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The conserved phase–field system with memory

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Standard conserved phase–field system in $\Omega \times (0, T)$:

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g \qquad (1)$$

$$\partial_t\chi - \Delta w = 0 \qquad (2)$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \qquad (3)$$

I.C. for ϑ, χ; B.C. for ϑ, χ, w

Ω = domain in $\mathbb{R}^3$ (bounded, etc.)
ϑ = relative temperature
χ = order parameter
w = chemical potential
ℓ = latent heat
g = source term

Homogeneous Neumann condition for w

$\overset{(2)}{\Longrightarrow}$ conserved, i.e.

$$\partial_t \int_\Omega \chi \, dx = 0$$

The above first equation

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g$$

comes from the energy balance

$$\partial_t(\vartheta + \ell\chi) + \operatorname{div}\mathbf{q} = g$$

and the **Fourier law** for the heat flux

$$\mathbf{q} = -k_0\nabla\vartheta \qquad (k_0 = 1)$$

Gurtin–Pipkin law:

$$\begin{aligned}\mathbf{q}(x,t) &= -\int_{-\infty}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\ &= -\int_{-\infty}^{0} k(t-s)\nabla\vartheta(x,s)\,ds \\ &\quad - \int_{0}^{t} k(t-s)\nabla\vartheta(x,s)\,ds \\ &= \mathbf{q}_0(x,t) - (k * \nabla\vartheta)(x,t).\end{aligned}$$

Hence the energy balance becomes

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = \text{new } g$$

accounting also for the **known** past history $\mathbf{q}_0$.

Typical assumptions on the **memory kernel**:

$k \in L^1(0,T)$ and

either smooth and $k(0) > 0$

or pos. type, i.e., $\int_0^t (k * v)(s)v(s)\,ds \geq 0 \quad \forall\, t, v.$

Prototype:

$$k_\varepsilon(t) = \frac{k_0}{\varepsilon}\exp(-t/\varepsilon)$$

where $k_0, \varepsilon > 0$. It comes from

$\varepsilon \partial_t \mathbf{q} + \mathbf{q} = -k_0 \nabla \vartheta$ Cattaneo–Maxwell law.

For small ε

$k_\varepsilon \sim k_0 \times \{\text{Dirac mass}\}$ and C-M $\sim$ Fourier

So, we consider the system

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \tag{1}$$
$$\partial_t\chi - \Delta w = 0 \tag{2}$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$
$$\text{Neumann B.C. for } k * \vartheta, \chi, w \tag{4}$$
$$\text{I.C. for } \vartheta, \chi \tag{5}$$

or some generalization of it.

Conserved case, without memory:

$$\partial_t(\vartheta + \ell\chi) - \Delta\vartheta = g$$
$$\partial_t\chi - \Delta w = 0$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta$$

I.C. and B.C.

or some generalization.

Caginalp
Alt, Palow
Kenmochi, Niezgódka
Brochet, Hilhorst, Novick-Cohen
. . . .
existence, uniqueness, attractors. . .

Wide literature on Cahn–Hilliard equation, e.g.,

Novick-Cohen (survey)
Kenmochi, Niezgódka, Pawlow (C-H + constraints)

Non-conserved case, with memory (G-P):

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g$$
$$\partial_t\chi + w = 0 \quad (\text{instead of} \quad \partial_t\chi - \Delta w = 0)$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta$$

I.C. and B.C.

or some generalization.

Aizicovici, Barbu
Colli, G., Grasselli
Colli, Laurençot
. . . .
existence, uniqueness, longtime behavior,
asymptotic analyses. . .

Other laws with memory for **q**
. . . .

Conserved case, with memory:

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \qquad (1)$$
$$\partial_t\chi - \Delta w = 0 \qquad (2)$$
$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \qquad (3)$$
$$\text{B.C. for } k * \vartheta, \chi, w \qquad (4)$$
$$\text{I.C. for } \vartheta, \chi \qquad (5)$$

Novick-Cohen:
existence, regularizing effect, partial uniqueness

- Colli, G., Laurençot, Novick-Cohen:
  uniqueness, longtime behavior
- Colli, G., Grasselli, Schimperna:
  existence, uniqueness, regularity
  for a generalized system
- Felli, Rocca:
  asymptotic analyses

Felli: limit as $\varepsilon \to 0$ of

$$\partial_t(\vartheta + \ell\chi) - \Delta(k_\varepsilon * \vartheta) = g \tag{1}$$

where $\quad k_\varepsilon(t) \sim \dfrac{k_0}{\varepsilon}\exp(-t/\varepsilon)$

$$\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

N.B.C. for $k_\varepsilon * \vartheta, \chi, w$. (4)

I.C. for ϑ, χ (5)

The limit problem is the conserved phase field system without memory.

Rocca: limit as $\mu \to 0$ of

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \tag{1}$$

$$\mu\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

N.B.C. for $k * \vartheta, \chi, w$. (4)

I.C. for ϑ, χ (5)

with some restriction on ℓ. The limit problem is obtained taking $\mu = 0$ (I.C. just for $\vartheta + \ell\chi$). In the limit the subsystem (2-3) can be simplified.

Recall (forgetting about I.C.)

$$\partial_t(\vartheta + \ell\chi) - \Delta(k * \vartheta) = g \tag{1}$$

$$\partial_t\chi - \Delta w = 0 \tag{2}$$

$$w := -\Delta\chi + \chi^3 - \chi - \ell\vartheta \tag{3}$$

$$\text{homogeneous N.B.C. for } k * \vartheta, \chi, w \tag{4}$$

$$H := L^2(\Omega), \quad V := H^1(\Omega),$$

$$W := \{v \in H^2(\Omega) : \partial_n v = 0\}$$

$$\implies \quad W \subset V \subset H \subset V' \subset W'$$

$$\langle Au, v\rangle := \int_\Omega (-\Delta u)v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega u(-\Delta v)$$

$$(u, v \in \text{ appropriate spaces})$$

$$\implies \quad A : W \to H, \quad A : V \to V', \quad A : H \to W'.$$

Then (1–4) becomes

$$(\vartheta + \ell\chi)' + A(k * \vartheta) = g$$

$$\chi' + Aw = 0$$

$$w := A\chi + \chi^3 - \chi - \ell\vartheta$$

and I.C. for ϑ, χ must be added.

Theorem 1. *Assume*

$$k \in L^1(0,T) \quad \textit{and} \quad k \textit{ of positive type, i.e.,}$$
$$\int_0^t (k * v)(s)v(s)\, ds \geq 0 \quad \forall\, t,\ v$$
$$g \in L^1(0,T;H), \quad \vartheta_0 \in H, \quad \chi_0 \in V.$$

Then the problem

$$(\vartheta + \ell\chi)' + A(k * \vartheta) = g$$
$$\chi' + Aw = 0$$
$$w := A\chi + \chi^3 - \chi - \ell\vartheta$$
$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0$$

has a unique solution (ϑ, χ) *satisfying*

$$\vartheta \in C^0([0,T];H) \cap H^1(0,T;W')$$
$$\chi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V')$$
$$w \in L^2(0,T;V). \ \blacksquare$$

Theorem 2. *HP's as above for any $T > 0$ and*

$$k \in L^1(0,\infty) \quad \text{and} \quad \int_0^\infty k(s)\,ds \neq 0$$
$$g \in L^1(0,\infty;H).$$

Let ω be the ω − limit in $V' \times H$ according to: $(\vartheta_\infty, \chi_\infty) \in \omega$ iff

$$\text{there exists} \quad \{t_n\} \nearrow +\infty \quad \text{such that}$$
$$\vartheta(t_n) \to \vartheta_\infty \quad \text{in } V'$$
$$\chi(t_n) \to \chi_\infty \quad \text{in } H.$$

Then $\omega \neq \emptyset$, compact, connected. Moreover, $(\vartheta_\infty, \chi_\infty) \in \omega$ implies:

- *ϑ_∞ is constant, namely*

$$\vartheta_\infty := \fint_\Omega \left(\vartheta_0 + \int_0^\infty g\,dt \right)$$

  and the whole $\{\vartheta(t)\}$ converges to ϑ_∞;
- *χ_∞ belongs to V and solves*

$$A\chi_\infty + \chi_\infty^3 - \chi_\infty - \ell\vartheta_\infty = \text{const.}$$
$$\int_\Omega \chi_\infty = \int_\Omega \chi_0. \ \blacksquare$$

Rewrite two equations as

$$\vartheta' + \ell\chi' + A(k * \vartheta) = g$$
$$w := A\chi + \chi^3 - \chi - \ell\vartheta.$$

Generalize as follows

$$\vartheta' + \overset{\overset{\lambda'(\chi)}{\uparrow}}{\ell}\chi' + A(k * \vartheta) = g$$

$$w := A\chi + \underbrace{\chi^3 - \chi}_{\downarrow} - \overset{\overset{\lambda'(\chi)}{\uparrow}}{\ell}\vartheta \qquad \underset{\downarrow}{\boxed{\frac{1}{4}(\chi^2 - 1)^2}}$$

$$\overbrace{\beta(\chi) + \sigma'(\chi)} = \frac{\partial}{\partial \chi}\big(\overbrace{j(\chi) + \sigma(\chi)} \big)$$

and obtain

$$(\vartheta + \lambda(\chi))' + A(k * \vartheta) = g$$
$$w := A\chi + \beta(\chi) + \sigma'(\chi) - \lambda'(\chi)\,\vartheta$$

or, more generally,

$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta, \qquad \xi \in \beta(\chi)$$

Hence, we consider the system

$$(\vartheta + \lambda(\chi))' + A(k * \vartheta) = f \tag{1}$$
$$\chi' + Aw = 0 \tag{2}$$
$$w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)\,\vartheta \tag{3}$$
$$\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) \tag{4}$$
$$\vartheta(0) = \vartheta_0, \quad \chi(0) = \chi_0 \tag{5}$$

where λ, β, σ are as general as possible.

Structure assumptions:

$k \in W^{2,1}(0,T)$ and $k(0) > 0$
$j : \mathbb{R} \to [0, +\infty]$ convex, proper, l.s.c.
$j(0) = 0, \quad \beta = \partial j \quad (\Longrightarrow \quad \beta(0) \ni 0)$
$\lambda, \sigma, \in C^1(\mathbb{R})$ and λ', σ' Lipschitz
maybe: also λ Lipschitz

New unknown. Replace ϑ by

$$u := 1 * \vartheta \qquad (\textit{freezing index})$$

Then the system becomes

$$\begin{aligned}
&(u' + \lambda(\chi))' + k(0)Au = f - A(k' * u) && (1)\\
&\chi' + Aw = 0 && (2)\\
&w = A\chi + \xi + \sigma'(\chi) - \lambda'(\chi)u' && (3)\\
&\chi \in D(\beta) \quad \text{and} \quad \xi \in \beta(\chi) && (4)\\
&u(0) = 0, \quad u'(0) = \vartheta_0, \quad \chi(0) = \chi_0 && (5)
\end{aligned}$$

and we look for solutions (u, χ, w, ξ) satisfying

$$\begin{aligned}
&u \in C^0([0,T];V) \cap C^1([0,T];H)\\
&\chi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V')\\
&j(\chi) \in L^\infty(0,T;L^1(\Omega))\\
&w \in L^2(0,T;V)\\
&\xi \in L^2(0,T;H)
\end{aligned}$$

Theorem 1. *Let all the above structure assumptions hold and assume*

$$f \in L^1(0,T;H) + W^{1,1}(0,T;V')$$
$$\vartheta_0 \in H, \quad \chi_0 \in V, \quad j(\chi_0) \in L^1(\Omega)$$
$$\fint_\Omega \chi_0 \in \operatorname{int} D(\beta).$$

Then there exists a unique (u,χ) and some (w,ξ) such that (u,χ,w,ξ) is a solution to (1–5) with the required regularity. ∎

Theorem 2. *HP's as in Thm 1 but the Lipschitz continuity of λ.*
Then the existence part still holds true. ∎

Theorem 3. *HP's as in Thm 1. If moreover*

$$f \in W^{1,1}(0,T;H) + W^{2,1}(0,T;V'), \quad f(0) \in H$$
$$\vartheta_0 \in V$$

then

$$u + 1 * \lambda(\chi) \in W^{2,\infty}(0,T;H) \cap W^{1,\infty}(0,T;V). \blacksquare$$

Theorem 4. *HP's as in Thm 1. If moreover*

$$f \in L^2(0,T;H) + W^{1,1}(0,T;V'), \quad \chi_0 \in D(\beta) \text{ a.e.}$$

and there exists ξ_0 *such that*

$$\xi_0 \in H, \qquad \xi_0 \in \beta(\chi_0) \text{ a.e.}$$
$$w_0 := A\chi_0 + \xi_0 + \sigma'(\chi_0) - \lambda'(\chi_0)\vartheta_0 \in V$$

then

$$\chi \in L^\infty(0,T;W) \cap H^1(0,T;V) \cap W^{1,\infty}(0,T;V'). \blacksquare$$