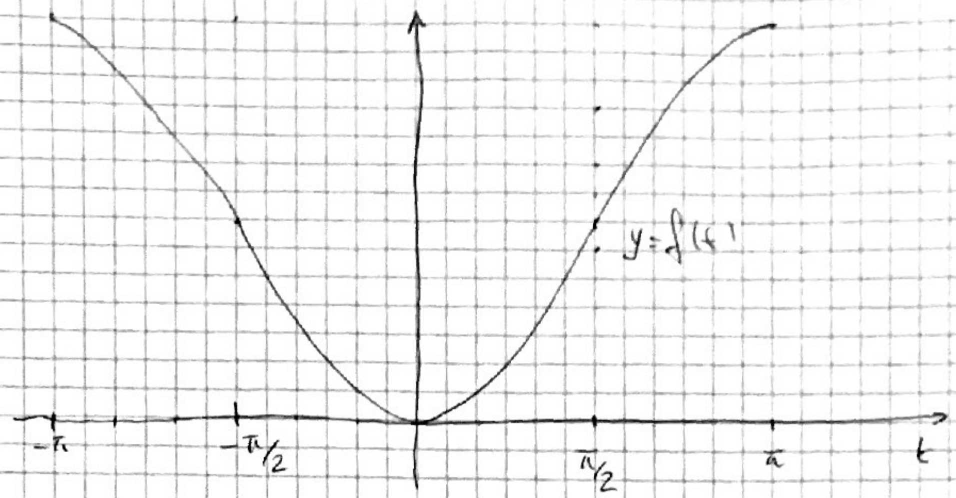
(1)



È chiaro che f è sviluppabile, poiché $f \in L^2(-\pi,\pi)$. Infatti

$$\int_{-\pi}^{\pi}|f(t)|^2dt = 2\int_0^{\pi} f(t)^2 dt = 2\int_0^{\pi/2} t^4 + 2\int_{\pi/2}^{\pi}\left[\frac{\pi^4}{4}+(\pi-t)^4 + \right.$$

$$\left. - 2\frac{\pi^2}{2}(\pi-t)^2\right]dt = 2\left[\frac{t^5}{5}\right]_0^{\pi/2} + 2\left[\frac{\pi^4}{4}t - \frac{(\pi-t)^5}{5} + \right.$$

$$\left. + \pi^2\frac{(\pi-t)^3}{3}\right]_{\pi/2}^{\pi} = 2\frac{\pi^5}{32\cdot 5} + 2\left[\frac{\pi^4}{4}\cdot\frac{\pi}{2}+\frac{\pi^5}{32\cdot 5} - \frac{\pi^2\pi^3}{8\cdot 3}\right]$$

$$= \frac{\pi^5}{80} + \frac{\pi^5}{4} + \frac{\pi^5}{80} - \frac{2\pi^5}{24}$$

$$= \frac{\pi^5}{240}\left[3+60+3-20\right] = \frac{46}{240}\pi^5$$

Abbiamo poi

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)dt = \frac{2}{\pi}\int_0^{\pi} f(t)dt$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} t^2 dt + \int_{\pi/2}^{\pi}\left[\frac{\pi^2}{2} - (\pi-t)^2\right]dt\right]$$

$$= \frac{2}{\pi}\left[\frac{t^3}{3}\Big|_0^{\pi/2} + \frac{\pi^2}{2}t\Big|_{\pi/2}^{\pi} + \frac{(\pi-t)^3}{3}\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi^3}{24}} + \frac{\pi^3}{4} - \cancel{\frac{\pi^3}{24}}\right] = \frac{\pi^2}{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} t^2 \cos nt\,dt + \int_{\pi/2}^{\pi}\left(\frac{\pi^2}{2}-(\pi-t)^2\right)\cos nt\,dt\right]$$

$$= \frac{2}{\pi}\left[t^2\frac{\sin nt}{n}\Big|_0^{\pi/2} - 2\int_0^{\pi/2} t\,\frac{\sin nt}{n} + \frac{\pi^2}{2}\frac{\sin nt}{n}\Big|_{\pi/2}^{\pi} - (\pi-t)^2\frac{\sin nt}{n}\Big|_{\pi/2}^{\pi} - 2\int_{\pi/2}^{\pi}(\pi-t)\,\frac{\sin nt}{n}\right]$$

$$= \frac{2}{\pi}\left[\cancel{\frac{\pi^2}{4}\frac{\sin n\pi/2}{n}} + 2t\,\frac{\cos nt}{n^2}\Big|_0^{\pi/2} - 2\int_0^{\pi/2}\frac{\cos nt}{n^2}\,dt - \cancel{\frac{\pi^2}{2}\frac{\sin n\pi/2}{n}} + \cancel{\frac{\pi^2}{4}\frac{\sin n\pi/2}{n}} + 2(\pi-t)\frac{\cos nt}{n^2}\Big|_{\pi/2}^{\pi} + 2\int_{\pi/2}^{\pi}\frac{\cos nt}{n^2}\,dt\right]$$

$$= \frac{2}{\pi}\left[\cancel{\pi\,\frac{\cos n\pi/2}{n^2}} - 2\,\frac{\sin nt}{n^3}\Big|_0^{\pi/2} - \cancel{\pi\,\frac{\cos n\pi/2}{n^2}} + 2\,\frac{\sin nt}{n^3}\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[-2\,\frac{\sin n\pi/2}{n^3} - 2\,\frac{\sin n\pi/2}{n^3}\right]$$

$$= -\frac{8}{\pi}\,\frac{\sin n\pi/2}{n^3}$$

In definitiva

$$S(t) = \frac{\pi^2}{4} - \sum_{n=1}^{\infty}\frac{8}{\pi}\,\frac{\sin n\frac{\pi}{2}}{n^3}\cos nt$$

La funzione è ovunque derivabile. Pertanto

$$\forall t \in \mathbb{R} \quad S(t) = f(t).$$

Dall'identità di Parseval

$$\int_{-\pi}^{\pi} |f(t)|^2\,dt = \pi\left[\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2\right]$$

$$\frac{46}{240}\pi^5 = \pi\left[\frac{1}{2}\frac{\pi^4}{4} + \sum_{n=1}^{\infty}\frac{64}{\pi^2}\frac{1}{n^6}\left|\sin n\frac{\pi}{2}\right|^2\right]$$

Osserviamo che

$$n = 2k \qquad a_{2k} = \sin k\pi = 0$$

$$n = 2k+1 \qquad a_{2k+1} = \sin(2k+1)\frac{\pi}{2} = (-)^k$$

Perciò

$$\frac{46}{240}\pi^4 = \frac{\pi^4}{8} + \frac{64}{\pi^2}\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6}$$

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} = \pi^4\left(\frac{46}{240} - \frac{30}{240}\right)\cdot\frac{\pi^2}{64}$$

$$= \frac{\pi^6}{\cancel{64}_4}\cdot\frac{\cancel{16}^1}{240}$$

$$\sum_{k=0}^{\infty}\frac{1}{(2k+1)^6} = \frac{\pi^6}{960}$$

Infine, per la convergenza puntuale $S(0) = f(0) = 0$.

Perciò

$$0 = \frac{\pi^2}{4} - \sum_{k=0}^{\infty}\frac{8}{\pi}\frac{(-)^k}{(2k+1)^3}$$

$$\sum_{k=0}^{\infty}\frac{(-)^k}{(2k+1)^3} = +\frac{\pi^3}{32}$$

---

(2) $$y'' + 4\lambda y' + 8\lambda^2 y = 2\left(x + \frac{1}{2\lambda}\right) \qquad \lambda \neq 0$$

$$\alpha^2 + 4\lambda\alpha + 4\lambda^2 = -4\lambda^2$$

$$(\alpha + 2\lambda)^2 = -4\lambda^2$$

$$\alpha = -2\lambda \pm 2\lambda i$$

Pertanto, l'integrale generale è

$$y = e^{-2\lambda x}\left[C_1\cos 2\lambda x + C_2\sin 2\lambda x\right] + y_p$$

Abbiamo

$$y_p = Ax + B \qquad y_p' = A \qquad y_p'' = 0$$

da cui

$$4\lambda A + 8\lambda^2 A x + 8\lambda^2 B = 2x + \frac{1}{\lambda}$$

$$8\lambda^2 A = 2 \qquad A = \frac{1}{4\lambda^2}$$

$$\frac{1}{\lambda} + 8\lambda^2 B = \frac{1}{\lambda} \Longrightarrow B = 0$$

Dunque

$$y = e^{-2\lambda x}\left[C_1 \cos 2\lambda x + C_2 \sin 2\lambda x\right] + \frac{x}{4\lambda^2}$$

Imponendo le condizioni ai limiti

$$\begin{cases} C_1 = 0 \\ e^{-2\lambda\pi}\left[C_1 \cos 2\lambda\pi + C_2 \sin 2\lambda\pi\right] = \frac{\pi}{4} - \frac{\pi}{4\lambda^2} \end{cases}$$

$$\begin{cases} C_1 = 0 \\ C_1 \cos 2\lambda\pi + C_2 \sin 2\lambda\pi = \frac{\pi}{4}\left(1 - \frac{1}{\lambda^2}\right) e^{2\lambda\pi} \end{cases}$$

$$\Delta(\lambda) = \begin{vmatrix} 1 & 0 \\ \cos 2\lambda\pi & \sin 2\lambda\pi \end{vmatrix} = \sin 2\lambda\pi$$

Pertanto, se $\sin 2\lambda\pi \neq 0$, ossia $2\lambda\pi \neq k\pi$, $\lambda \neq \frac{k}{2}$ $(k \neq 0)$ la soluzione esiste unica. Abbiamo

$$C_1 = 0 \qquad \left(\lambda \neq \frac{k}{2}, \quad k \in \mathbb{Z}\right)$$

$$C_2 = \frac{\frac{\pi}{4}\,\frac{\lambda^2 - 1}{\lambda^2}\, e^{2\lambda\pi}}{\sin 2\lambda\pi}$$

Se $\lambda = \frac{k}{2}$ otteniamo

$$\begin{cases} C_1 = 0 \\ C_1 \cos k\pi + C_2 \sin k\pi = \frac{\pi}{4}\left(1 - \frac{4}{k^2}\right) e^{k\pi} \end{cases}$$

$$\begin{cases} C_1 = 0 \\ (-1)^k C_1 = \frac{\pi}{4} \frac{k^2-4}{k^2} e^{k\pi} \end{cases}$$

Pertanto, se $k = \pm 2$ abbiamo infinite soluzioni

$$\begin{cases} C_1 = 0 \\ C_2 \text{ arbitrario} \end{cases}$$

Se $k \in \mathbb{Z}$, ma $k \neq \pm 2$ il sistema è impossibile

---

(3) $f = (2x + y)^2$

$g = \frac{x^2}{9} + \frac{y^2}{16} - 1 = 0$

Parametrizziamo il vincolo

$$\begin{cases} x = 3 \cos t \\ y = 4 \sin t \end{cases} \qquad t \in [0, 2\pi]$$

Corrispondentemente

$f(t) = (6 \cos t + 4 \sin t)^2 \qquad t \in [0, 2\pi]$

$f'(t) = 2(6 \cos t + 4 \sin t)(-6 \sin t + 4 \cos t)$

$f'(t) = 0 \qquad (6 \cos t + 4 \sin t)(-6 \sin t + 4 \cos t) = 0$

$6 \cos t + 4 \sin t = 0 \qquad \operatorname{tg} t = -\frac{3}{2}$

$$\begin{cases} \sin t = -3/\sqrt{13} \\ \cos t = 2/\sqrt{13} \end{cases} \quad \text{oppure} \quad \begin{cases} \sin t = 3/\sqrt{13} \\ \cos t = -2/\sqrt{13} \end{cases}$$

$\Downarrow \qquad\qquad \Downarrow$

$A\left(\frac{6}{\sqrt{13}}, -\frac{12}{\sqrt{13}}\right) \qquad B\left(-\frac{6}{\sqrt{13}}, \frac{12}{\sqrt{13}}\right)$

In corrispondenza $f = 0$ che è il minimo assoluto.

Da

$$-6\sin t + 4\cos t = 0$$

$$tg\, t = \frac{2}{3}$$

$$\begin{cases} \sin t = 2/\sqrt{13} \\ \cos t = 3/\sqrt{13} \end{cases} \quad \text{oppure} \quad \begin{cases} \sin t = -2/\sqrt{13} \\ \cos t = -3/\sqrt{13} \end{cases}$$

In corrispondenza $f = \left(2\cdot\frac{9}{\sqrt{13}} + \frac{8}{\sqrt{13}}\right)^2 = \left(-\frac{18}{\sqrt{13}} - \frac{8}{\sqrt{13}}\right)^2$

$= \frac{1}{13}676$ che è il massimo assoluto.

---

(4) $2z^3 + y^3 + x^4 - yz - 2z = 0$

$$f = 2z^3 + y^3 + x^4 - yz - 2z \qquad f \in C^\infty(\mathbb{R}^3)$$

Inoltre

$$f(0,0,1) = 2 - 2 = 0$$

$$f_x = 4x^3 \qquad f_x(0,0,1) = 0$$

$$f_y = 3y^2 - z \qquad f_y(0,0,1) = -1$$

$$f_z = 6z^2 - y - 2 \qquad f_z(0,0,1) = 4$$

Pertanto il Teorema di Dini garantisce l'univoca risolubilità rispetto a z. Il piano tangente ha equazione

$$-y + 4(z-1) = 0$$

Inoltre il vettore $(f_x(0,0,1), f_y(0,0,1), f_z(0,0,1)) = (0,-1,4)$ ha la direzione della normale. Pertanto la retta normale a Σ in P ha equazione

$$\begin{cases} x = 0 \\ y = -t \\ z = 1 + 4t \end{cases} \qquad t \in \mathbb{R}.$$