SOLUZIONI APPELLO 19/7/24

A1 $f(x) = \cos(2x)\,(1+2x)^{-1/2}$

per $x \to 0$

$\cos 2x = 1 - \frac{(2x)^2}{2} + o(x^3) = 1 - 2x^2 + o(x^3)$

$(1+2x)^{-1/2} = 1 - \frac{1}{2}(2x) + \underbrace{\left[\left(-\frac{1}{2}\right)\cdot\left(-\frac{3}{2}\right)\right]}_{2}(2x)^2 + \underbrace{\left[\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\right]}_{6}(2x)^3 + o(x^3)$

$= 1 - x + \frac{3}{2}x^2 - \frac{5}{2}x^3 + o(x^3)$

$P_2(x_0) \Rightarrow \left(1 - 2x^2 + o(x^3)\right)\left(1 - x + \frac{3x^2}{2} - \frac{5}{2}x^3 + o(x^3)\right)$

$= 1 - x + \frac{3x^2}{2} - \frac{5}{2}x^3 + o(x^3) - 2x^2 + 2x^3 =$

$= 1 - x - \frac{x^2}{2} - \frac{x^3}{2} + o(x^3)$

$$\boxed{P_2(x_0) = 1 - x - \frac{x^2}{2} - \frac{x^3}{2}}$$

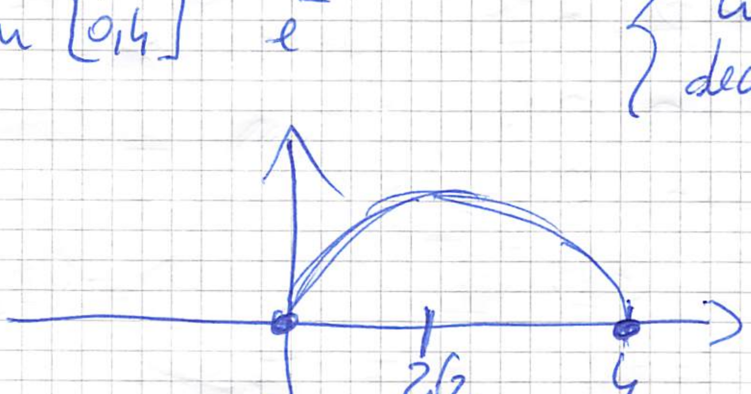
A2 $f(x) = 7|x|\sqrt{16-x^2}$ f pari $f : [-4,4] \to \mathbb{R}$ quindi la studio solo in $[0,4]$ $f(x) = 7x\sqrt{16-x^2} \geq 0$

f è continua e derivabile qui. $f(0)=0$ $f(4)=0$.

$f'(x) = 7\sqrt{16-x^2} + 7x\frac{-2x}{2\sqrt{16-x^2}} = 7\left\{\frac{16-x^2-x^2}{\sqrt{16-x^2}}\right\} \geq 0$

$f' \geq 0 \iff 16 - 2x^2 = 2(8-x^2) \geq 0 \iff -2\sqrt{2} \leq x \leq 2\sqrt{2}$

$\Rightarrow f$ in $[0,4]$ è $\begin{cases} \text{crescente in } [0, 2\sqrt{2}] \\ \text{decrescente in } [2\sqrt{2}, 4] \end{cases}$



$x_0 = 2\sqrt{2}$ pto MASSIMO ASSOLUTO

$f(2\sqrt{2}) = 7 \cdot 2\sqrt{2}\sqrt{16-8} = 7 \cdot 8 = 56$ MAX $= 56$

$x_1 = 0, 4$ PTI di minimo assoluto MIN $= 0$

A3) $L = \lim\limits_{x\to 0} \dfrac{2\sin x - 2x\cos x}{7\tan x - 7x} = \lim\limits_{x\to 0} \dfrac{2\sin x - 2x\cos x}{7\frac{\sin x}{\cos x} - 7x}$

$= \lim\limits_{x\to 0} \dfrac{2\sin x\cos x - 2x\cos^2 x}{7\sin x - 7x\cos x} = \left[\dfrac{0}{0}\right]$

$\sin x = x - \dfrac{x^3}{6} + o(x^3)$

$\cos x = 1 - \dfrac{x^2}{2} + o(x^2)$

$\dfrac{2\sin x\cos x - 2x\cos^2 x}{7\sin x - 7x\cos x} = \cancel{2\left(x - \frac{x^3}{6} + o(x^3)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right) - 2x\left(1 - \frac{x^2}{2} + o\right)^2}$

$= \dfrac{2\left(x - \frac{x^3}{6} + o(x^3)\right)\left(1 - \frac{x^2}{2} + o(x^2)\right) - 2x\left(1 - \frac{x^2}{2} + o(x^2)\right)^2}{7\left(x - \frac{x^3}{6} + o(x^3)\right) - 7x\left(1 - \frac{x^2}{2} + o(x^2)\right)} =$

$= \dfrac{\cancel{2x} - \frac{x^3}{3} - \frac{2x^3}{2} + o(x^3) - \cancel{2x} + 2x^3 + o(x^3)}{\cancel{7x} - \frac{7}{6}x^3 + o(x^3) - \cancel{7x} + \frac{7}{2}x^3 + o(x^3)} = \dfrac{\frac{2}{3}x^3 + o(x^3)}{\frac{7}{3}\cancel{\frac{14}{6}}x^3 + o(x^3)}$

$L = \lim\limits_{x\to 0} \dfrac{\frac{2}{3}x^3 + o(x^3)}{\frac{7}{3}x^3 + o(x^3)} = \boxed{\dfrac{2}{7}}$

$\lim\limits_{n\to +\infty} \dfrac{(-1)^n \sin n + \ln(1 + \cos^2 n) + 7e^n}{\cosh\left(\frac{2}{n}\right) + 14\sinh(n)} =$

$= \lim\limits_{n\to +\infty} \dfrac{7e^n + o(e^n)}{\frac{14}{2}e^n + o(e^n)} = \boxed{1}$

$\boxed{A4}$ $f(x) = 6\sin\left(\log(1+e^x) + 6\arctan x - \log 2\right)$

$f(0) = 0$

$$f'(x) = 6\cos\left(\log(1+e^x) + 6\arctan x - \log 2\right) \cdot \left(\frac{e^x}{1+e^x} + \frac{6}{1+x^2}\right)$$

$$f'(0) = 6\,\underbrace{\cos(0)}_{=1} \cdot \left(\frac{1}{2} + 6\right) = 6 \cdot \frac{13}{2} = 39$$

$T(x) = 39x$

$\boxed{A5}$ $u'' + 4u' + 4u = 0$ Pol. Caratt. $\underbrace{\lambda^2 + 4\lambda + 4}_{(\lambda+2)^2} = 0$

$\lambda_{1,2} = -2 \pm \sqrt{4-4}$

$\lambda_1 = \lambda_2 = -2$

$$\boxed{u_0(t) = c_1 e^{-2t} + c_2 t e^{-2t}}$$ (poiché doppie!)

Sol part. $u'' + 4u' + 4u = e^{-2t} + e^{2t}$.

Metodo somiglianza + princ. sovrapposizione

Sol particolare per $u'' + 4u' + 4u = e^{-2t}$

$u_p = At^2 e^{-2t}$ con $A \in \mathbb{R}$ (il t^2 appare perché $\lambda_1 = \lambda_2 = -2$)

$u_p' = e^{-2t}(-2At^2 + 2At)$

$u_p'' = e^{-2t}\left(-2A(-2t^2 + 2t) - 4At + 2A\right)$

$$u_p'' + 4u_p' + 4u_p = e^{-2t}\left\{\cancel{4At^2} - \cancel{8At} + 2A \cancel{- 8At^2} + \cancel{8At} \cancel{+ 4At^2}\right\} =$$

$$= 2A e^{-2t} \underset{\text{IMPONGO}}{=} 1 \qquad A = \frac{1}{2}$$

$u_{p,1}(t) = \frac{1}{2} t^2 e^{-2t}$

Cerco sol particolare per $u''+4u'+4u=e^{2t}$

$u_p=Ae^{2t}$ perché 2 non è radice pol. caratt.

$u_p'=2Ae^{2t}$ $\quad u_p''=4Ae^{2t}$

$u_p''+4u_p'+4u_p=e^{2t}A\{4+8+4\}\underset{\text{IMPONGO}}{=}e^{2t}$

$A=\frac{1}{16}$ $\qquad u_{p,2}=\frac{1}{16}e^{2t}$

SOL. PART: $\boxed{u_p(t)=\frac{1}{2}t^2e^{-2t}+\frac{1}{16}e^{2t}}$

$\boxed{A6}$ $z^2=8\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right)=8\,e^{i\frac{5}{6}\pi}$

$z=\rho e^{i\theta}$ $\qquad z^2=\rho^2e^{2\theta i}$

$\begin{cases}\rho^2=8\\ 2\theta=\frac{5}{6}\pi+2k\pi \quad k\in\mathbb{Z}\end{cases}\Rightarrow\begin{cases}\rho=2\sqrt{2}\\ \theta=\frac{5}{12}\pi+k\pi \quad k=0,1\end{cases}$

$\boxed{z_k=2\sqrt{2}\,e^{i\left(\frac{5}{12}\pi+k\pi\right)} \quad k=0,1}$

$z^2+i\bar{z}-\operatorname{Im}z=-8$ $\qquad z=x+yi$

$x^2+2xyi-y^2+\cancel{y}+ix-\cancel{y}=-8$ $\qquad \bar{z}=x-yi \quad i\bar{z}=ix+y$

$\operatorname{Im}z=y$

$\begin{cases}x^2-y^2=-8 & (1)\\ 2xy+x=0\end{cases}$ $\qquad x(2y+1)=0 \rightarrow x=0,\ y=\frac{1}{2}$

$x=0\rightarrow(1)$ $\qquad -y^2=-8$ $\qquad y=\pm2\sqrt{2}$

$y=\frac{1}{2}\longrightarrow(1)$ $\qquad x^2-\frac{1}{4}=-8$ $\qquad x^2=-8+\frac{1}{4}$ Nessuna sol. ($x\in\mathbb{R}$!)

IN CONCLUSIONE $\boxed{z_{1,2}=\pm2\sqrt{2}\,i}$

A7 $I = \int_6^8 (x-2)\ln(x-4)\,dx =$ PARTI $u = \ln(x-4)$ $u' = \frac{1}{x-4}$ $v = \frac{(x-2)^2}{2}$ $v' = x-2$

$$= \frac{(x-2)^2}{2}\ln(x-4)\Big|_6^8 - \int_6^8 \frac{(x-2)^2}{2(x-4)}\,dx =$$

$$= 18\ln 4 - 8\ln(2) - \frac{1}{2}\int_6^8 \frac{(x-4+2)^2}{(x-4)}\,dx =$$

$$= 18\ln(2^2) - 8\ln 2 - \frac{1}{2}\int_6^8 \frac{(x-4)^2 + 4(x-4) + 4}{x-4}\,dx =$$

$$= 36\ln 2 - 8\ln 2 - \frac{1}{2}\int_6^8 x-4\,dx - \frac{\cancel{4}}{\cancel{2}}2\int_6^8 1\,dx - \frac{\cancel{4}}{\cancel{2}}2\int_6^8 \frac{1}{x-4}\,dx$$

$$= 28\ln 2 - \frac{1}{2}\frac{x^2}{2}\Big|_6^8 + \cancel{4} - \cancel{4} - 2\ln(x-4)\Big|_6^8 =$$

$$= 28\ln 2 - 16 + 9 - 2\ln 4 + 2\ln 2 =$$

$$= 28\ln 2 - 2\ln 2 - 7 = 26\ln 2 - 7$$

A8 $$\sum_{n=1}^{+\infty} \frac{\left[\ln\left(1+\frac{2}{n}\right)\right]^2}{\arctan(2n)\left[\sinh\frac{1}{n} - \sin\frac{1}{n}\right]^\alpha}$$

Confronto asintotico:
per $n \to +\infty$

$$\left(\ln\left(1+\frac{2}{n}\right)\right)^2 \sim \frac{4}{n^2}$$

$$\arctan(2n) \longrightarrow \frac{\pi}{2} \quad \text{per } n \to +\infty$$

$$\left(\sinh\frac{1}{n} - \sin\frac{1}{n}\right)^\alpha = \left[\cancel{\frac{1}{n}} + \frac{1}{6}\left(\frac{1}{n}\right)^3 - \cancel{\frac{1}{n}} + \frac{1}{6}\frac{1}{n^3} + o\left(\frac{1}{n^3}\right)\right]^\alpha$$

$$\sim \frac{1}{3^\alpha}\frac{1}{n^{3\alpha}}$$

quindi basta studiare (le costanti si trascurano)

$$\sum_{n=1}^{+\infty} \frac{1}{n^2}\cdot\frac{1}{1/n^{3\alpha}} = \sum \frac{1}{n^{2-3\alpha}} < +\infty$$

$$\iff 2-3\alpha > 1 \iff \boxed{\alpha < 1/3}$$

B1 $g(x) = o(e^x)$ $\qquad f(x) \sim x$ per $x \to +\infty$

$$\lim_{x\to+\infty} \frac{f(x)\,g(x)}{e^{2x}} = \lim_{x\to+\infty} \underbrace{\frac{f(x)}{e^x}}_{\to 0} \cdot \underbrace{\frac{g(x)}{e^x}}_{\to 0} = 0$$

quindi risp corretta $\boxed{C}$

B2 $\exists$ finito $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{|x-x_0|}$

$\Longrightarrow \lim_{x\to x_0} f(x) = f(x_0)$ Caratterizzazione continuità in x_0

quindi risp. corretta è $\boxed{D}$

B3 $f : (0,+\infty) \to \mathbb{R}$ cont, $g(x) = e^{f(x)} \geq 0$

per proprietà esponenziale.

Risp. corretta $\boxed{B}$

Controesempi alle altre ? $f(x) = x$ per A, C, D :

e^x non ha né max né min in $(0,+\infty)$

B4 $\lim_{x\to+\infty} f(x) = 1$

Cambio variabile $y = \frac{1}{x} \to 0^+$ $\quad \lim_{y\to 0^+} f\left(\frac{1}{y}\right) = 1$

quindi risposta $\boxed{B}$

B5 $\sum \frac{\cos(n\pi)}{n^{\pi/4} + \ln n}$ $\qquad \cos(n\pi) = (-1)^n$

e $n^{\pi/4} + \ln n \sim n^{\pi/4}$

quindi la serie NON conv. assolutamente $\qquad \left(\frac{\pi}{4} \in (0,1)\right)$

ma converge semplicemente per Leibniz.

Risposta corretta $\boxed{D}$

$\boxed{B6}$ $a_n = \sin\left(n\frac{\pi}{2}\right)\sinh n$

$$\sin\left(n\frac{\pi}{2}\right) = \begin{cases} 0 & \text{se } n \text{ PARI} \\ (-1)^k & \text{se } n \text{ DISPARI} \quad n=2k+1 \end{cases}$$

$\sinh n \longrightarrow +\infty$ per $n \longrightarrow +\infty$

dunque $\nexists \lim_{n\to+\infty} a_n$ risposta esatta $\boxed{B}$

$\boxed{B7}$ $f\in C^2$ $f(x) = 1+x^2+o(x^2)$ per $x\to 0 \Rightarrow$

$f(0)=1 \quad f'(0)=0 \quad f''(0)=2$

$\Rightarrow x_0 = 0$ è pto di minimo relativo/locale

risp esatta: $\boxed{C}$

(Essendo locale lo sviluppo, non possiamo sapere se è minimo globale/assoluto).

$\boxed{B8}$ $F(x) = \int_0^x |t|\,dt \qquad F'(x) = |x| \geq 0$

quindi F è crescente (e derivabile)

risp esatta $\boxed{B}$

[illegible]