&= \int_{Q_t}\vartheta_1\{\lambda(\chi_1)-\lambda(\chi_2)-\lambda'(\chi_1)\chi\} + \int_{Q_t}\vartheta_2\{\lambda(\chi_2)-\lambda(\chi_1)+\lambda'(\chi_2)\chi\}\\
&\le c\int_{Q_t}(\vartheta_1+\vartheta_2)|\chi|^2 \le c\int_0^t \|\vartheta_1(s)+\vartheta_2(s)\|_4\|\chi(s)\|_4\|\chi(s)\|_2\,ds\\
&\le c\int_0^t \|\vartheta_1(s)+\vartheta_2(s)\|_V\big(\|\nabla\chi(s)\|_H+\|\chi(s)\|_H\big)\|\chi(s)\|_H\,ds\\
&\le \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \|\vartheta_1(s)+\vartheta_2(s)\|_V^2\|\chi(s)\|_H^2\,ds\\
&\quad + c\int_0^t \|\vartheta_1(s)+\vartheta_2(s)\|_V\|\chi(s)\|_H^2\,ds\\
&\le \delta\int_{Q_t}|\nabla\chi|^2 + c_\delta\int_0^t \big(1+\|\vartheta_1(s)\|_V^2+\|\vartheta_2(s)\|_V^2\big)\,\|\chi(s)\|_H^2\,ds.
\end{aligned}
$$

At this point, we choose δ small enough and apply the extended version of the Gronwall lemma [8, pp. 156-157]. We obtain

$$
\begin{aligned}
&\int_Q(\ln\vartheta_1-\ln\vartheta_2)\,\vartheta + \int_\Omega|(1*\nabla u)(t)|^2 + \int_\Omega|\chi(t)|^2 + \int_Q|\nabla\chi|^2 + \int_Q\xi\chi\\
&\le c\,\exp\big(c\varphi(\vartheta_1,\vartheta_2)\big)\Big\{\psi(\vartheta_1,\vartheta_2)\|\vartheta_\Gamma\|_{L^1(0,T;H^{-1/2}(\Gamma))} + \|R\|^2_{L^1(0,T;H)} + \|\eta_0\|^2_H\Big\}
\end{aligned}
\tag{5.2}
$$

where we have set

$$
\begin{aligned}
\varphi(\vartheta_1,\vartheta_2) &:= 1+\|\vartheta_1\|^2_{L^2(0,T;V)}+\|\vartheta_2\|^2_{L^2(0,T;V)}\\
\psi(\vartheta_1,\vartheta_2) &:= \|\ln\vartheta_1\|_{L^\infty(0,T;H)}+\|\ln\vartheta_2\|_{L^\infty(0,T;H)}.
\end{aligned}
\tag{5.3}
$$

In the particular case of the same sets of data, the right hand side of (5.2) vanishes, and this implies that the two solutions have the same u and χ, whence the same ϑ. By comparison in (2.21), we see that the last component ξ is uniquely determined as well. This shows uniqueness.

To prove Theorem 2.2, we come back to the above situation of two sets of data and assume that all of their norms related to assumptions (2.9–12) are bounded by a constant M. Because of uniqueness, the solutions we are considering coincide with the ones we have constructed in our existence and stability result already proved in the previous section. Hence, the quantities (5.3) are bounded by a constant M', as specified in the last part of Theorem 2.1. Therefore, inequality (5.2) yields (2.25), for some constant M'' fulfilling all the desired requirements.

6. Further properties of the solution

This section is devoted to further regularity. We first prove Theorem 2.3, by assuming (2.26) in addition to (2.9–12). We use the notation

$$\|v\|_{q,p,Q_t} := \|v\|_{L^p(0,t;L^q(\Omega))} \quad \text{and} \quad \|v\|_{q,p} := \|v\|_{q,p,Q}$$

for $p, q \in [1, \infty]$. We perform further a priori estimates on the solutions $(\vartheta_\varepsilon, \chi_\varepsilon)$ to the approximating problems (3.8–10). However, to simplify the notation, we do not stress the dependence on ε of some of the function we define.

First regularity estimate. Our first aim is proving (2.27). We start from the inequality

$$\begin{aligned} &\|v\|_{\bar{q},\bar{p},Q_t} \le c\|v\|_{L^2(0,t;V)\cap L^\infty(0,t;H)} \\ &\quad \text{for} \quad 2 \le \bar{q} \le 6, \quad 2 \le \bar{p} \le \infty, \quad \frac{1}{\bar{p}} + \frac{3}{2\bar{q}} = \frac{3}{4} \end{aligned} \tag{6.1}$$

which holds for every $v \in L^2(0,T;V) \cap L^\infty(0,T;H)$ and $t \in (0,T)$. In (6.1), c depends on the values of $\bar{p}$ and $\bar{q}$ as well. For the sake of simplicity, we allow the values of all constants termed c to depend even on the exponent of the L^p- type spaces we consider whenever such exponents are fixed. A general estimate that yields (6.1) in the three-dimensional case is proved in [24, pp. 74–75] for functions vanishing on Σ and for a general Ω. There, the value of c is computed exactly. However, it is easy to see that the same proof holds for $V-$ valued functions provided that Ω is smooth. In such a case, c depends on the constant of the "full" Gagliardo-Nirenberg inequality, hence on the smoothness of Ω.

In order to prove (2.27), we essentially follow [24, pp. 194-201], where the variational solution to a general linear parabolic equation is considered and the time average of the solution enters the test function used. In our case, we do not need any average since χ_ε is smooth (see (3.7)), but our equation is nonlinear and, despite of that, we want to obtain an estimate that is uniform with respect to ε. For any positive integer n and for $r \ge 1$ we set

$$v_n := \pm \left(\min\left\{\chi^\pm, n\right\}\right)^{2r-1} \quad \text{and} \quad w_n := \left(\min\left\{\chi^\pm, n\right\}\right)^r.$$

For the sake of convenience, we define also

$$\psi_n(s) := \int_0^s \left(\min\left\{s_1^{\pm}, n\right\}\right)^{2r-1} ds_1 \quad \text{for } s \in \mathbb{R} \quad \text{and} \quad f_\varepsilon := -\sigma'(\chi_\varepsilon) - \lambda_\varepsilon'(\chi_\varepsilon)\vartheta_\varepsilon.$$

Then, we multiply (3.9) by v_n and integrate over Q_t. We obtain

$$\int_{Q_t} \partial_t \chi_\varepsilon \, v_n + \int_{Q_t} \nabla\chi_\varepsilon \cdot \nabla v_n + \int_{Q_t} \beta_\varepsilon(\chi_\varepsilon) v_n = \int_{Q_t} f_\varepsilon v_n. \tag{6.2}$$

The first integral is given by

$$\int_{Q_t} \partial_t \chi_\varepsilon \, v_n = \int_{Q_t} \partial_t \psi_n(\chi_\varepsilon) = \int_\Omega \psi_n(\chi_\varepsilon(t)) - \int_\Omega \psi_n(\chi_0).$$

Noting that the following inequalities hold

$$\frac{1}{2r}\left(\min\left\{s^{\pm}, n\right\}\right)^{2r} \le \psi_n(s) \le \frac{1}{2r}|s|^{2r} \quad \text{for every } s \in \mathbb{R}$$

we deduce the estimates

$$\int_\Omega \psi_n(\chi_\varepsilon(t)) \ge \frac{1}{2r}\int_\Omega |w_n(t)|^2 \quad \text{and} \quad \int_\Omega \psi_n(\chi_0) \le \frac{1}{2r}\int_\Omega |\chi_0|^{2r}.$$

The second integral of (6.2) is easy to handle. We have indeed

$$\int_{Q_t} \nabla\chi_\varepsilon \cdot \nabla v_n = \frac{2r-1}{r^2}\int_{Q_t} |\nabla w_n|^2.$$

The third term is nonnegative since $\beta_\varepsilon(\chi_\varepsilon)v_n \ge 0$ a.e. in Q. In fact, in view of (2.1), β_ε is monotone and $\beta_\varepsilon(0) = 0$, whence $\beta_\varepsilon(\chi_\varepsilon)$ and χ_ε have the same sign. Then, we deal with the right hand side. Using the Hölder inequality, we have

$$\begin{aligned} \int_{Q_t} f_\varepsilon v_n &\le \int_{Q_t} |f_\varepsilon||v_n| = \int_{Q_t} |f_\varepsilon|\, w_n^{2-1/r} \\ &\le \|f_\varepsilon\|_{q,2}\, \|w_n^{2-1/r}\|_{q',2,Q_t} = \|f_\varepsilon\|_{q,2}\, \|w_n\|_{\bar q,\bar p,Q_t}^{2-1/r} \end{aligned}$$

where we have defined q, q', $\bar p$, and $\bar q$ by

$$\frac{1}{q} = \frac{1}{2r} + \frac{1}{3}, \quad \frac{1}{q'} = 1 - \frac{1}{q}, \quad \bar p = 4 - \frac{2}{r}, \quad \text{and} \quad \bar q = 2q' - \frac{q'}{r}.$$

We note that $q < 3$ and recall that f_ε is estimated in $L^2(0,T;L^3(\Omega))$ by (4.18). Moreover, our choice of $\bar p$ and $\bar q$ fulfils the compatibility conditions listed in (6.1).

Hence, we can apply (6.1) itself. Observing that $w_n \in C^0([0,T];H)$ by (3.7) and using the elementary Young inequality (2.29), we deduce that

$$\begin{aligned}
&\int_{Q_t} f_\varepsilon\, v_n \le c\|w_n\|_{L^2(0,t;V)\cap L^\infty(0,t;H)}^{2-1/r} \le \delta\|w_n\|_{L^2(0,t;V)\cap L^\infty(0,t;H)}^2 + c_\delta \\
&\le \delta\int_{Q_t}|\nabla w_n|^2 + \delta\int_{Q_t}|w_n|^2 + \delta\sup_{0\le s\le t}\|w_n(s)\|_H^2 + c_\delta \\
&\le \delta\int_{Q_t}|\nabla w_n|^2 + \delta(T+1)\sup_{0\le s\le t}\|w_n(s)\|_H^2 + c_\delta\,.
\end{aligned}$$

Collecting (6.2) and all the inequalities we have obtained, choosing δ small enough, and applying the generalized Gronwall lemma, we conclude that $\{w_n\}$ is bounded in $L^2(0,T;V)\cap L^\infty(0,T;H)$. Then, letting n tend to ∞, we infer that $|\chi_\varepsilon|^r$ belongs to such a space and precisely that

$$\{|\chi_\varepsilon|^r\} \text{ is bounded in } L^2(0,T;V)\cap L^\infty(0,T;H)\,. \tag{6.3}$$

As we can assume χ_ε converging to χ a.e. in Q at least for a subsequence, due to the strong convergences proved at the end of Section 4, we infer that the weak limit induced by estimate (6.3) coincides with $|\chi|^r$. Hence, we obtain both conditions (2.27), since $r \ge 1$ is arbitrary.

Second regularity estimate. Now, we prove (2.28). Hence, we fix $p \in [1,2)$. Clearly, we can assume $p > 1$, hence $p \in (1,2)$. We set for convenience

$$F_\varepsilon := R + k * \Delta u_\varepsilon + \partial_t \lambda_\varepsilon(\chi_\varepsilon)$$

and claim that the following estimate holds

$$\|F_\varepsilon\|_{L^2(0,T;L^r(\Omega))} \le c_r \quad \text{for every } r < 2\,. \tag{6.4}$$

Indeed, we have

$$F_\varepsilon = R + k(0)(1 * \Delta u_\varepsilon) + k' * 1 * \Delta u_\varepsilon + \lambda_\varepsilon'(\chi_\varepsilon)\partial_t \chi_\varepsilon$$

and we can use (2.9) for the first term, (2.4) and (4.22) for the convolutions, and (3.7), (6.3), and (4.14) for the last product. On the other hand, (3.8) reads

$$\partial_t \operatorname{Ln}_\varepsilon(\vartheta_\varepsilon) - k_0 \Delta u_\varepsilon = F_\varepsilon. \tag{6.5}$$

We define for any positive integer $n > \vartheta^*$

$$\vartheta_n := \min\left\{\vartheta_\varepsilon^+, n\right\}$$

and multiply (6.5) by $\vartheta_n^{2p-1} - \vartheta_{\mathcal{H}}^{2p-1}$. Noting that such a test function vanishes on Σ due to (3.6) and (2.15), we integrate over Q_t and obtain

$$\begin{aligned}
&\int_{Q_t} \partial_t \vartheta_\varepsilon \ \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon)\, \vartheta_n^{2p-1} + k_0 \int_{Q_t} \nabla\vartheta_\varepsilon \cdot \nabla\vartheta_n^{2p-1} \\
&= \int_{Q_t} \partial_t \,\mathrm{Ln}_\varepsilon(\vartheta_\varepsilon)\, \vartheta_{\mathcal{H}}^{2p-1} + k_0 \int_{Q_t} \nabla u_\varepsilon \cdot \nabla\vartheta_{\mathcal{H}}^{2p-1} \\
&\quad + k_0 \int_{Q_t} \nabla\vartheta_{\mathcal{H}} \cdot \nabla\vartheta_n^{2p-1} + \int_{Q_t} F_\varepsilon \vartheta_n^{2p-1} - \int_{Q_t} F_\varepsilon \vartheta_{\mathcal{H}}^{2p-1}. \qquad (6.6)
\end{aligned}$$

We want to estimate the first term from below. We have

$$\int_{Q_t} \partial_t \vartheta_\varepsilon \ \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon)\, \vartheta_n^{2p-1} = \int_{Q_t} \partial_t \psi_n(\vartheta_\varepsilon) = \int_\Omega \psi_n(\vartheta_\varepsilon(t)) - \int_\Omega \psi_n(\vartheta_{0\varepsilon})$$

where $\psi_n : \mathbb{R} \to \mathbb{R}$ is defined by means of the formula

$$\psi_n(r) := \int_0^r \mathrm{Ln}_\varepsilon'(s) \left(\min\left\{s^+, n\right\}\right)^{2p-1} ds \quad \text{for } r \in \mathbb{R}\,.$$

Therefore, we need an estimate from below for ψ_n. With the notation (4.6–7), we see that $\rho_\varepsilon(0) \to 0$ and easily have for every $r \geq 0$ and for $\varepsilon > 0$ small enough

$$\mathrm{Ln}_\varepsilon'(r) \geq \ln_\varepsilon'(r) = \frac{1}{\rho_\varepsilon(r) + \varepsilon} \geq \frac{1}{r + \rho_\varepsilon(0) + \varepsilon} \geq \frac{1}{r+1} \qquad (6.7)$$

whence also

$$\psi_n(r) \geq \varphi\left(\min\{r, n\}\right) \quad \text{where} \quad \varphi(r) := \int_0^r \frac{s^{2p-1}}{s+1}\, ds \quad \text{for } r > 0\,.$$

Now, we estimate the function φ from below. We have for $r > 1$

$$\varphi(r) \geq \int_1^r \frac{s^{2p-1}}{s+1}\, ds \geq \int_1^r \frac{s^{2p-1}}{2s}\, ds = \frac{1}{2(2p-1)}\, r^{2p-1} - c.$$

As a similar estimate trivially holds for $0 \leq r \leq 1$ as well, we obtain

$$\int_\Omega \psi_n(\vartheta_\varepsilon(t)) \geq \int_\Omega \varphi(\vartheta_n(t)) \geq \frac{1}{2(2p-1)} \int_\Omega |\vartheta_n(t)|^{2p-1} - c$$

for ε small enough. On the other hand, one easily sees that

$$\int_\Omega \psi_n(\vartheta_{0\varepsilon}) \leq c$$

owing to Lemma 4.2 and to (3.4). Therefore, we conclude that

$$\int_{Q_t} \partial_t \vartheta_\varepsilon \ \mathrm{Ln}_\varepsilon'(\vartheta_\varepsilon)\, \vartheta_n^{2p-1} \geq \frac{1}{2(2p-1)} \int_\Omega |\vartheta_n(t)|^{2p-1} - c.$$

The second term of (6.6) is easier to handle. We have indeed

$$k_0 \int_{Q_t} \nabla\vartheta_\varepsilon \cdot \nabla\vartheta_n^{2p-1} = k_0 \int_{Q_t} \nabla\vartheta_n \cdot \nabla\vartheta_n^{2p-1} = \frac{k_0(2p-1)}{p^2} \int_\Omega |\nabla\vartheta_n^p|^2.$$

The first term on the right hand side is treated by integration by parts. We obtain

$$\begin{aligned} &\int_{Q_t} \partial_t \operatorname{Ln}_\varepsilon(\vartheta_\varepsilon)\, \vartheta_{\mathcal H}^{2p-1} \\ &= \int_\Omega \operatorname{Ln}_\varepsilon(\vartheta_\varepsilon(t))\vartheta_{\mathcal H}^{2p-1}(t) - \int_\Omega \operatorname{Ln}_\varepsilon(\vartheta_{0\varepsilon})\vartheta_{\mathcal H}^{2p-1}(0) - \int_{Q_t} \operatorname{Ln}_\varepsilon(\vartheta_\varepsilon)\partial_t\vartheta_{\mathcal H}^{2p-1} \\ &\le c\|\operatorname{Ln}_\varepsilon(\vartheta_\varepsilon)\|_{L^\infty(0,T;H)} + c \le c \end{aligned}$$

thanks to (4.23) and (2.15). Next, with the help of (4.14) we have

$$k_0 \int_{Q_t} \nabla u_\varepsilon \cdot \nabla\vartheta_{\mathcal H}^{2p-1} \le k_0\|u_\varepsilon\|_{L^2(0,T;V)}\|\vartheta_{\mathcal H}^{2p-1}\|_{L^2(0,T;V)} \le c.$$

In order to deal with the next term of (6.6), we let $q := (2p-1)/(p-1)$ and $r := 2(2p-1)$. Observing that $(1/2) + (1/q) + (1/r) = 1$, we can use the Hölder inequality this way

$$\begin{aligned} &k_0 \int_{Q_t} \nabla\vartheta_{\mathcal H} \cdot \nabla\vartheta_n^{2p-1} = \frac{k_0(2p-1)}{p} \int_{Q_t} \vartheta_n^{p-1}\nabla\vartheta_n^p \cdot \nabla\vartheta_{\mathcal H} \\ &\le \int_0^t \|\vartheta_n^{p-1}(s)\|_q\|\nabla\vartheta_n^p(s)\|_2\|\nabla\vartheta_{\mathcal H}(s)\|_r\, ds \\ &\le \delta \int_{Q_t} |\nabla\vartheta_n^p|^2 + c_\delta \int_0^t \|\vartheta_n^{p-1}(s)\|_q^2\|\nabla\vartheta_{\mathcal H}(s)\|_r^2\, ds \\ &\le \delta \int_{Q_t} |\nabla\vartheta_n^p|^2 + c_\delta \int_0^t \|\vartheta_n^{2p-1}(s)\|_1^{2/q}\|\nabla\vartheta_{\mathcal H}(s)\|_r^2\, ds. \end{aligned}$$

Noting that $2/q \le 1$ and $r \le 6$ and owing to the Sobolev inequality, we get

$$k_0 \int_{Q_t} \nabla\vartheta_{\mathcal H} \cdot \nabla\vartheta_n^{2p-1} \le \delta \int_{Q_t} |\nabla\vartheta_n^p|^2 + c_\delta \int_0^t \left(1 + \|\vartheta_n^{2p-1}(s)\|_1\right) \|\vartheta_{\mathcal H}(s)\|_{H^2(\Omega)}^2\, ds.$$

We observe that the function $\|\vartheta_{\mathcal H}(\,\cdot\,)\|_{H^2(\Omega)}^2$ belongs to $L^1(0,T)$ because of (2.26) and (2.14). In order to deal with the second last term, we set $q := (2p-1)/(p-1)$ as above and define $\nu > 1$ by means of $(1/\nu) + (1/q) + (1/6) = 1$. Then, it turns out that $\nu < 2$, whence F_ε is bounded in $L^2(0,T;L^\nu(\Omega))$ by (6.4). Hence, we have

$$\begin{aligned} &\int_{Q_t} F_\varepsilon\vartheta_n^{2p-1} \le \int_{Q_t} |F_\varepsilon|\vartheta_n^{p-1}\vartheta_n^p \\ &\le \int_0^t \|F_\varepsilon(s)\|_\nu\|\vartheta_n^{p-1}(s)\|_q\|\vartheta_n^p(s)\|_6\, ds \\ &\le \delta \int_0^t \|\vartheta_n^p(s)\|_6^2\, ds + c_\delta \int_0^t \|F_\varepsilon(s)\|_\nu^2\|\vartheta_n^{p-1}(s)\|_q^2\, ds \\ &\le \delta \int_0^t \|\vartheta_n^p(s)\|_6^2\, ds + c_\delta \int_0^t \left(1 + \|\vartheta_n^{2p-1}(s)\|_1\right) \|F_\varepsilon(s)\|_\nu^2\, ds \end{aligned}$$

where we have used the same argument as before as far as q is concerned. Hence, we have to estimate the first integral of the last line. Accounting for the Sobolev inequality (2.32) once more, we have

$$\begin{aligned}
&\int_0^t \|\vartheta_n^p(s)\|_6^2\, ds \le M_\Omega \int_0^t \|\vartheta_n^p(s)\|_V^2\, ds \\
&\le c\int_{Q_t} |\nabla\vartheta_n^p|^2 + c\int_0^t \Big(\int_\Omega \vartheta_n^{p-1/2}(s)\vartheta_n^{1/2}(s)\Big)^2 ds \\
&\le c\int_{Q_t} |\nabla\vartheta_n^p|^2 + c\|\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))}\int_0^t \|\vartheta_n^{2p-1}(s)\|_1\, ds. \qquad (6.8)
\end{aligned}$$

Let us check that $\|\vartheta_n\|_{L^\infty(0,T;L^1(\Omega))}$ can be estimated. Using the definition (4.1) of I_ε and the inequality (6.7), for $r \ge 0$ we have indeed

$$\mathrm{I}_\varepsilon(r) = \int_0^r s\,\mathrm{Ln}_\varepsilon'(s)\, ds \ge \int_0^r \frac{s}{s+1}\, ds = r - \ln(r+1) \ge \frac{r}{2} - c.$$

We infer that

$$\int_\Omega |\vartheta_n(t)| \le \int_\Omega \vartheta_\varepsilon^+(t) \le 2\int_\Omega \mathrm{I}_\varepsilon(\vartheta_\varepsilon^+(t)) + c \le c$$

for a.a. $t \in (0,T)$ due to (4.14). Therefore, we conclude that

$$\begin{aligned}
&\int_{Q_t} F_\varepsilon \vartheta_n^{2p-1} \\
&\le \delta M_\Omega \int_{Q_t} |\nabla\vartheta_n^p|^2 + c\int_0^t \|\vartheta_n^{2p-1}(s)\|_1\, ds + c_\delta \int_0^t \big(1 + \|\vartheta_n^{2p-1}(s)\|_1\big)\, \|F_\varepsilon(s)\|_\nu^2\, ds.
\end{aligned}$$

Finally, the last term of (6.6) is easily estimated as follows

$$-\int_{Q_t} F_\varepsilon \vartheta_{\mathcal{H}}^{2p-1} \le (\vartheta^*)^{2p-1}\|F_\varepsilon\|_{L^1(Q)} \le c$$

thanks to (2.15) and (6.4). At this point, we collect (6.6) and all the inequalities we have derived and choose δ small enough. Then, we apply the Gronwall lemma and obtain the following estimate (cf. also (6.8))

$$\|\vartheta_n\|_{L^\infty(0,T;L^{2p-1}(\Omega))} + \|\vartheta_n^p\|_{L^2(0,T;V)} \le c.$$

From that it is easy to deduce a similar estimate for ϑ_ε^+, namely

$$\|\vartheta_\varepsilon^+\|_{L^\infty(0,T;L^{2p-1}(\Omega))} + \|(\vartheta_\varepsilon^+)^p\|_{L^2(0,T;V)} \le c \qquad (6.9)$$

simply using the basic properties of the Lebesgue theory. However, we cannot yet derive (2.28), since (6.9) involves new nonlinearities. We need some technical lemmas.

Lemma 6.1. *The inverse function of* $\ln_\varepsilon$ *is given by the formula* $\ln_\varepsilon^{-1}(s) = e^s + \varepsilon s$ *for every* $s \in \mathbb{R}$. ∎

Proof. Let $s \in \mathbb{R}$ and $r := e^s + \varepsilon s$. Then, $\rho_\varepsilon(r) = e^s$ by (4.7), whence (4.6) yields

$$\ln_\varepsilon(r) = \frac{r - \rho_\varepsilon(r)}{\varepsilon} = s$$

and we conclude. ∎

Lemma 6.2. *Let* $q \geq 1$ *and* $z \in L^2(Q)$ *be such that* $e^{qz} \in L^2(Q)$ *and set*

$$z_\varepsilon := ((\ln_\varepsilon^{-1}(z))^+)^q.$$

Then $z_\varepsilon \to e^{qz}$ *strongly in* $L^2(Q)$. ∎

Proof. The previous lemma yields $z_\varepsilon = ((e^z + \varepsilon z)^+)^q$, and we immediately see that $z_\varepsilon \to e^{qz}$ a.e. in Q. On the other hand, assuming $\varepsilon \in (0,1)$, we have

$$(e^z + \varepsilon z)^+ \leq e^z + \varepsilon z^+ \leq e^z + z^+ \leq 2e^z \quad \text{whence } z_\varepsilon \leq 2^q e^{qz} \quad \text{a.e. in } Q.$$

Therefore, we can apply the Lebesgue dominated convergence theorem. ∎

Conclusion. By (6.9), we have that

$$(\vartheta_\varepsilon^+)^p \to \vartheta_p \quad \text{weakly in } L^2(0,T;V) \tag{6.10}$$

for some $\vartheta_p \in L^2(0,T;V)$, at least for a subsequence, and we show that $\vartheta_p = \vartheta^p$. To this aim, we introduce $\alpha, \alpha_\varepsilon : \mathbb{R} \to \mathbb{R}$ by means of the formulas

$$\alpha(r) := e^{pr} \quad \text{and} \quad \alpha_\varepsilon(r) := ((\ln_\varepsilon^{-1}(r))^+)^p$$

and observe that they are monotone and continuous. Thus, we can see them as maximal monotone graphs in $\mathbb{R} \times \mathbb{R}$. As the equality $\vartheta_p = \vartheta^p$ is equivalent to $\vartheta_p = \alpha(\ln \vartheta)$, we only need to prove that

$$\int_{Q_t} (\vartheta_p - \alpha(z))(\ln \vartheta - z) \geq 0 \quad \text{for every } z \in L^2(Q) \text{ such that } e^{pz} \in L^2(Q). \tag{6.11}$$

So, we fix such a z and define $z_\varepsilon := \alpha_\varepsilon(z)$ and $\ell_\varepsilon := \ln_\varepsilon(\vartheta_\varepsilon)$. Then, we have

$$\int_{Q_t} \big((\vartheta_\varepsilon^+)^p - z_\varepsilon\big)(\ell_\varepsilon - z) = \int_{Q_t} (\alpha_\varepsilon(\ell_\varepsilon) - \alpha_\varepsilon(z))(\ell_\varepsilon - z) \geq 0$$

due to the monotonicity of α_ε, and we obtain (6.11) whenever we can pass to the limit in the above inequality. In fact, the following convergences

$$(\vartheta_\varepsilon^+)^p - \vartheta_{\mathcal{H}}^p \to \vartheta_p - \vartheta_{\mathcal{H}}^p \quad \text{weakly in } L^2(0,T;V_0)$$
$$z_\varepsilon \to e^{pz} \quad \text{strongly in } L^2(Q)$$
$$\ell_\varepsilon \to \ln \vartheta \quad \text{weakly in } L^2(Q) \text{ and strongly in } L^2(0,T;V_0')$$

actually hold, as we immediately see. Indeed, the first one is ensured by (6.10), the second one is given by Lemma 6.2 with $q = p$, and the last ones follow from (4.30) and (4.27), respectively. So, arguing as in (4.31), we have $\vartheta_p = \vartheta^p$, and this concludes the proof of the first of conditions (2.28).

Finally, let us come to the second of (2.28). To this aim, we observe that (4.14) implies that

$$\|\vartheta_\varepsilon^+\|_{L^2(0,T;V)} \leq c$$

so that we can argue exactly as before and apply Lemma 6.2 with $q = 1$. This yields

$$\vartheta_\varepsilon^+ \to \vartheta \quad \text{weakly in } L^2(0,T;V).$$

Therefore, the regularity to be shown follows from the first of estimates (6.9).

Acknoledgements. The authors would like to acknowledge financial support from the MIUR-COFIN 2004 research program on “Mathematical modelling and analysis of free boundary problems” and from the IMATI of CNR, Pavia, Italy.

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