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# On a phase transition model of Penrose–Fife type

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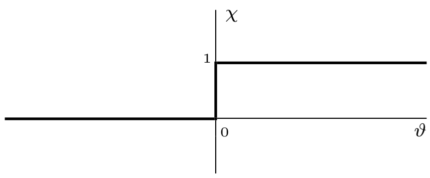
Two-phase Stefan problem ($x \in \Omega \subset \mathbb{R}^3$, $t \in (0,T)$)

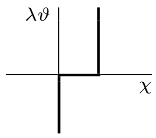
| | |
|---|---|
| • $\partial_t e + \operatorname{div} \mathbf{q} = g$ | energy balance |
| $e = e(x,t)$ | internal energy |
| $\mathbf{q} = \mathbf{q}(x,t)$ | heat flux |
| $g = g(x,t)$ | heat source |
| $e = \vartheta + \lambda\chi$ | constitutive law for $e$ |
| $\mathbf{q} = -\nabla\vartheta$ | Fourier law |
| $\vartheta = \vartheta_{abs} - \vartheta_c = \vartheta(x,t)$ | relative temperature |
| $\chi = \chi(x,t)$ | phase parameter |
| $\lambda = \text{cnst} > 0$ | latent heat |
| • $\chi \in \mathcal{H}(\vartheta)$ | $\chi$ = proportion of the solide phase |

where $\mathcal{H}$ is the Heaviside graph




Introduction 1

*Typical phase field model*




First note that

$$\chi \in \mathcal{H}(\vartheta) \Longleftrightarrow \mathcal{H}^{-1}(\chi) \ni \lambda\vartheta$$

Then relax ($\mu, \nu > 0$ small constants).
Phase relaxation (Visintin 1985)

$$\mu\partial_t\chi + \mathcal{H}^{-1}(\chi) \ni \lambda\vartheta$$

Phase field

$$\mu\partial_t\chi - \nu\Delta\chi + \mathcal{H}^{-1}(\chi) \ni \lambda\vartheta$$

Allen–Cahn dynamics

$$\mu\partial_t\chi - \nu\Delta\chi + \mathcal{W}'(\chi) = \lambda\vartheta$$

where $\mathcal{W}$ is a double well potential like

$$\mathcal{W}(\chi) = \chi^2(1-\chi)^2.$$

Both phase field models are included in

$$\begin{aligned} &\partial_t e + \operatorname{div} \mathbf{q} = g \\ &e = \vartheta + \lambda \chi, \quad \mathbf{q} = -\nabla \vartheta \\ &\mu \partial_t \chi - \nu \Delta \chi + \partial j(\chi) + \sigma'(\chi) \ni \lambda \vartheta \end{aligned}$$

where

$$\begin{aligned} &j : \mathbb{R} \to [0, +\infty] \quad && \text{convex, proper, l.s.c.} \\ &\sigma : \mathbb{R} \to \mathbb{R} && \text{smooth, } \sigma' \text{ Lipschitz} \end{aligned}$$

Phase dynamics is the gradient flow governed by the free energy functional

$$\mathcal{F}_\vartheta(\chi) = \frac{\nu}{2} \int_\Omega |\nabla \chi|^2 + \int_\Omega \Big( j(\chi) + \sigma(\chi) - \lambda \vartheta \chi \Big)$$

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Very wide literature, see lots of references in

M. Brokate & J. Sprekels:
*Hysteresis and phase transitions*
Springer-Verlag, 1996

A. Visintin:
*Models of phase transitions*
Birkhäuser, 1996

Trouble of previous models

- linearization near $\vartheta_{abs} = \vartheta_c$
- no thermodynamical consistency

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Penrose–Fife 1990

- no linearization
- thermodynamical consistency
- $\vartheta = \vartheta_{abs} =$ **absolute temperature**: $\vartheta > 0$

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Generalized model

$$\partial_t e + \operatorname{div} \mathbf{q} = g$$

$$e = \vartheta + \lambda \chi$$

some constitutive law for $\mathbf{q}$

$$\mu \partial_t \chi - \nu \Delta \chi + \partial j(\chi) + \sigma'(\chi) \ni \frac{\lambda}{\vartheta_c} - \frac{\lambda}{\vartheta}$$

where $j$ and $\sigma$ as above

**5** —— **References**

- particular cases
- modified problems
- choice of the boundary conditions

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**Well-posedness**

Colli, Horn, Kenmochi, Laurençot, Niezgódka, Sprekels, Zheng, . . . (several combinations)

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**Other directions**:

long-time behavior, asymptotic analyses, memory, numerical approach, Cahn-Hilliard dynamics. . .

**References** —— **5**

## 6 References

*Zheng 1992*: well-posedness
*Kenmochi, Niezgódka 1993 and 1994*: well-posedness
*Horn 1993*: numerical scheme
*Sprekels 1993*: well-posedness
*Sprekels, Zheng 1993*: well-posedness
*Kenmochi, Niezgódka 1994*: well-posedness
*Laurençot 1994*: well-posedness
*Colli, Sprekels 1995*: asymptotic analysis
*Laurençot 1995*: well-posedness
*Horn, Laurençot, Sprekels 1996*: well-posedness
*Novick-Cohen, Zheng 1996*: stationary solutions
*Klein 1998*: time discretization
*Colli, Laurençot 1998*: well-posedness
*Colli, Sprekels 1998*: memory
*Krejčí, Sprekels 1998*: connections with hysteresis and $\nu$=0
*Shirakawa, Ito, Yamazaki, Kenmochi 1998*: well-posedness and long time behavior
*Laurençot 1998*: well-posedness with Fourier law
*Colli, Laurençot, Sprekels 1999*: well-posedness
*Colli, Sprekels 1999*: well-posedness with Fourier law and $\nu$=0
*Kenmochi, Kubo 1999*: well-posedness
*Klein 1999*: time discretization
*Ito, Kenmochi 2000*: inertial sets
*Stefanelli 2001*: time discretization
*Colli, Grasselli, Ito 2002*: parabolic-hyperbolic related problems
*Recupero 2002*: memory
*Shen, Zheng 2002*: actractors
*Ito, Kenmochi, Kubo* (to appear): well-posedness, also C-H dyn
*Rocca* (to appear): well-posedness with C-H dyn and memory
*Rocca, Schimperna* (to appear): C-H dyn and Fourier law, well-posedness
*Rocca, Schimperna* (to appear): C-H dyn and memory, asymptotics
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## References 6

- Colli–Laurençot 1998 (CL)
- Colli–Laurençot–Sprekels 1999 (CLS)

The constitutive law for $\mathbf{q}$ has the form

$$\mathbf{q} = -\nabla\alpha(\vartheta)$$

where

$$\alpha : (0, +\infty) \to \mathbb{R}$$

is **strictly monotone, concave, onto**
suitable behavior near 0 and $+\infty$
(thermodynamical consistency ensured)

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Equivalent forms

$$\mathbf{q} = -k_1(\vartheta)\nabla\vartheta \qquad \text{and} \qquad \mathbf{q} = k_2(\vartheta)\nabla(1/\vartheta)$$

where

$$k_1(\vartheta) = \alpha'(\vartheta) \quad \text{and} \quad k_2(\vartheta) = \vartheta^2\alpha'(\vartheta)$$

Take $\lambda = \mu = \nu = 1$ and set $\beta := \partial j$. Then for a new $\sigma$ the problem reads

$$\begin{aligned}
&\partial_t(\vartheta + \chi) - \Delta u = g \\
&u = \alpha(\vartheta) \\
&\partial_t\chi - \Delta\chi + \xi + \sigma'(\chi) = -\frac{1}{\vartheta} \\
&\xi \in \beta(\chi)
\end{aligned}$$

Boundary conditions:

| | |
|---|---|
| $\partial_n u + cu =$ given | (third type b.c., $c > 0$) |
| $\partial_n \chi = 0$ | (homogeneous Neumann b.c.) |

Initial conditions for $\vartheta$ e $\chi$.

**CL:** existence with
a very general $\alpha$

**CLS:** regularity and
uniqueness among smooth sol's
with a much more particular $\alpha$

**Assumptions of CL:**
(besides strict monotonicity and concavity)

here any $a > 0$

$$\begin{array}{lll} \alpha(\vartheta) \approx -1/\vartheta^a \quad \text{or} \quad \alpha(\vartheta) \approx \ln\vartheta & (\vartheta \to 0^+) \\ \alpha(\vartheta) \quad \text{“between”} \quad \ln\vartheta \text{ and } \vartheta & (\vartheta \to +\infty) \end{array}$$

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**Assumptions of CLS:**

$$\begin{array}{ll} \alpha(\vartheta) \approx -1/\vartheta & (\vartheta \to 0^+) \\ \alpha(\vartheta) \approx \vartheta & (\vartheta \to +\infty) \end{array}$$

Joint paper with

Andrea Marson (Padova)

Math. Meth. Appl. Sci. 2003

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- assumptions on the structure
  as general as possible
  (fill the gap, if possible)
- **Dirichlet** conditions for $u$ corresponding to a given $\vartheta_{abs} > 0$ on the bdry as a limit case

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**Assumptions between CL and CLS:**

$$\alpha(\vartheta) \approx -1/\vartheta \qquad (\vartheta \to 0^+)$$
$$\alpha(\vartheta) \quad \text{“between”} \quad \ln\vartheta \text{ and } \vartheta \qquad (\vartheta \to +\infty)$$

- Existence (third type converges to Dirichlet)
- Regularity and uniqueness among smooth sol's
  (also for third type b.c.: CLS improved)

**Convergence and existence**

- penalize the Dirichlet condition
- the penalized problem is a third type b.v.p.
- a priori estimates and weak compactness
- strong compactness and usual monotonicity methods

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**Regularity and uniqueness**

- CLS argument with modification

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**Mathematical tools**

- use of elementary properties of $\alpha$

**Precise assumptions on $\alpha$**
(besides strict monotonicity and concavity)

$$r^2\alpha'(r) = 1 + o(1) \qquad \text{as } r \to 0^+$$
$$r^d\alpha'(r) = c_\infty + o(1) \quad \text{as } r \to +\infty \quad (c_\infty > 0)$$
with $0 \le d \le 1$ ($d \approx$ distance from Fourier)

Notation ($r > 0$)

$$\alpha(r) = -\frac{1}{r} + \ell(r) \quad (\ell = \text{remainder})$$

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- $$\big(\alpha(r) - \alpha(s)\big)\left(\frac{1}{s} - \frac{1}{r}\right) \ge c\left(\frac{1}{s} - \frac{1}{r}\right)^2 \qquad c > 0$$
- $\ell^2(r) \le \delta\alpha^2(r) + c_\delta\big(1 + \widehat{\alpha}(r)\big) \qquad (\widehat{\alpha}' = \alpha)$
- The function $\ell \circ \alpha^{-1}$ is Lipschitz continuous in $\mathbb{R}$
- If $o(1)$ becomes $O(r)$ as $r \to 0^+$, then

$$|(\ell \circ \alpha^{-1})'(s)| \le c\sqrt{(\alpha^{-1})'(s)}$$

**e.b.** $$\langle \partial_t(\vartheta_\varepsilon(t) + \chi_\varepsilon(t)), v\rangle + \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla v + \frac{1}{\varepsilon}\int_\Gamma (u_\varepsilon(t) - u_\Gamma(t))v = \int_\Omega g(t)v$$

$\forall$ admissible $v$, a.e. in $(0,T)$

**ph.d.** $$\partial_t \chi_\varepsilon - \Delta\chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon) = -\frac{1}{\vartheta_\varepsilon}$$

Neumann b.c. for $\chi$

$\vartheta_\varepsilon(0) = \vartheta_0, \quad \chi_\varepsilon(0) = \chi_0$

Joint paper with

P. Colli, E. Rocca, G. Schimperna (Pavia)

in preparation

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- assumptions as general as possible
- **Neumann** conditions for $u$

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**Assumptions on $\alpha$ as above**

$$\alpha(\vartheta) \approx -1/\vartheta \qquad (\vartheta \to 0^+)$$
$$\alpha(\vartheta) \quad \text{“between”} \quad \underset{d=1}{\underset{\uparrow}{\ln\vartheta}} \text{ and } \underset{d=0}{\underset{\uparrow}{\vartheta}} \qquad (\vartheta \to +\infty)$$

**Further assumption**, unfortunately

$$D(\beta) = \mathbb{R} \quad \text{and} \quad \text{growth conditions at infinity}$$

in particular, no constraints on $\chi$

- Existence (third type converges to Neumann)
- Improvement of the uniqueness proof:
  uniqueness of the nonsmooth solution
  (it works also for other b.c.)
- Regularity and uniqueness among smooth sol's
  (as in GM)

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**Convergence and existence**

- approximate the Neumann condition with third type b.c.
- a priori estimates and weak compactness
- strong compactness and usual monotonicity methods
- tricky point: main a priori estimate

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**New uniqueness proof**

- modification of previous arguments
- assumptions on $\ell$ furtherly reinforced

**Idea for the main a priori estimate**

Recall

$$\begin{aligned} r^2\alpha'(r) &= 1 + o(1) && \text{as } r \to 0^+ \\ \alpha(r) &= -\frac{1}{r} + \ell(r) \\ \ell(r) &= o(1/r) && \text{as } r \to 0^+ \\ r^d\alpha'(r) &= c_\infty + o(1) && \text{as } r \to +\infty \\ 0 &\le d \le 1 \end{aligned}$$

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$\Omega \subset \mathbb{R}^3$ open, bounded, connected, smooth
$\Gamma = \partial\Omega, \quad Q = \Omega \times (0, T)$
$V = H^1(\Omega) \hookrightarrow H = L^2(\Omega) \hookrightarrow V'$
$H^2_n = \{v \in H^2(\Omega) : \ \partial_n v = 0\}$
reasonable assumptions on data

$$\vartheta_\varepsilon \in L^\infty(0,T;H) \cap H^1(0,T;V') \cap \dots$$
$$\chi_\varepsilon \in L^2(0,T;H_n^2) \cap H^1(0,T;H)$$
$$u_\varepsilon \in L^2(0,T;V), \quad \xi_\varepsilon \in L^2(Q)$$
$$\vartheta_\varepsilon > 0 \quad \text{a.e. in } Q, \quad 1/\vartheta_\varepsilon \in L^2(0,T;V)$$
$$u_\varepsilon = \alpha(\vartheta_\varepsilon), \quad \xi_\varepsilon \in \beta(\chi_\varepsilon) \quad \text{a.e. in } Q$$

**e.b.** $$\langle \partial_t(\vartheta_\varepsilon(t) + \chi_\varepsilon(t)), v\rangle + \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla v + \varepsilon \int_\Gamma u_\varepsilon(t) v = \int_\Omega g(t) v + \int_\Gamma h(t) v \quad \forall\, v \in V, \quad \text{a.e. in } (0,T)$$

**ph.d.** $$\partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon) = -\frac{1}{\vartheta_\varepsilon} \quad \text{a.e. in } Q$$

$$\vartheta_\varepsilon(0) = \vartheta_0, \quad \chi_\varepsilon(0) = \chi_0$$

**Limit problem:**
same regularity and similar equations
with $\varepsilon = 0$ in (e.b.)

$$\int_0^t (\text{e.b.})\Big|_{t=s,\ v=\vartheta_\varepsilon(s)+u_\varepsilon(s)-\text{cnst}} ds$$
$$+ \int_{Q_t} (\text{ph.d.}) \times \partial_t \chi_\varepsilon \, dx \, ds$$

where $Q_t = \Omega \times (0,t)$.

- main terms (lhs)

$$\frac{1}{2}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_\Omega \widehat{\alpha}(\vartheta(t)) \qquad \left(\widehat{\alpha}' = \alpha\right)$$
$$+ \int_{Q_t} \nabla u_\varepsilon \cdot \nabla \vartheta_\varepsilon + \int_{Q_t} |\nabla u_\varepsilon|^2$$
$$+ \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + \frac{1}{2}\int_\Omega |\nabla \chi_\varepsilon(t)|^2 + \int_\Omega j(\chi_\varepsilon(t))$$

- to be compensated (don't worry)

$$\int_{Q_t} \partial_t \chi_\varepsilon \, u_\varepsilon \quad (\text{lhs}) \qquad \text{and} \qquad - \int_{Q_t} \frac{\partial_t \chi_\varepsilon}{\vartheta_\varepsilon} \quad (\text{rhs})$$

- "source" terms and "easy" terms (rhs)

- Source terms on the right hand side

$$\int_{Q_t} gv + \int_{\Gamma\times(0,t)} hv \qquad (v = \vartheta_\varepsilon + u_\varepsilon - \mathrm{cnst})$$

**trouble !!!**

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**Trouble 1**

$$\int_{\Gamma\times(0,t)} h\vartheta_\varepsilon$$

The left hand side can help just with

$$\frac{1}{2}\int_\Omega |\vartheta_\varepsilon(t)|^2 + \int_\Omega \widehat{\alpha}(\vartheta(t))$$
$$+ \int_{Q_t} \nabla u_\varepsilon \cdot \nabla \vartheta_\varepsilon + \int_{Q_t} |\nabla u_\varepsilon|^2$$

**Lemma.** *Assume*

$$v \in L^2(\Omega), \quad v > 0, \quad \textit{and} \quad \nabla\alpha(v) \in L^2(\Omega)^3$$

*and set*

$$d_\bullet = \frac{4}{1+3d} \quad \textit{for} \quad 0 \le d \le 1\,.$$

*Then the trace of* $v$ *belongs to* $L^{d_\bullet}(\Gamma)$ *and*

$$\|v\|_{L^{d_\bullet}(\Gamma)} \le \delta\, \|\nabla\alpha(v)\|_H^2 + c_\delta \left(1 + \|v\|_H^2\right)$$

*for any* $\delta > 0$ *and some* $c_\delta = c(\delta, \Omega, \alpha)$.

**Proof.** Use the Gagliardo trace theorem and the Hölder and Sobolev inequalities.

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So, reasonable assumptions on $h$ yield
($h \in L^\infty(\Gamma \times (0,T))$ works for any $d \in [0,1]$)

$$\int_{\Gamma\times(0,t)} h\vartheta_\varepsilon \le \delta \int_{Q_t} |\nabla u_\varepsilon|^2 + c_\delta \int_{Q_t} |\vartheta_\varepsilon|^2 + c_\delta$$

and this can be controlled by the left hand side.

**Trouble 2**

$$\int_{Q_t} g u_\varepsilon + \int_{\Gamma\times(0,t)} h u_\varepsilon \le c(g,h)\,\|u_\varepsilon\|_{L^2(0,T;V)}$$

with the **full** $V$-norm, while the l.h.s. contains only the **seminorm**. So, we have to be careful.

Setting for convenience

$$\langle f(t), v\rangle := \int_\Omega g(t)v + \int_\Gamma h(t)v \quad \text{for} \quad v\in V$$

our trouble becomes

$$\int_0^t \langle f(s), u_\varepsilon(s)\rangle \, ds$$

We are going to use the Poincaré inequality

$$\|u_\varepsilon(t) - m_\varepsilon(t)\|_V^2 \le c \int_\Omega |\nabla u_\varepsilon(t)|^2$$

where $m_\varepsilon$ is the mean value

$$m_\varepsilon(t) = \fint_\Omega u_\varepsilon(t)$$

We split and estimate our integral this way

$$\begin{aligned}
&\int_0^t \langle f(s), u_\varepsilon(s)\rangle \, ds \\
&= \int_0^t \langle f(s), u_\varepsilon(s) - m_\varepsilon(s)\rangle \, ds \\
&\quad + \int_0^t m_\varepsilon(s) \langle f(s), 1\rangle \, ds \\
&\le \delta \int_{Q_t} |\nabla u_\varepsilon|^2 + c_\delta \|f\|^2_{L^2(0,T;V')} \\
&\quad + \|f\|_{L^\infty(0,T;V')} \|1\|_V \int_0^t |m_\varepsilon(s)| \, ds
\end{aligned}$$

(even better. . . )

We have to estimate the last integral

We split the mean value this way

$$
\begin{aligned}
m_\varepsilon(t) = & \fint_\Omega u_\varepsilon(t) = \fint_\Omega \alpha(\vartheta_\varepsilon(t)) \\
= & - \fint_\Omega \frac{1}{\vartheta_\varepsilon(t)} + \fint_\Omega \ell(\vartheta_\varepsilon(t))
\end{aligned}
$$

Hence (forget the remainder, please)

$$
\int_0^t |m_\varepsilon(s)|\, ds \leq \int_0^t \left( \fint_\Omega \frac{1}{\vartheta_\varepsilon(s)} \right) ds + \dots
$$

The last mean value is computed via (ph.d.)

$$
-\frac{1}{\vartheta_\varepsilon} = \partial_t \chi_\varepsilon - \Delta \chi_\varepsilon + \xi_\varepsilon + \sigma'(\chi_\varepsilon)
$$

Take the mean value and use $\partial_n \chi = 0$

$$\fint_\Omega \frac{1}{\vartheta_\varepsilon} \leq c \int_\Omega |\partial_t \chi_\varepsilon| + c \int_\Omega |\xi_\varepsilon| + c \int_\Omega (1 + |\chi_\varepsilon|)$$

Finally integrate over $(0, t)$

$$\begin{aligned} &\int_0^t \left( \fint_\Omega \frac{1}{\vartheta_\varepsilon(s)} \right) ds \\ &\leq \delta \int_{Q_t} |\partial_t \chi_\varepsilon|^2 + c_\delta + c \int_{Q_t} (1 + |\chi_\varepsilon|^2) \\ &\quad + c \int_{Q_t} |\xi_\varepsilon| \qquad (\text{recall} \quad \xi_\varepsilon \in \beta(\chi_\varepsilon)) \end{aligned}$$

Everything works but the last integral. It should be controlled by the term

$$\int_\Omega j(\chi_\varepsilon(t))$$

on the l.h.s. We need the further assumption

$$|s| \leq c\,(1 + j(r)) \quad \forall\, r \in \mathbb{R} \quad \forall\, s \in \beta(r)$$

in order to use Gronwall.

**Uniqueness proof**

**e.b.** $$\langle \partial_t(\vartheta(t) + \chi(t)), v\rangle + \int_\Omega \nabla u_\varepsilon(t) \cdot \nabla v = \int_\Omega g(t)v + \int_\Gamma h(t)v \quad \forall\, v \in V, \quad \text{a.e. in } (0,T)$$

**ph.d.** $$\partial_t \chi - \Delta \chi + \xi + \sigma'(\chi) = -\frac{1}{\vartheta} \quad \text{a.e. in } Q$$

$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0$$

Integrate the e.b. in $t$

**i.e.b.** $$\int_\Omega (\vartheta + \chi)v + \int_\Omega \nabla \left( \textstyle\int_0^t u \right) \cdot \nabla v = \langle \text{known}, v\rangle$$

Write eqn's for two solution and take the difference.
I set $\vartheta = \vartheta_1 - \vartheta_2$, etc., or write diff$\{\ldots\}$. Then

**$\delta$ i.e.b.** $$\int_\Omega \vartheta v + \int_\Omega \chi v + \int_\Omega \nabla \left( \textstyle\int_0^t u \right) \cdot \nabla v = 0$$

**$\delta$ ph.d.** $$\partial_t \chi - \Delta \chi + \xi + \text{diff}\{\sigma'(\chi_i)\} = \text{diff}\{-1/\vartheta_i\}$$

Now

$$\int_0^t (\delta\,\text{i.e.b.})(\text{s})\Big|_{v\,=\,u(s)}\,ds + \int_{Q_t} (\delta\,\text{ph.d.}) \times \chi$$

We obtain

$$\begin{aligned} &\int_{Q_t} \vartheta\, u + \int_{Q_t} \chi u + \frac{1}{2}\int_\Omega \Big|\nabla \textstyle\int_0^t u\Big|^2 \\ &\quad + \frac{1}{2}\int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla\chi|^2 + \int_{Q_t} \xi\chi \\ &= \int_{Q_t} \text{diff}\{-1/\vartheta_i\}\chi - \text{easy term} \end{aligned}$$

Two possibilities

use $u$ and write $\vartheta_i = \alpha^{-1}(u_i)$
use $\vartheta$ and write $u_i = \alpha(\vartheta_i)$

Previous argument (CLS and GM): use $u$.
This leeds to the integral

$$\int_{Q_t} (1 + u_1^2 + u_2^2)\chi^2$$

Playing with Hölder, one sees that Gronwall works with smooth $u_i$, namely

$$u_i \in L^\infty(0, T; L^6(\Omega))$$

Such a smoothness needs a regularity result.
Assuming just the first reinforcement

$$r^2\alpha'(r) = 1 + O(r) \quad \text{as } r \to 0^+$$

and the data to be smoother, one proves

$$u_i \in L^\infty(0, T; V)$$

and uses the 3D Sobolev inclusion $V \subseteq L^6(\Omega)$

Use $\vartheta$, instead

Recall

$$u_i = \alpha(\vartheta_i) \quad \text{and} \quad \alpha(r) = -\frac{1}{r} + \ell(r)$$

Reinforce a little more, namely

$$\ell \text{ Lipschitz continuous near } 0$$

whence $\ell$ globally Lipschitz continuous.

$$\begin{aligned}
&\int_{Q_t} \vartheta\, u + \underbrace{\int_{Q_t} \chi u}_{(1)} \\
&\quad + \frac{1}{2} \int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla \chi|^2 + \ldots \\
&= \underbrace{\int_{Q_t} \operatorname{diff}\{-1/\vartheta_i\} \chi}_{(2)} + \ldots
\end{aligned}$$

Compensate (1) and (2) and use Lipschitz

$$\int_{Q_t} \vartheta\, u + \frac{1}{2} \int_\Omega |\chi(t)|^2 + \int_{Q_t} |\nabla\chi|^2$$
$$\leq \int_{Q_t} |\operatorname{diff}\{\ell(\vartheta_i)\}||\chi| + \dots$$
$$\leq c \int_{Q_t} |\vartheta||\chi| + \dots$$

We have

$$\vartheta u = \vartheta \operatorname{diff}\{\alpha(\vartheta_i)\} \geq \delta_0 \, \frac{\vartheta^2}{1 + \vartheta_1^d + \vartheta_2^d}$$

Hence we use the elementary inequality

$$|\vartheta||\chi| \leq \frac{\delta_0}{2} \, \frac{\vartheta^2}{1 + \vartheta_1^d + \vartheta_2^d} + \frac{1}{2\delta_0} \, (1 + \vartheta_1^d + \vartheta_2^d)\chi^2$$

and have

$$\frac{\delta_0}{2} \int_{Q_t} \frac{\vartheta^2}{1 + \vartheta_1^d + \vartheta_2^d} + \frac{1}{2} \, \|\chi(t)\|_H^2 + \int_{Q_t} |\nabla\chi|^2$$
$$\leq c \int_{Q_t} (1 + \vartheta_1^d + \vartheta_2^d)\chi^2$$

Then OK if $d = 0$.
Assume $0 < d \leq 1$. We still want to apply Gronwall.

Write

$$\int_{Q_t} \vartheta_i^d \chi^2 = \int_0^t \int_\Omega \vartheta_i^d \chi \chi$$

and play with Hölder

$$p, q \geq 1 \quad \text{and} \quad \frac{1}{p} + \frac{2}{q} = 1$$

$$p := \frac{2}{d} \in [2, \infty) \quad \text{and} \quad q := \frac{4}{2-d} \in (2, 4]$$

Then

$$\int_{Q_t} \vartheta_i^d \chi^2 \leq \int_0^t \left\| \vartheta_i^d \right\|_{L^p(\Omega)} \|\chi\|_{L^q(\Omega)}^2$$

$$= \int_0^t \|\vartheta_i\|_H^d \|\chi\|_{L^q(\Omega)}^2 \leq c \int_0^t \|\chi\|_{L^4(\Omega)}^2$$

since $\vartheta_i \in L^\infty(0, T; H)$ and $q \leq 4$.
The compact embedding $V \subset L^4(\Omega)$ yields

$$\|v\|_{L^4} \leq \delta \|\nabla v\|_H + c_\delta \|v\|_H$$

and we obtain

$$\int_{Q_t} (\vartheta_1^d + \vartheta_2^d) \chi^2 \leq \frac{1}{2} \int_{Q_t} |\nabla \chi|^2 + C \int_0^t \|\chi\|_H^2$$

Cahn–Hilliard type dynamics

$$\partial_t(\vartheta + \chi) - \Delta\alpha(\vartheta) = g$$
$$\partial_t\chi - \Delta w = 0$$
$$w \in -\Delta\chi + \partial j(\chi) + \sigma'(\chi) - \frac{1}{\vartheta_c} + \frac{1}{\vartheta}$$

homogeneous Neumann b.c. for $\chi$ and $w$

some b.c. for either $\vartheta$ or $u$

initial conditions for $\vartheta$ and $\chi$

with $\alpha$ **as before** (essentially)

- Main difference in the a priori estimates

  some $\|\cdot\|_H$ replaced by $\|\cdot\|_{V'}$

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- existence result for the third type problem for $u$ ($0 \le d < 1$, with A. Marson)
- existence for the Dirichlet problem should work as in GM
- uniqueness works as before

  $$\|v\|_{L^4} \le \delta\,\|\nabla v\|_H + c_\delta\,\|v\|_{V'}$$

- dealing with Neumann conditions... (hope)