(1) $\begin{cases} x = \sqrt{\vartheta - \vartheta^2}\cos\vartheta \\ y = \sqrt{\vartheta - \vartheta^2}\sin\vartheta \end{cases}$

$$dx = \left(\frac{1-2\vartheta}{2\sqrt{\vartheta-\vartheta^2}}\cos\vartheta - \sqrt{\vartheta-\vartheta^2}\sin\vartheta\right)d\vartheta$$

$$dy = \left(\frac{1-2\vartheta}{2\sqrt{\vartheta-\vartheta^2}}\sin\vartheta + \sqrt{\vartheta-\vartheta^2}\cos\vartheta\right)d\vartheta$$

$$\int_\Gamma x\,dy - y\,dx = \int_0^1 \left\{\left[\frac{1-2\vartheta}{2}\sin\vartheta\cos\vartheta + (\vartheta-\vartheta^2)\cos^2\vartheta\right] + \right.$$

$$\left. + \left[-\frac{1-2\vartheta}{2}\cos\vartheta\sin\vartheta + (\vartheta-\vartheta^2)\sin^2\vartheta\right]\right\}d\vartheta = \int_0^1 (\vartheta-\vartheta^2)\,d\vartheta =$$

$$= \left[\frac{\vartheta^2}{2} - \frac{\vartheta^3}{3}\right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

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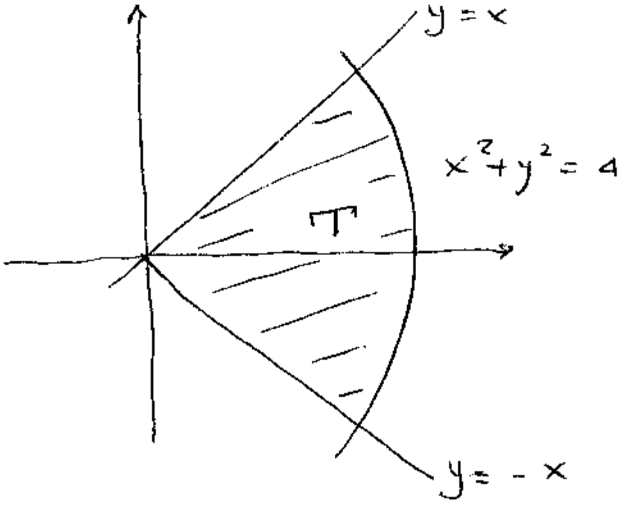
(2) $$d\sigma_2 = \sqrt{1 + z_x^2 + z_y^2}\,dx\,dy = \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy$$



$$\int_\Sigma y^2\,d\sigma_2 =$$

$$= \int_T y^2 \sqrt{1 + 4x^2 + 4y^2}\,dx\,dy$$

$T: \begin{cases} 0 \le \rho \le 2 \\ -\frac{\pi}{4} \le \vartheta \le \frac{\pi}{4} \end{cases}$

$$= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\vartheta \int_0^2 \rho^2 \sin^2\vartheta \sqrt{1 + 4\rho^2}\,\rho\,d\rho$$

$$= \left(\begin{array}{ll} 1 + 4\rho^2 = s^2 & \\ 8\rho\,d\rho = 2s\,ds & \\ \rho = 0 & s = 1 \\ \rho = 2 & s = \sqrt{17} \end{array}\right) = \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \sin^2\vartheta\,d\vartheta \int_1^{\sqrt{17}} \frac{s^2-1}{4}\cdot s\,\frac{s}{4}\,ds$$

$$= \frac{1}{16} \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1-\cos 2\vartheta}{2} \, d\vartheta \int_1^{\sqrt{17}} (s^4 - s^2) ds$$

$$= \frac{1}{32} \left[\vartheta - \frac{\sin 2\vartheta}{2} \right]_{-\pi/4}^{\pi/4} \left[\frac{s^5}{5} - \frac{s^3}{3} \right]_1^{\sqrt{17}}$$

---

③ Si tratta di una equazione a variabili separabili
Quindi

$$\int e^{-y} dy = \int x \, dx$$

$$-e^{-y} = \frac{x^2}{2} + C \qquad : \text{ integrale generale}$$

Imponendo la condizione iniziale abbiamo

$$-1 = C$$

Dunque

$$-e^{-y} = -1 + \frac{x^2}{2}$$

$$e^{-y} = 1 - \frac{x^2}{2}$$

$$-y = \ln\left(1 - \frac{x^2}{2}\right)$$

$$y = -\ln\left(1 - \frac{x^2}{2}\right)$$

---

④ Se consideriamo la retta $y = 0$ abbiamo

$$f|_{y=0} = 0$$

Se consideriamo la retta $y = x$ abbiamo

$$f\Big|_{y=x} = \frac{x}{2x} = \frac{1}{2}$$

Poiché su due direzioni diverse la funzione assume valori costanti e diversi, f non può essere continua

---

(5) $$\sum_{n=0}^{\infty} \frac{x^{3n+4}}{n+1} = \sum_{n=0}^{\infty} \frac{x^{3n+3}}{n+1} \cdot x =$$

$$= x \sum_{n=0}^{\infty} \frac{(x^3)^{n+1}}{n+1}$$

Se poniamo $x^3 = t$, siamo ricondotti allo studio della serie di potenze $\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1}$. Si verifica facilmente che la serie converge in $I = [-1, 1)$ e quindi, ritornando alla variabile x, avremo il medesimo intervallo di convergenza.

Inoltre

$$\sum_{n=0}^{\infty} \frac{t^{n+1}}{n+1} = \sum_{n=0}^{\infty} \int_0^t s^n \, ds = \int_0^t \sum_{n=0}^{\infty} s^n ds$$

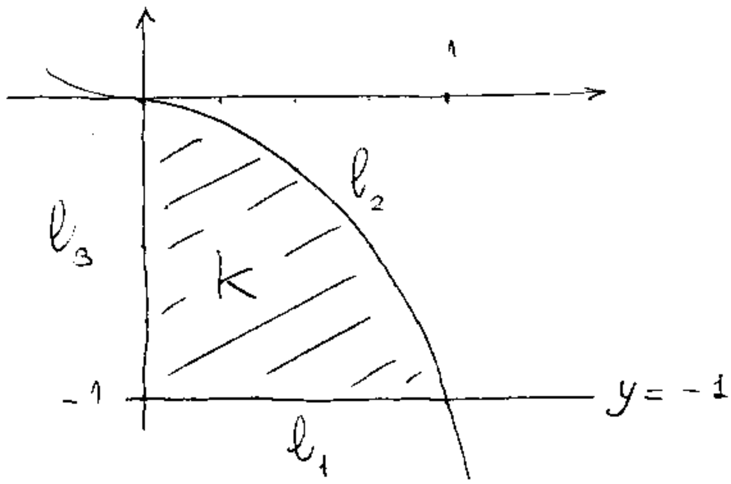
$$= \int_0^t \frac{1}{1-s} ds = - \ln(1-s)\Big|_0^t$$

$$= - \ln(1-t)$$

Risostituendo, concludiamo che $\forall\, x \in [-1, 1)$ risulta

$$\sum_{n=0}^{\infty} \frac{x^{3n+4}}{n+1} = -x \ln(1-x^3).$$

(6) $f = x^3 - y^3$



$\underline{CN}\ \begin{cases} \frac{\partial f}{\partial x} = 0 \\ \frac{\partial f}{\partial y} = 0 \end{cases}$

$\begin{cases} 3x^2 = 0 \\ -3y^2 = 0 \end{cases} \implies O(0,0)$

Poiché $O \in \partial K$, non abbiamo massimi e/o minimi relativi liberi e, dunque, gli estremi assoluti saranno assunti sul bordo. Abbiamo

$\ell_1 : \quad y = -1 \qquad 0 \leq x \leq 1$

$\ell_2 : \quad y = -x^3 \qquad 0 \leq x \leq 1$

$\ell_3 \quad \cancel{y}\, x = 0 \qquad -1 \leq y \leq 0$

$f|_{\ell_1} = x^3 + 1 \qquad m_1 = 1 \qquad M_1 = 2$

$f|_{\ell_3} = -y^3 \qquad m_3 = 0 \qquad M_3 = 1$

$f|_{\ell_2} = x^3 + x^9 = g(x) \qquad x \in [0, 1]$

$g'(x) = 3x^2 + 9x^8 = 3x^2(1 + 3x^6)$

Poiché $g' \geq 0 \quad \forall x \in \mathbb{R}$ concludiamo che

$m_2 = 0 \qquad M_2 = 2$

Dunque $\quad M_{ass} = 2 \quad , \quad m_{ass} = 0$

---

(7) $\quad e^{x-y+z} - \cos(x+y-z) - \sin(x-y-z) = 0$

$f = e^{x-y+z} - \cos(x+y-z) - \sin(x-y-z) \in C^\infty(\mathbb{R}^3)$

Inoltre

$$f(0,0,0) = e^0 - \cos 0 - \sin 0 = 1 - 1 = 0$$

$$f_z = e^{x-y+z} - \sin(x+y-z) + \cos(x-y-z)$$

$$f_z(0,0,0) = e^0 + \cos 0 = 2$$

Infine

$$f_x = e^{x-y+z} + \sin(x+y-z) - \cos(x-y-z)$$

$$f_y = -e^{x-y+z} + \sin(x+y-z) + \cos(x-y-z)$$

Le ipotesi del Teo di Dini sono tutte verificate e possiamo quindi concludere che esistono $B_r(0,0)$, $J_{r'}(0)$ e $g: B_r(0,0) \to J_r(0)$ con $g \in C^\infty(B_r(0,0))$, tale che

$$g(0,0) = 0$$
$$f(x, y, g(x,y)) \equiv 0 \qquad \forall (x,y) \in B_r(0,0).$$

Quanto al piano tangente, abbiamo

$$\pi_{tg}: \quad f_x(0,0,0)\,x + f_y(0,0,0)\,y + f_z(0,0,0)\,z = 0$$

Poiché $f_x(0,0,0) = f_y(0,0,0) = 0$, abbiamo

$$\pi_{tg}: \quad z = 0$$

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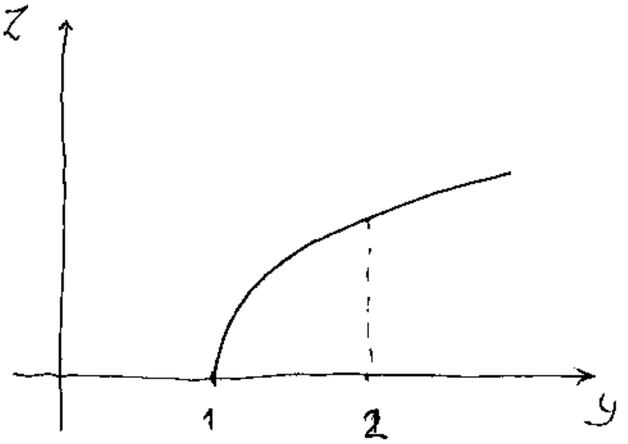
(8)



$$z = \sqrt{y-1} \qquad y \in [1,2]$$

$$y = z^2 + 1 \qquad z \in [0,1]$$

Dunque

$$r(z) = z^2 + 1$$

Quindi

$$V = \int_0^1 A(z)\,dz = \pi \int_0^1 r^2(z)\,dz$$

$$= \pi \int_0^1 (z^2+1)^2\,dz = \pi \int_0^1 (z^4+2z^2+1)\,dz$$

$$= \pi \left[\frac{z^5}{5} + \frac{2}{3} z^3 + z \right]_0^1 = \pi \left[\frac{1}{5} + \frac{2}{3} + 1 \right].$$