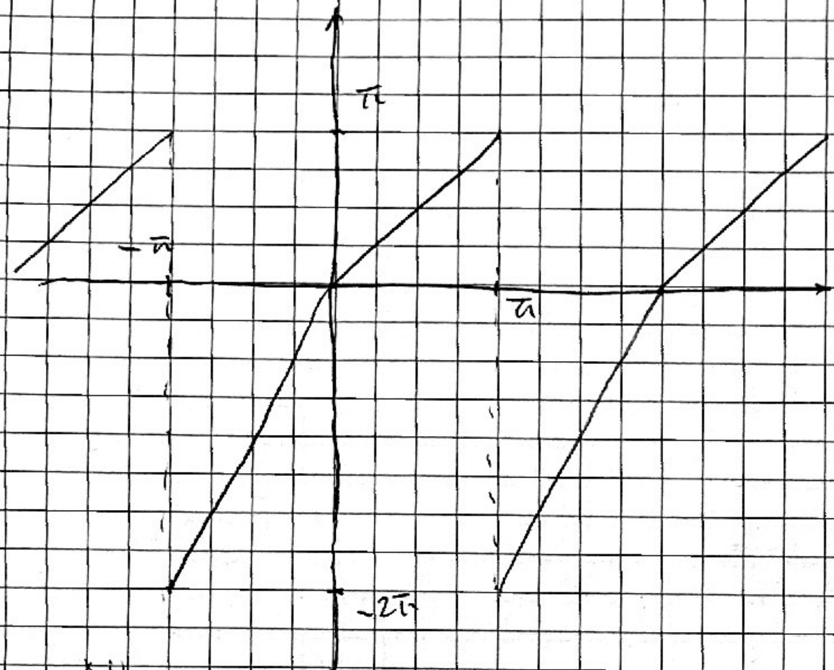
(1) $f(t) = \begin{cases} 2t & t \in [-\pi, 0) \\ t & t \in [0, \pi) \end{cases}$



È evidente che f appartiene a $L^2(-\pi,\pi)$ e pertanto è sviluppabile. Infatti

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = \int_{-\pi}^{0} 4t^2 dt + \int_0^{\pi} t^2 dt = \frac{4}{3} t^3 \Big|_{-\pi}^{0} + \frac{t^3}{3}\Big|_0^{\pi} = \frac{5}{3}\pi^3$$

Abbiamo poi

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} 2t\,dt + \int_0^{\pi} t\,dt\right]$$

$$= \frac{1}{\pi}\left[t^2 \Big|_{-\pi}^{0} + \frac{t^2}{2}\Big|_0^{\pi}\right] = \frac{1}{\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} 2t\cos nt\,dt + \int_0^{\pi} t\cos nt\,dt\right]$$

$$= \frac{1}{\pi}\left[\cancel{2t\frac{\sin nt}{n}\Big|_{-\pi}^{0}} - 2\int_{-\pi}^{0}\frac{\sin nt}{n}dt + \cancel{t\frac{\sin nt}{n}\Big|_0^{\pi}} + \right.$$

$$\left. - \int_0^{\pi}\frac{\sin nt}{n}dt\right] = \frac{1}{\pi}\left[2\frac{\cos nt}{n^2}\Big|_{-\pi}^{0} + \frac{\cos nt}{n^2}\Big|_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[2\frac{1-(-)^n}{n^2} + \frac{(-)^n - 1}{n^2}\right] = \frac{1-(-)^n}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt = \frac{1}{\pi}\left[\int_{-\pi}^{0} 2t\sin nt\,dt + \int_{0}^{\pi} t\sin nt\,dt\right]$$

$$= \frac{1}{\pi}\left[-2t\frac{\cos nt}{n}\Big|_{-\pi}^{0} + \cancel{2\int_{-\pi}^{0}\frac{\cos nt}{n}\,dt} - t\frac{\cos nt}{n}\Big|_{0}^{\pi} + \right.$$

$$\left. + \cancel{\int_{0}^{\pi}\frac{\cos nt}{n}\,dt}\right] = \frac{1}{\pi}\left(2(-\pi)\frac{(-)^n}{n} - \pi\frac{(-)^n}{n}\right)$$

$$= \frac{3(-)^{n+1}}{n}$$

Pertanto

$$S(t) = -\frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{1-(-)^n}{\pi n^2}\cos nt + \frac{3(-)^{n+1}}{n}\sin nt$$

Osserviamo che

1) $\forall\, t \in ((2k-1)\pi, 2k\pi) \cup (2k\pi, (2k+1)\pi)$ f è continua e derivabile in t e, dunque $S(t) = f(t)$

2) $\forall\, t = 2k\pi$ f è continua in t ed esistono finite la derivata dx e sx. Pertanto, ancora, $S(t) = f(t)$

3) $\forall\, t = (2k+1)\pi$ f ha un salto finito e le pendenze destra e sinistra sono finite. Pertanto $S(t) = \frac{f(t^+) + f(t^-)}{2} = -\frac{\pi}{2}$

Dall'identità di Parseval sappiamo che

$$\int_{-\pi}^{\pi} |f(t)|^2\,dt = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2\right]$$

Dunque

$$\frac{5}{3}\pi^3 = \pi\left[\frac{\pi^2/4}{2} + \sum_{n=1}^{\infty}\left(\frac{1-(-)^n}{\pi n^2}\right)^2 + \frac{9}{n^2}\right]$$

Osserviamo che

$$\sum_{n=1}^{\infty} \left(\frac{1-(-)^n}{\pi n^2} \right)^2 = \sum_{k=0}^{\infty} \left(\frac{2}{\pi (2k+1)^2} \right)^2$$

$$= \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

Otteniamo

$$\left(\frac{5}{3}\pi^3 - \frac{\pi^3}{8} \right) \frac{1}{\pi} - 9 \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

$$\frac{40-3}{24} \pi^2 - 9 \frac{\pi^2}{6} = \frac{4}{\pi^2} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

$$\frac{\pi^2}{4} \left(\frac{37-36}{24} \right) \pi^2 = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

$$\frac{\pi^4}{96} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^4}$$

Infine, se ci poniamo in $t=0$ abbiamo

$$f(0) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \frac{1-(-)^n}{\pi n^2}$$

$$0 = -\frac{\pi}{4} + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

$$\frac{\pi^2}{8} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^2}$$

---

(2) $z' = Az$ $\qquad A = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$

$\det(\lambda I - A) = 0 \qquad \begin{vmatrix} \lambda - 1 & -2 \\ -5 & \lambda + 2 \end{vmatrix} = 0$

$$\lambda^2 - \lambda + 2\lambda - 2 - 10 = 0$$

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda + 4)(\lambda - 3) = 0 \qquad \begin{array}{l} \lambda_1 = -4 \\ \lambda_2 = 3 \end{array}$$

Per $\lambda_1 = -4$ determiniamo $h_1 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$\begin{bmatrix} -5 & -2 \\ -5 & -2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$5\alpha + 2\beta = 0 \qquad \begin{cases} \alpha = 2 \\ \beta = -5 \end{cases}$$

Per $\lambda_2 = 3$ determiniamo $h_2 = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

$$\begin{bmatrix} 2 & -2 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2\alpha - 2\beta = 0 \qquad \begin{cases} \alpha = 1 \\ \beta = 1 \end{cases}$$

Pertanto, la soluzione generale è

$$\underline{z} = \begin{bmatrix} 2e^{-4x} & e^{3x} \\ -5e^{-4x} & e^{3x} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

o anche, posto $\underline{z} = \begin{bmatrix} u \\ v \end{bmatrix}$

$$\begin{cases} u = 2c_1 e^{-4x} + c_2 e^{3x} \\ v = -5c_1 e^{-4x} + c_2 e^{3x} \end{cases}$$

In $x=0$ abbiamo

$$\begin{cases} 2C_1 \cancel{e} + C_2 = 1 \\ -5C_1 + C_2 = 1 \end{cases} \qquad 7C_1 = 0 \qquad \begin{matrix} C_1 = 0 \\ C_2 = 1 \end{matrix}$$

Quindi, la soluzione cercata è

$$\begin{cases} u = e^{3x} \\ v = e^{3x} \end{cases}$$

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(3) $f = 2x^4 + y^2$ È evidente che $f \in C^\infty(\mathbb{R}^2)$

Abbiamo $\nabla f = 0$ $\begin{cases} 8x^3 = 0 \\ 2y = 0 \end{cases} \Longrightarrow (0,0)$ è l'unico punto stazionario.

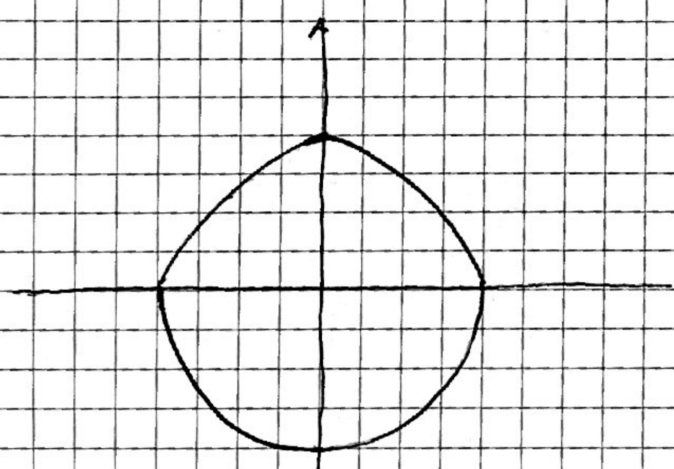
Poichè $f \geq 0$ in $\mathbb{R}^2$ e $f(0,0) = 0$
è chiaro che $(0,0)$ è punto di minimo locale ed assoluto.

Consideriamo, ora, $D = \{(x,y) \in \mathbb{R}^2 : -\sqrt{1-x^2} \leq y \leq 1-x^2\}$

$-\sqrt{1-x^2} = y$

$1 - x^2 = y^2$

$x^2 + y^2 = 1$



Γ_1 $y = 1 - x^2$ $x \in [-1,1]$

Γ_2 $y = -\sqrt{1-x^2}$ $x \in [-1,1]$

Visto che $(0,0)$ è punto di minimo assoluto, è sufficiente cercare il massimo assoluto, che è sicuramente assunto ~~in~~ su Γ_1 oppure su Γ_2

$$f\big|_{\Gamma_2} = 2x^4 + 1 - x^2 \qquad x \in [-1,1]$$

$$g_2(x) = 2x^4 - x^2 + 1$$

$$g_2' = 8x^3 - 2x = 2x(4x^2-1)$$

$$g_2' \geq 0$$

$-1 \quad - \quad -\frac{1}{2} \quad + \quad 0 \quad - \quad \frac{1}{2} \quad + \quad 1$

$x = \pm \frac{1}{2}$ minimo locale $\qquad g_2\left(\pm\frac{1}{2}\right) = 2\frac{1}{16} - \frac{1}{4} + 1 = \frac{7}{8}$

$x = 0$ massimo locale $\qquad g_2(0) = 1$

$x = \pm 1 \qquad g_2(\pm 1) = 2$

$$M_2 = 2$$

$$f\big|_{\Gamma_1} = 2x^4 + (1-x^2)^2 \qquad x \in [-1,1]$$

$$= 2x^4 + x^4 - 2x^2 + 1 = 3x^4 - 2x^2 + 1$$

$$g_1(x) = 3x^4 - 2x^2 + 1$$

$$g_1'(x) = 12x^3 - 4x = 4x(3x^2-1)$$

$$g_1'(x) \geq 0$$

$-1 \quad - \quad -1/\sqrt{3} \quad + \quad - \quad 1/\sqrt{3} \quad + \quad +1$

$x = 0$ massimo locale $g_1(0) = 1$

$x = \pm \frac{1}{\sqrt{3}}$ minimo locale $g_1\left(\pm \frac{1}{\sqrt{3}}\right) = 3\frac{1}{9} - \frac{2}{3} + 1 = \frac{2}{3}$

$x = \pm 1$ $g_1(\pm 1) = 2$

$M_1 = 2$

Pertanto

$$m_{ass} = 0$$

$$M_{ass} = 2$$

---

(4) $\Phi(y) = \int_0^{\pi} \left((y')^2 - 4y^2 + y\cos x\right) dx$

Scriviamo l'equazione di Eulero-Lagrange

$$\frac{d}{dx}\frac{\partial}{\partial y'}\left((y')^2 - 4y^2 + y\cos x\right) - \frac{\partial}{\partial y}\left((y')^2 - 4y^2 + y\cos x\right) = 0$$

$$\frac{d}{dx}\left[2y'\right] - \left[-8y + \cos x\right] = 0$$

$$2y'' + 8y = \cos x$$

$$y'' + 4y = \frac{1}{2}\cos x$$

Risolto

$$y = C_1 \cos 2x + C_2 \sin 2x + y_p$$

$$y_p = A \sin x + B \cos x \qquad y_p' = A\cos x - B\sin x$$

$$y_p'' = -A\sin x - B\cos x$$

$$-A\sin x - B\cos x + 4A\sin x + 4B\cos x = \frac{1}{2}\cos x$$

$$3A = 0 \qquad A = 0$$

$$3B = \frac{1}{2} \qquad B = \frac{1}{6}$$

Quindi

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{6}\cos x$$

Imponendo $y(\pi) = 1$ abbiamo

$$1 = C_1 \cos 2\pi + \cancel{C_2 \sin 2\pi} + \frac{1}{6}\cos \pi$$

$$1 = C_1 - \frac{1}{6} \qquad C_1 = \frac{7}{6}$$

Pertanto, le estremali cercate sono

$$y = \frac{7}{6}\cos 2x + C_2 \sin 2x + \frac{1}{6}\cos x$$