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NONLINEAR PARABOLIC PROBLEMS

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Phase transition with memory

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Energy balance for phase transition:

$$\partial_t e + \operatorname{div} \mathbf{q} = \text{known source term} \tag{1}$$

where

$e = \vartheta + \chi$	internal energy (coefficients $= 1$)
ϑ	relative temperature
χ	order parameter
$\mathbf{q}$	heat flux

Then, (1) becomes

$$\partial_t(\vartheta + \chi) + \operatorname{div} \mathbf{q} = \text{known source term} \tag{2}$$

We need two more equations

$$\text{relationship between } \mathbf{q} \text{ and } \vartheta \tag{3}$$

$$\text{relationship between } \vartheta \text{ and } \chi \tag{4}$$

and initial and boundary conditions.

The classical Stefan problem is derived by the equations

$$\begin{aligned} \partial_t(\vartheta+\chi)+\operatorname{div}\mathbf{q} &= \text{known term} \quad \textit{(energy balance)} \\ \mathbf{q} &= -k_0\nabla\vartheta \quad \textit{(Fourier law)} \\ \chi &\in H(\vartheta) \end{aligned}$$

where $k_0 = \text{constant} > 0$ and

$$H = \begin{cases} \text{the Heaviside graph} \\ \text{the sign graph} \end{cases}$$

according to our convenience.
We obtain the well–knonw problem

$$\begin{aligned} &\partial_t(\vartheta+\chi) - k_0\Delta\vartheta = \text{known term} \\ &\chi \in H(\vartheta) \end{aligned}$$

Rewrite the Stefan problem in the form

$$\begin{aligned} &\partial_t(\vartheta + \chi) - k_0 \Delta \vartheta = \text{known term} \\ &H^{-1}(\chi) \ni \vartheta \end{aligned} \tag{1}$$

Replace (1) by a differential equation

$$\mu \partial_t \chi + H^{-1}(\chi) \ni \vartheta \tag{2}$$

phase relaxation model

$$\mu \partial_t \chi - \nu \Delta \chi + H^{-1}(\chi) \ni \vartheta \tag{3}$$

phase field model (i.e., with diffusion)

where μ, ν are small parameters.
Alternative models require

$$W'(\chi) \qquad \text{in place of} \qquad H^{-1}(\chi)$$

where W is a ($W-$shaped) double well potential. Typical case:

$$\begin{aligned} &W(r) = \frac{c}{4}(r^2 - 1)^2 \qquad (c > 0 \text{ constant}) \\ &W'(r) = cr^3 - cr \end{aligned}$$

Relaxed models 3

More generally

$$\underbrace{\mu\partial_t\chi \ \Big[-\nu\Delta\chi\Big] + H^{-1}(\chi) \ \Big[\text{or } W'(\chi)\Big] \ni \vartheta}$$

$$\downarrow$$

$$\overbrace{\mu\partial_t\chi \ \Big[-\nu\Delta\chi\Big] + \beta(\chi) - \gamma(\chi) \ni \vartheta}$$

where β maximal monotone graph in $\mathbb{R} \times \mathbb{R}$ and γ smooth function. Examples:

$\beta = H^{-1}$	and	$\gamma = 0$	(*standard Stefan case*)
$\beta(r) = cr^3$	and	$\gamma(r) = cr$	(*standard double well case*)

For instance, the standard phase field model is

$$\partial_t(\vartheta + \chi) - k_0\Delta\vartheta = \text{known term}$$
$$\mu\partial_t\chi - \nu\Delta\chi + W'(\chi) = \vartheta$$

where W is a standard double well potential.

Wide literature, starting with

Visintin and Frémond–Visintin (phase relaxation)
Caginalp and Fix (phase field)

See, e.g., the book “Models of phase transition” by Visintin.

The framework for Stefan problems and phase field systems is *“nonlinear parabolic PDE’s”*.

Recall the *Fourier law* for the heat flux

$$\mathbf{q} = -k_0 \nabla\vartheta, \qquad k_0 = \text{constant} > 0 \tag{1}$$

Introduce **memory** (linearly):

either $$\mathbf{q}(x,t) = \mathbf{q}_0(x,t) - k_0\nabla\vartheta(x,t) - (k * \nabla\vartheta)(x,t) \tag{2}$$

Coleman–Gurtin law

or $$\mathbf{q}(x,t) = \mathbf{q}_0(x,t) - (k * \nabla\vartheta)(x,t) \tag{3}$$

Gurtin–Pipkin law

where $(a * b)(t) = \int_0^t a(t-s)\, b(s)\, ds$, in general

$\mathbf{q}_0 =$ past history

$k =$ function of time, only

Typical memory kernel: $k(t) = \dfrac{c}{\tau} \exp(-t/\tau) \qquad (t > 0)$

where $\tau > 0$ small parameter. It comes from

$$\tau\partial_t \mathbf{q} + \mathbf{q} = -c\nabla\vartheta \qquad \textit{Cattaneo–Maxwell law}$$

Coupling with an equation for the order parameter, we obtain a system involving both

PDE's and *integro–differential equations.*

The energy balance

$$\partial_t(\vartheta + \chi) + \operatorname{div} \mathbf{q} = \text{known source term}$$

becomes

$$\begin{aligned} \partial_t(\vartheta + \chi) - k_0 \Delta\vartheta - \Delta(k * \vartheta) &= \text{known} \quad \textit{Coleman–Gurtin} \\ \partial_t(\vartheta + \chi) - \Delta(k * \vartheta) &= \text{known} \quad \textit{Gurtin–Pipkin} \end{aligned}$$

If k is a decreasing exponential, we have

$$\begin{aligned} \text{Coleman–Gurtin} &\longrightarrow \text{parabolic equation} \\ \text{Gurtin–Pipkin} &\longrightarrow \text{hyperbolic equation} \end{aligned}$$

with respect to ϑ.

Review reference:

Joseph–Preziosi, *Heat waves*, Rev. Modern Phys. (1989).

1) Existence, uniqueness, regularity for

$$\begin{aligned}&\partial_t(\vartheta+\chi)\ \Big[-\Delta\vartheta\Big]-\Delta(k*\vartheta)=\text{known}\\&\mu\partial_t\chi-\nu\Delta\chi+\beta(\chi)-\gamma(\chi)\ni\vartheta\end{aligned}$$

and for some generalization of this system
(further nonlinear terms).

2) Asymptotic analysis as either μ or ν tend to 0: degenerate limit problems.

3) Asymptotic analysis as k tends to the Dirac mass. In this case, the limit problem is parabolic, namely

$$\begin{aligned}&\partial_t(\vartheta+\chi)-\Delta\vartheta=\text{known}\\&\mu\partial_t\chi-\nu\Delta\chi+\beta(\chi)-\gamma(\chi)\ni\vartheta\end{aligned}$$

4) Long time behavior of the solution.

5) Different relaxation terms, e.g.

$$\begin{aligned}&\partial_t(\vartheta+h*\chi)-\Delta(k*\vartheta)=\text{known}\\&\Big[\mu\partial_t\chi-\nu\Delta\chi\Big]+\beta(\chi)-\gamma(\chi)\ni\vartheta\end{aligned}$$

where h is a memory kernel too.

Results **7**

Several combinations of

Aizicovici
Barbu
Bonetti
Bonfanti
Chadam
Colli
Grasselli
Laurençot
Luterotti
Showalter
Walkington
Yin
G.G.
. . . .

Penrose–Fife with memory

(strongly nonlinear in ϑ)

Conserved phase field with memory

(fourth order equation for χ)

Two types of assumptions on k:

1) k *a kernel of positive type*, i.e.

$$\left.\begin{aligned} &k \in L^1_{\mathrm{loc}}[0,+\infty) \quad \text{(at least)} \quad \text{and} \\ &\int_0^t (k * v)(s)\, v(s)\, ds \geq 0 \quad \forall t > 0, \quad \forall v \in L^2(0,t) \end{aligned}\right\} \tag{1}$$

2) k is smooth and

$$k(0) > 0. \tag{2}$$

Any decreasing exponential fulfills both properties. Sufficient conditions for (1) are indeed:

$$k \text{ is positive, decreasing, and convex.}$$

The modified heat equation becomes

$$\partial_t(\vartheta + \chi) - \Delta(k * \vartheta) = \text{known} \tag{1}$$

Assume k smooth and introduce

$$u(x,t) := (1 * \vartheta)(x,t) = \int_0^t \vartheta(x,s)\,ds \quad \textit{(freezing index).}$$

Then (1) becomes

$$\partial_t^2 u - \Delta(k * \partial_t u) = \text{known} - \partial_t \chi$$

On the other hand, we have

$$k * \partial_t u = \partial_t(k * u) = k(0)\,u + k' * u$$

whence

$$\partial_t^2 u - k(0)\Delta u = \text{known} - \partial_t \chi - \Delta(k' * u).$$

Therefore:

$$\begin{array}{rcl} k(0) > 0 & \Longrightarrow & \text{hyperbolic principal part} \\ k \text{ smooth} & \Longrightarrow & \Delta(k' * u) \sim \text{lower order term} \end{array}$$

Similar behavior with respect to the new state variable

$$w = u + 1 * \chi = 1 * (\vartheta + \chi).$$

Assume

$$k_\varepsilon \to \text{Dirac mass} \qquad \text{as } \varepsilon \to 0.$$

Hence (formally)

$$\begin{gathered}\partial_t(\vartheta_\varepsilon + \chi_\varepsilon) - \Delta(k_\varepsilon * \vartheta_\varepsilon) = f \\ \Big\downarrow \\ \partial_t(\vartheta + \chi) - \Delta\vartheta = f\end{gathered}$$

and

$$\begin{gathered}\mu\partial_t\chi_\varepsilon - \nu\Delta\chi_\varepsilon + \beta(\chi_\varepsilon) - \gamma(\chi_\varepsilon) \ni \vartheta_\varepsilon \\ \Big\downarrow \\ \mu\partial_t\chi - \nu\Delta\chi + \beta(\chi) - \gamma(\chi) \ni \vartheta\end{gathered}$$

Goal: make this argument rigorous.

Theorem (Colli-G-Grasselli). *The above result holds if*

$$k_\varepsilon(t) = \frac{1}{\varepsilon}\exp(-t/\varepsilon) \tag{1}$$

More generally, we can add a perturbation term tending to 0 *in an appropriate norm.*

Idea for the kernel (1).

Choose

$$w_\varepsilon := 1 * (\vartheta_\varepsilon + \chi_\varepsilon) \qquad \text{and} \qquad w := 1 * (\vartheta + \chi)$$

as state variables. Then

$$\partial_t^2 w_\varepsilon - \Delta(k_\varepsilon * w_\varepsilon) = \dots \tag{2}$$
$$\partial_t w - \Delta(1 * w) = \dots \tag{3}$$

where χ_ε and χ enter the rhs and have to be treated using the second equations.

Apply the operator $v \mapsto \varepsilon v + 1 * v$ to (2):

$$\varepsilon\partial_t^2 w_\varepsilon + \partial_t w_\varepsilon - \Delta\big((\varepsilon k_\varepsilon + 1 * k_\varepsilon) * w_\varepsilon\big) = \dots$$

and observe that (1) implies

$$\varepsilon k_\varepsilon + 1 * k_\varepsilon = 1.$$

Hence we obtain

$$\varepsilon\partial_t^2 w_\varepsilon + \partial_t w_\varepsilon - \Delta(1 * w_\varepsilon) = \dots$$

and we can compare with (3).

Assume the Coleman–Gurtin law

$$\mathbf{q}(x,t) = \mathbf{q}_0(x,t) - \nabla\vartheta(x,t) - (k * \nabla\vartheta)(x,t)$$

assume k of positive type

$$\int_0^t (k * v)(s)\, v(s)\, ds \geq 0 \quad \forall t > 0, \quad \forall v \in L^2(0,t)$$

and write the modified heat equation (no χ):

$$\partial_t\vartheta - \Delta\vartheta - \Delta(k * \vartheta) = \text{known}$$

Multiply by ϑ and integrate over $\Omega \times (0,t)$

$$\frac{1}{2}\int_\Omega |\vartheta(t)|^2 + \int_0^t \int_\Omega |\nabla\vartheta|^2 + \underbrace{\int_\Omega \int_0^t (k * \nabla\vartheta) \cdot \nabla\vartheta}_{\geq 0} = \dots$$

Thus, the standard parabolic estimate still holds.

Goal: Try and do the same for Stefan problems.

It is better to use $e = \vartheta + \chi$ as state variable.

$$\begin{aligned}
&\partial_t e - \Delta\vartheta - \Delta(k * \vartheta) = \text{known} && (1)\\
&e \in \alpha(\vartheta) \qquad \text{or} \qquad \vartheta \in \alpha^{-1}(e) \\
&\text{Standard case:} \qquad \alpha = \text{identity} + \text{Heaviside}
\end{aligned}$$

Try and multiply by either ϑ or e. Set

$$\alpha = \partial\varphi \qquad \text{whence} \qquad \alpha^{-1} = \partial\varphi^*$$

and observe that (formally)

$$\vartheta\,\partial_t e = \alpha^{-1}(e)\,\partial_t e = \partial\varphi^*(e)\,\partial_t e = \partial_t\varphi^*(e).$$

Hence, multiplying (1) by ϑ and e

$$\int_\Omega \varphi^*(e(t)) + \int_0^t\int_\Omega |\nabla\vartheta|^2 + \underbrace{\int_\Omega\int_0^t (k * \nabla\vartheta)\cdot\nabla\vartheta}_{\geq 0} = \dots$$

$$\frac{1}{2}\int_\Omega |e(t)|^2 + \underbrace{\int_0^t\int_\Omega \nabla\vartheta\cdot\nabla e}_{\geq 0} + \underbrace{\int_\Omega\int_0^t (k * \nabla\vartheta)\cdot\nabla e}_{?} = \dots$$

Problems: Must e be $L^2(\Omega)$ − valued?
What about the boundary terms (integration by parts)?

The most difficult case (among standard ones) is the following: keep α as general as possible (maximal monotone) and assume third type boundary condition, i.e.

$$\mathbf{q} \cdot \mathbf{n} = \vartheta - \vartheta_\Gamma \qquad \text{on } \Gamma := \partial\Omega$$

where ϑ_Γ is prescribed, i.e.

$$\partial_n(\vartheta + k * \vartheta) + \vartheta = \vartheta_\Gamma \tag{2}$$

Damlamian–Kenmochi: (2) and any α, but no memory
Aizicovici–Colli–Grasselli: any α, but Dirichlet b.c.
Colli–Grasselli: (2), but α sublinear
Barbu–Colli–G–Grasselli: general case

Theorem (B-C-G-G). *Main assumption: k smooth.*
i) Existence and uniqueness for any maximal monotone α.
ii) $e \in C_w([0, +\infty); L^1(\Omega))$ and $e \in \alpha(\vartheta)$ a.e. in $\Omega \times (0, +\infty)$ if α is everywhere defined.

Main further assumption: $k \in L^1(0, +\infty)$ of positive type.
iii) Long time behavior for ϑ if α^{-1} is Lipschitz.
iv) Long time behavior for e under stronger assumptions.

i) Idea for existence and uniqueness. Recall:

$$\begin{aligned}&\partial_t e - \Delta\vartheta - \Delta(k * \vartheta) = \text{known}\\&\partial_n(\vartheta + k * \vartheta) + \vartheta = \vartheta_\Gamma\end{aligned}$$

Variational formulation: for any $v \in H^1(\Omega)$

$$\langle \partial_t e, v\rangle + \underbrace{\int_\Omega \nabla\vartheta \cdot \nabla v + \int_\Gamma \vartheta v}_{\langle A\vartheta, v\rangle} + \underbrace{\int_\Omega \nabla(k * \vartheta) \cdot \nabla v}_{\langle B(k*\vartheta), v\rangle} = \text{r.h.s.}$$

Hence, the abstract equation in $H^1(\Omega)'$

$$e' + A\vartheta + B(k * \vartheta) = \text{some } f \tag{3}$$

coupled with

$$\text{generalization of} \quad e \in \alpha(\vartheta) \tag{4}$$

meaningful if e is just $H^1(\Omega)' -$ valued
and equivalent to $e \in \alpha(\vartheta)$ a.e. if e is $L^2(\Omega) -$ valued.

If $k = 0$, Damlamian–Kenmochi transform (3–4) into

$$e' + \partial J(e) \ni f$$

in the framework of $H^1(\Omega)'$, where

$$J : H^1(\Omega)' \to (-\infty, +\infty]$$

is a convex l.s.c. functional related to α.

For a general (smooth) k, we argue as follows

$$\begin{aligned} e' + A\vartheta + B(k * \vartheta) &= f \\ A\vartheta + (kBA^{-1}) * A\vartheta &= -e' + f \quad \text{(solving for } A\vartheta) \\ A\vartheta = \mathcal{G}(e') &= -e' + \mathcal{F}(e) \\ e' + A\vartheta &= \mathcal{F}(e) \end{aligned}$$

where $\mathcal{F}(e)$ contains some convolution, and couple this equation with the generalized condition $e \in \alpha(\vartheta)$ essentially as Damlamian–Kenmochi do. We obtain the integrodifferential abstract equation in $H^1(\Omega)'$

$$e' + \partial J(e) \ni \mathcal{F}(e)$$

and use a fix point argument in the space $C^0([0,T]; H^1(\Omega)')$.

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e is an $H^1(\Omega)'$ – valued generalized solution.

After (ii), previous form of the problem, whence:

$$\begin{gathered} k \text{ of positive type} \\ \downarrow \\ \text{a priori estimates} \\ \downarrow \\ \text{long time behavior} \end{gathered}$$

ii) Idea for solution defined a.e. to

$$e' + \partial J(e) \ni \mathcal{F}(e)$$

In simpler situations e is $L^2(\Omega)-$valued. This is not our case.

$$D(\alpha) = \mathbb{R} \quad \longrightarrow \quad \text{modified Brézis argument}$$
$$H^{-1}(\Omega) \cap L^1(\Omega) \quad \longrightarrow \quad H^1(\Omega)' \cap L^1(\Omega)$$

As $H^1(\Omega)' \cap L^1(\Omega)$ is meaningless, we define:

$$u \in H^1(\Omega)' \quad \textit{belongs to} \quad H^1(\Omega)' \cap L^1(\Omega)$$
$$\Longleftrightarrow$$
$$\langle u, v\rangle = \int_\Omega wv \quad \forall v \in H^1(\Omega) \cap L^\infty(\Omega)$$
$$\textit{for some (unique)} \quad w \in L^1(\Omega) \quad (\text{we set } w = u).$$

Moreover, we introduce

$$\widetilde{J}(u) = \begin{cases} \int_\Omega \varphi^*(u) & \text{if } u \in H^1(\Omega)' \cap L^1(\Omega) \\ & \text{and } \varphi^*(u) \in L^1(\Omega) \\ +\infty & \text{otherwise for } v \in H^1(\Omega)'. \end{cases}$$

Using

$$D(\alpha) = \mathbb{R} \qquad \Longrightarrow \qquad \lim_{|r| \to +\infty} \frac{\varphi^*(r)}{|r|} = +\infty$$

we prove that $\widetilde{J}$ is l.s.c. and that $J = \widetilde{J}$.
Hence

$$J(e(t)) \le C \quad \text{becomes} \quad \int_\Omega \varphi^*(e(t)) \le C$$

This yields $e(t) \in H^1(\Omega)' \cap L^1(\Omega)$ and weak $L^1(\Omega)-$continuity.