Soluzioni Analisi 1

17/7/26

A1 $f(x)=\dfrac{x^2-1}{x^4+15}$ $f:\mathbb{R}\to\mathbb{R}$ PARI

- È subito chiaro che $f(\pm 1)=0$ e
  $f\geq 0$ in $(-\infty,-1]\cup[1,+\infty)$
  $f\leq 0$ in $[-1,1]$
  Inoltre $\lim_{x\to\pm\infty} f(x)=0$
- Studiamo f':

$$f'(x)=\frac{2x(x^4+15)-4x^3(x^2-1)}{(x^4+15)^2}\geq 0$$

$$\iff 2x^5-4x^5+4x^3+30x\geq 0$$

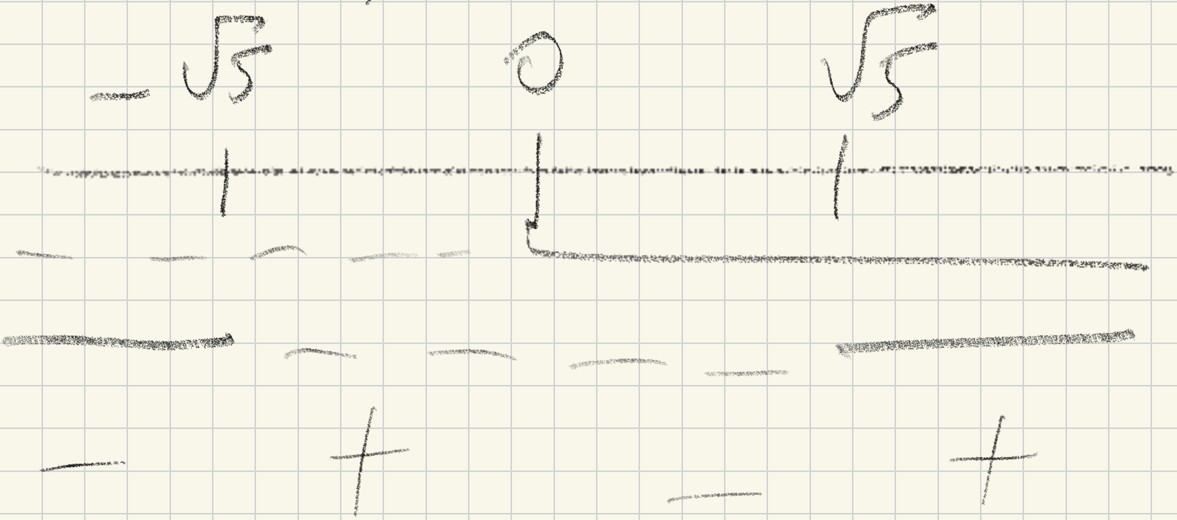
$$\iff x^5-2x^3-15x\leq 0$$

Studiamo $x^4-2x^2-15=0$

$$x^2 = 1 \pm \sqrt{1+15} = \begin{cases} 5 \\ -3 \end{cases}$$

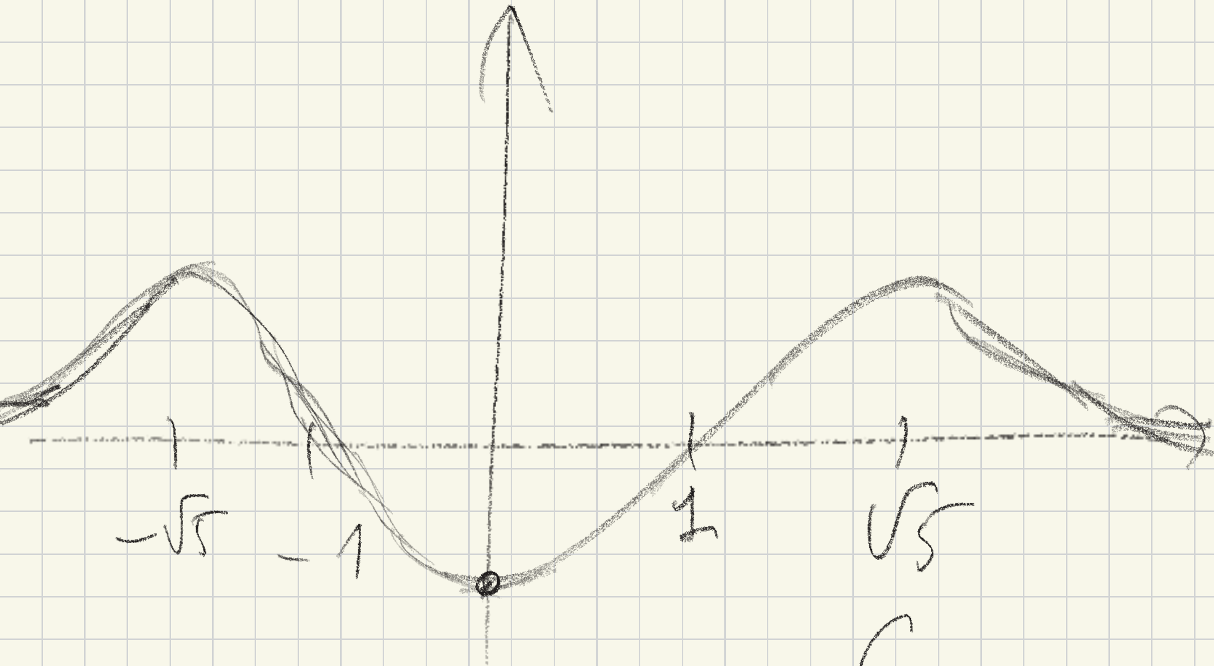
Quindi dobbiamo studiare

$$x(x^2-5)(x^2+3) \le 0$$



Quindi $f'(x) \ge 0$ in

$$(-\infty, -\sqrt{5}) \cup (0, \sqrt{5})$$



PTI DI MAX: $x_{1,2} = \pm\sqrt{5}$

PTI DI MIN: $x_0 = 0$

$$MAX = f(\pm\sqrt{5}) = \frac{5-1}{25+15} = \frac{1}{10}$$

$$MIN = f(0) = -\frac{1}{15}$$

A2|

$$\lim_{x\to 0} \frac{e^{\alpha+3x-4x^2} - e^{\alpha}}{\frac{\sqrt{2}}{2} - \cos\left(\frac{\pi}{4}+\sqrt{2}x\right)} =$$

$$= \lim_{x\to 0} \frac{e^{\alpha}\left\{e^{3x-4x^2} - 1\right\}}{\frac{\sqrt{2}}{2}\left\{1 - \frac{\cos\frac{\pi}{4} - \left(\sin\frac{\pi}{4}\right)\cdot\sqrt{2}x + o(x)}{\frac{\sqrt{2}}{2}}\right\}} =$$

Sviluppo $\cos\left(\frac{\pi}{4}+\sqrt{2}x\right)$ per $x\to 0$

$$= \lim_{x\to 0} \frac{e^{\alpha}\left\{3x - 4x^2 + o(x)\right\}}{\frac{\sqrt{2}}{2}\left\{\cancel{1} - \cancel{1} + \sqrt{2}x + o(x)\right\}} = 3e^{\alpha}$$

Devo imporre $3e^{\alpha} = 2$

$$\Rightarrow \alpha = \log\left(\frac{2}{3}\right)$$

A3

$$z^2+6zi\cancel{-1}=\sqrt{2}\,e^{\frac{7}{4}\pi i}=\sqrt{2}\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}i\right)$$

$$=1\cancel{-i}$$

$$\Leftrightarrow z^2+6iz-1=0$$

$$z_{1/2}=-3i\pm\sqrt{-9+1}=$$

$$=-3i\pm 2\sqrt{2}i$$

$$=\left(-3\pm 2\sqrt{2}\right)i$$

$\boxed{4b)}$

$$\lim_{n \to +\infty} \left(\frac{n^6 + n^5 - n^5 + n^2 + n^4}{n^6 + n^5} \right)^{n^2} =$$

$$= \lim_{n \to +\infty} \left(1 + \frac{-n^5 + n^4 + n^2}{n^6 + n^5} \right)^{n^2} =$$

$$-\frac{1}{n}$$

$$= \lim_{n \to +\infty} \Big[\underbrace{\left(1 - \frac{1}{n}\right)^{n}}_{\to\, e^{-1}} \Big]^{\overbrace{n}^{\to\, +\infty}} = 0$$

$$\lim_{x \to 0} \frac{\tan x - \sin x + \ln^4(1+x)}{2\arctan^3(x)} \qquad \ln^4(1+x) = o(x^3)$$

$$=$$

$$\tan x = x + \frac{x^3}{3} + o(x^3)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$= \lim_{x \to 0} \frac{\cancel{x} + \frac{x^3}{3} + o(x^3) - \cancel{x} + \frac{x^3}{6} + o(x^3) + o(x^3)}{2x^3 + o(x^3)}$$

$$= \frac{1}{4}$$

A5

$$I=\int_1^3 \sqrt{2x+4}\, e^{\sqrt{2x+4}}\, dx = \quad \text{SOSTITUZ.}$$

$t=\sqrt{2x+4}$

$dt = \frac{\cancel{2}}{\cancel{2}\sqrt{2x+4}}\, dx$

$\Rightarrow dx = t\,dt$

$x=1 \quad t=\sqrt{6}$

$x=3 \quad t=\sqrt{10}$

$$= \int_{\sqrt{6}}^{\sqrt{10}} t^2 e^t \, dt =$$

PARTI

$$= t^2 e^t \Big|_{\sqrt{6}}^{\sqrt{10}} - 2\int_{\sqrt{6}}^{\sqrt{10}} t e^t =$$

PARTI

e^t	e^t
t^2	$2t$
e^t	e^t
t	1

PARTI

$$= 10e^{\sqrt{10}} - 6e^{\sqrt{6}} - 2te^t \Big|_{\sqrt{6}}^{\sqrt{10}}$$

$$+2e^t \Big|_{\sqrt{6}}^{\sqrt{10}} =$$

$$I = e^{\sqrt{10}}\left\{10 - 2\sqrt{10} + 2\right\} + e^{\sqrt{6}}\left\{-6 + 2\sqrt{6} - 2\right\}$$

$\boxed{A6}$ $f(x) = e^{(x-2)} \sin(x-2) \cos^3(x-2)$

$$e^{x-2} = 1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + o\left((x-2)^3\right)$$

μ

$x \to 2$

$$\sin(x-2) = (x-2) - \frac{(x-2)^3}{6} + o\left((x-2)^3\right)$$

$$\cos^3(x-2) = \left(1 - \frac{(x-2)^2}{2} + o\left((x-2)^3\right)\right)^3$$

$$= \left(1 - \frac{3}{2}(x-2)^2 + o\left((x-2)^3\right)\right)$$

Quindi:

$$f(x) = \left\{1 + (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{6} + o\left((x-2)^3\right)\right\}$$

$$\cdot \left\{(x-2) - \frac{(x-2)^3}{6} + o\left((x-2)^3\right)\right\} \cdot$$

$$\cdot \left\{1 - \frac{3}{2}(x-2)^2 + o\left((x-2)^3\right)\right\} =$$

$$= \left\{ (x-2) + (x-2)^2 + \frac{(x-2)^3}{2} - \frac{(x-2)^3}{6} + o(x-2)^3 \right\}$$

$$\cdot \left\{ 1 - \frac{3}{2}(x-2)^2 + o\left((x-2)^3\right) \right\}$$

$$= (x-2) + (x-2)^2 + \frac{(x-2)^3}{3} - \frac{3}{2}(x-2)^3$$

$$+ o\left((x-2)^3\right)$$

$$= (x-2) + (x-2)^2 - \frac{7}{6}(x-2)^3 + o\left((x-2)^3\right)$$

quindi

$$P_3(x,2) = (x-2) + (x-2)^2 - \frac{7}{6}(x-2)^3$$

$\boxed{A7}$

$$\begin{cases} y'(x) = 4(x-1)\,y(x) + \ln(x+2)\,e^{2(x-1)^2} \\ y(1) = 3 + 3\ln x \end{cases}$$

$$y'(x) - 4(x-1)\,y(x) = \ln(x+2)\; e^{2(x-1)^2}$$

$$\alpha(x) = -4(x-1) \qquad A(x) = \int_1^x -4(t-1)\,dt$$

$$= -2(x-1)^2$$

$$\underbrace{e^{-2(x-1)^2}\big(y'(x) - 4(x-1)\,y(x)\big)} = \ln(x+2)$$

$$\frac{d}{dx}\left\{ y(x)\, e^{-2(x-1)^2} \right\} = \ln(x+2)$$

Integro fra 1 e x:

$$y(x)\, e^{-2(x-1)^2} - y(1) = \int_1^x \ln(t+2)\,dt$$

PART

$$= t\ln(t+2)\Big|_1^x - \int_1^x \frac{t}{t+2}\,dt$$

$$\begin{array}{cc} 1 & t \\ \ln(t+2) & \frac{1}{t+2} \end{array}$$

$$= x\ln(x+2) - \ln(3) - \int_1^x 1 - \frac{2}{t+2}\,dt$$

$$= x\ln(x+2) - \ln 3 - (x-1) + 2\ln(x+2) - 2\ln 3$$

$$= (x+2)\ln(x+2) - 3\ln 3 - (x-1)$$

$$y(x) = e^{2(x-1)^2}\left\{ 3 + \cancel{3\ln 3} - \cancel{3\ln 3} - x + 1 + (x+2)\ln(x+2) \right\}$$

$$= e^{2(x-1)^2}\left\{ 4 - x + (x+2)\ln(x+2) \right\}$$

A8 $\sum \arctan\left(\frac{1}{n^\alpha}\right)\left(\frac{1}{n^{1/7}}-\ln\left(1+\frac{1}{n^{1/7}}\right)\right)^2$

crit. confronto asintotico:

$\arctan\frac{1}{n^\alpha}\sim\frac{1}{n^\alpha}$ $\quad n\to+\infty$ $(\alpha>0)$

$\ln\left(1+\frac{1}{n^{1/7}}\right)=\frac{1}{n^{1/7}}-\frac{1}{2n^{2/7}}+o\left(\frac{1}{n^{2/7}}\right)$

$\Rightarrow$ la serie converge se e solo se converge

$$\sum\frac{1}{n^\alpha}\left(\cancel{\frac{1}{n^{1/7}}}-\cancel{\frac{1}{n^{1/7}}}+\frac{1}{2n^{2/7}}\right)^2=$$

$$=\sum\frac{1}{n^{\alpha+4/7}}<+\infty \iff \boxed{\alpha+\frac{4}{7}>1}$$

B1) Siccome $f: [-a,a] \to \mathbb{R}$ pari $\Rightarrow f(a) = f(-a)$

e quindi si può applicare Rolle a f (cont. in $[-a,a]$, deriv. in $(-a,a)$)

$\Rightarrow \exists\, c \in (-a,a)$ t.c. $f'(c) = 0$.

$$g'(x) = \cos(f(x)+1) \cdot f'(x) \Rightarrow$$

$\exists\, c \in (-a,a)$ t.c. $g'(c) = 0$.

quindi risposta corretta

$\boxed{C}$

B2) se $f(t) = o(t)$ per $t \to +\infty$

allora $\left(t = \frac{1}{x}\right)$

$$f\left(\frac{1}{x}\right) = o\left(\frac{1}{x}\right) \text{ per } x \to 0^+$$

ovvero $\lim_{x \to 0^+} \frac{f(1/x)}{1/x} = \lim_{x \to 0^+} x f\left(\frac{1}{x}\right) = 0$

quindi risposta corretta $\boxed{A}$

B3)

$$P_1^h(x,0) = h(0) + h'(0)x$$

$$h(0) = \frac{f(0)}{g(0)}$$

$$h'(0) = \frac{f'(0)g(0) - f(0)g'(0)}{g^2(0)}$$

Grazie alle ipotesi su f, g:

$f(0) = a$ $f'(0) = b$

$g(0) = 1$ $g'(0) = \beta$

$\Rightarrow P_1^h(x,0) = a + (b - a\beta)x$

risposta corretta $\boxed{A}$

B4) g è monotone strett. decrescente, risposta $\boxed{D}$, perché composizione di una funzione strett. crescente (f^2) e di una strett. decrescente $\left(z \to \frac{1}{1+z}\right)$

B5 Siccome se

(a_n) è monotona

allora $\lim_{n \to +\infty} a_n$ esiste (finito o infinito)

è anche vera la contronominale

se $\lim_{n \to +\infty} a_n$ non esiste,

allora (a_n) NON è monotona.

quindi risposta corretta $\boxed{D}$

B6) Se $\sum a_n$ diverge

e $a_n \leq 0$, allora $\sum a_n = -\infty$

$c_n = \max\{-a_n, 0\} = -a_n$

$\Rightarrow \sum c_n = \sum -a_n = +\infty$

diverge!

Quindi risposta corretta $\boxed{A}$

(Per esercizio cercate i controesempi alle altre opzioni!)

37) Per le proprietà dei limiti di composizioni e della continuità (di f e ln)

$$\lim_{x \to 0} f(x) = f(0) > 0$$

(perché $f: \mathbb{R} \to (0, +\infty)$)

e

$$\lim_{x \to 0} \ln(f(x)) = \ln(f(0))$$

risposta corretta $\boxed{D}$

B8) Per proprietà additività integrale:

$$\int_c^b f(x)\,dx = \int_a^b f(x)\,dx + \int_c^a f(x)\,dx$$

quindi risposte corrette

$\boxed{B}$